Extreme Value Analysis of the Haezendonck–Goovaerts Risk Measure with a General Young Function

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Abstract

For a risk variable X and a normalized Young function $\varphi(\cdot)$, the Haezendonck–Goovaerts risk measure for X at level $q \in (0, 1)$ is defined as $H_q[X] = \inf_{x \in \mathbb{R}} (x + h)$, where h solves the equation $\mathrm{E}\left[\varphi\left((X - x)_+/h\right)\right] = 1 - q$ if $\Pr(X > x) > 0$ or is 0 otherwise. In a recent work, we implemented an asymptotic analysis for $H_q[X]$ with a power Young function for the Fréchet, Weibull and Gumbel cases separately. A key point of the implementation was that h can be explicitly solved for fixed x and q, which gave rise to the possibility to express $H_q[X]$ in terms of x and q. For a general Young function, however, this does not work anymore and the problem becomes a lot harder. In the present paper, we extend the asymptotic analysis for $H_q[X]$ to the case with a general Young function and we establish a unified approach for the three extreme value cases. In doing so, we overcome several technical difficulties mainly due to the intricate relationship between the working variables x, h and q.

Keywords: asymptotics; (extended) regular variation; Haezendonck–Goovaerts risk measure; max-domain of attraction; Young function

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1   Introduction

Let X be a real-valued random variable, representing a risk variable in loss-profit style, with a distribution function $F = 1 - \overline{F}$ on $\mathbb{R} = (-\infty, \infty)$. Let $\varphi(\cdot)$ be a normalized Young

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function; that is, $\varphi(\cdot)$ is a nonnegative and convex function on $\mathbb{R}_+ = [0, \infty)$ with $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(\infty) = \infty$. Due to its convexity, the Young function $\varphi(\cdot)$ is continuous and strictly increasing on $\{t \in \mathbb{R}_+ : \varphi(t) > 0\}$. See Krasnosel'skiĭ and Rutickiĭ (1961) and Neveu (1975) for related discussions of Young functions. Recall that the Orlicz heart associated with $\varphi(\cdot)$ is defined as

$$L_0^\varphi = \{X : \mathrm{E}\left[\varphi(c\,|X|)\right] < \infty \text{ for all } c > 0\};$$

see, e.g. page 77 of Rao and Ren (1991). By the convexity of $\varphi(\cdot)$ we know that L_0^φ is a convex set. Thus, for $X \in L_0^\varphi$, the expectation $\mathrm{E}\left[\varphi\left((X-x)_+/h\right)\right]$, which frequently appears in the sequel, is finite for every $x \in \mathbb{R}$ and $h > 0$. Here, and throughout the paper, for a real number x we write $x_+ = x1_{(x \geq 0)}$ for its positive part, with 1_A denoting the indicator of an event A.

For a Young function $\varphi(\cdot)$ and a risk variable $X \in L_0^\varphi$, let $h = h(x, q)$ be the unique solution to the equation

$$\mathrm{E}\left[\varphi\left(\frac{(X-x)_+}{h}\right)\right] = 1 - q, \qquad q \in (0, 1), \tag{1.1}$$

if $\overline{F}(x) > 0$ and let $h(x, q) = 0$ if $\overline{F}(x) = 0$. For $q \in (0, 1)$, the Haezendonck–Goovaerts (HG) risk measure for X is defined as

$$H_q[X] = \inf_{x \in \mathbb{R}} (x + h(x, q)) = x_* + h_*, \tag{1.2}$$

where (x_*, h_*) denotes the minimizer. By Proposition 3(b,d) of Bellini and Rosazza Gianin (2012), the minimizer (x_*, h_*) always exists for all $q \in (0, 1)$, and it is unique if $\varphi(\cdot)$ is strictly convex. In particular, the minimizer x_* is called the Orlicz quantile of X by them.

This risk measure was first introduced by Haezendonck and Goovaerts (1982) based on the Swiss premium calculation principle induced by the Orlicz norm and was revisited by Goovaerts et al. (2004). Since recently, it has attracted increasing attention from researchers; see Bellini and Rosazza Gianin (2008a, 2008b, 2012), Krätschmer and Zähle (2011), Nam et al. (2011), Goovaerts et al. (2012), Mao and Hu (2012), Tang and Yang (2012), Cheung and Lo (2013) and Ahn and Shyamalkumar (2014), among others. As pointed out by Bellini and Rosazza Gianin (2012), the HG risk measure is a law invariant and coherent risk measure for a convex Young function $\varphi(\cdot)$. The simplest case is when $\varphi(s) = s_+$, reducing the HG risk measure to the well-known Tail Value-at-Risk.

In the recent work of Tang and Yang (2012), we implemented an asymptotic analysis for $H_q[X]$ with a power Young function for the Fréchet, Weibull and Gumbel cases separately. The assumption of a power Young function essentially simplified the problem. Precisely speaking, through equation (1.1) the variable h can be expressed in terms of x and q. Plugging this expression into (1.2), the computation of the HG risk measure reduced to a one-variable minimization problem.

The aim of this paper is to extend the asymptotic study of the HG risk measure to a general Young function. The aimed extension leads to a lot of technical difficulties. For a general Young function, an explicit formula for h in terms of x and q cannot be deduced from (1.1) anymore, and subsequently the approach of Tang and Yang (2012) fails. In the present paper, we apply extreme value theory to derive asymptotics for the HG risk measure at a high confidence level for the risk variable X following a distribution function from the max-domain of attraction of the generalized extreme value distribution, namely $F \in \mathrm{MDA}(G_\gamma)$. Besides a general Young function, we chase a unified treatment for the Fréchet, Gumbel and Weibull cases.

Throughout this paper, denote by $\hat{x} \leq \infty$ the upper endpoint of the risk variable X. As discussed in Section 1 of Tang and Yang (2012), we only need to consider the non-trivial case that $\Pr(X = \hat{x}) = 0$. Coincidently, this is implied by our assumption that $F \in \mathrm{MDA}(G_\gamma)$. Recalling (1.2), for the power Young function case the equivalence between the limits $x_* \to \hat{x}$ and $q \uparrow 1$ easily follows from Lemma 2.2 of Tang and Yang (2012). In the present general Young function case, we shall prove that $x_* \to \hat{x}$ still holds as $q \uparrow 1$, which is a key step for later derivations. Another key step of the present work is to prove that, for a distribution function $F \in \mathrm{MDA}(G_\gamma)$ with an auxiliary function $a(\cdot)$, the ratio of $a(1/\overline{F}(x_*))$ and h_* converges to a positive constant as $q \uparrow 1$. This limit of the ratio eventually reduces the asymptotic analysis of the HG risk measure to a one-variable minimization problem again.

The rest of the paper consists of six sections. Section 2 prepares some preliminaries on regular variation, extended regular variation, and the max-domain of attraction of the generalized extreme value distribution. In Section 3 we establish a general result for the HG risk measure with a general Young function. This result is the starting point of the later asymptotic analysis. Sections 4–5 focus on the relationships between the three working variables x, h and q through the auxiliary function $a(\cdot)$, which are key steps in the proof of the main result. Section 6 shows the main result, which represents a unified asymptotic solution of the HG risk measure with a general Young function for the Fréchet, Gumbel and Weibull cases. Finally, Section 7 numerically examines the accuracy of the asymptotic formulas obtained in the paper.

2 Preliminaries

2.1 Notational conventions

For two positive functions $f(\cdot)$ and $g(\cdot)$, we write $f(\cdot) \sim g(\cdot)$ if the ratio of the left-hand side (LHS) and right-hand side (RHS) converges to 1; that is, $\lim f(\cdot)/g(\cdot) = 1$. We also write $f(\cdot) \lesssim g(\cdot)$ if $\limsup f(\cdot)/g(\cdot) \leq 1$ and write $f(\cdot) \asymp g(\cdot)$ if $0 < \liminf f(\cdot)/g(\cdot) \leq \limsup f(\cdot)/g(\cdot) < \infty$.

By saying that ε is a small positive number we mean that ε is an arbitrarily small but

fixed positive number. For example, with this understanding, if $a < b$ then the inequality $a + \varepsilon < b$ will be used without explanation.

We often use the letter C to denote an absolute positive constant, which does not depend on the working variables such as x, h and q, but which may vary from place to place so that relations such as $C + 1 = C$, $2C = C$ and $C/C = C$ make sense.

The càglàd inverse function of a non-decreasing function $f(\cdot)$ on $\mathbb{R}$ is denoted by $f^{\leftarrow}(y) = \inf\{x \in \mathbb{R} : f(x) \geq y\}$ for $y \in \mathbb{R}$, where we follow the convention that $\inf \emptyset = +\infty$. As usual, the left and right derivatives of a function $f(\cdot)$ on $\mathbb{R}$, which exist if $f(\cdot)$ is convex, are denoted by $f'_-(\cdot)$ and $f'_+(\cdot)$, respectively.

2.2 Regular variation

A positive measurable function $f(\cdot)$ is said to be regularly varying at $t_0 = 0+$ or ∞ with index $\alpha \in \mathbb{R}$, denoted by $f(\cdot) \in \mathrm{RV}_\alpha(t_0)$, if

$$\lim_{t \to t_0} \frac{f(st)}{f(t)} = s^\alpha, \qquad s > 0.$$

The class $\mathrm{RV}_0(t_0)$ consists of functions slowly varying at t_0. Moreover, a positive measurable function $f(\cdot)$ is said to be rapidly varying at $t_0 = 0+$ or ∞, denoted by $f(\cdot) \in \mathrm{RV}_\infty(t_0)$ or $1/f(\cdot) \in \mathrm{RV}_{-\infty}(t_0)$, if

$$\lim_{t \to t_0} \frac{f(st)}{f(t)} = \infty, \qquad s > 1.$$

The following Potter's bounds for regularly varying functions are well known; see Theorem 1.5.6 of Bingham et al. (1987):

Lemma 2.1 *Let $f(\cdot) \in \mathrm{RV}_\alpha(t_0)$ with $t_0 = 0+$ or ∞ and $\alpha \in \mathbb{R}$. Then for arbitrarily small $0 < \varepsilon < 1$ and all s, t such that both t and st are sufficiently close to t_0, it holds that*

$$(1 - \varepsilon)\left(s^{\alpha+\varepsilon} \wedge s^{\alpha-\varepsilon}\right) \leq \frac{f(st)}{f(t)} \leq (1 + \varepsilon)\left(s^{\alpha+\varepsilon} \vee s^{\alpha-\varepsilon}\right).$$

Fixing t to some constant in the above yields the following: for arbitrarily fixed $\delta > 0$, there is some large constant $C > 0$ such that the inequalities

$$\frac{1}{C}\left(s^{\alpha+\varepsilon} \wedge s^{\alpha-\varepsilon}\right) \leq f(s) \leq C\left(s^{\alpha+\varepsilon} \vee s^{\alpha-\varepsilon}\right) \tag{2.1}$$

hold uniformly for all $0 < s \leq 1/\delta$ provided $t_0 = 0+$ or for all $s > \delta$ provided $t_0 = \infty$.

The following lemma is a restatement of Proposition 0.8(V) of Resnick (1987):

Lemma 2.2 *Let $f(\cdot)$ be a non-decreasing function on $\mathbb{R}_+$ with $f(\infty) = \infty$. Then $f(\cdot) \in \mathrm{RV}_\alpha(\infty)$ with $\alpha \in [0, \infty]$ if and only if $f^{\leftarrow}(\cdot) \in \mathrm{RV}_{1/\alpha}(\infty)$, where we follow the conventions $1/0 = \infty$ and $1/\infty = 0$.*

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2.3 Extended regular variation

The concept of extended regular variation is useful for our unified derivation. By definition, a positive measurable function $f(\cdot)$ is said to be extended regularly varying at ∞ with index $\gamma \in \mathbb{R}$, denoted by $f(\cdot) \in \mathrm{ERV}_\gamma$, if there is an auxiliary function $a(\cdot) > 0$ such that, for all $s > 0$,

$$\lim_{t\to\infty} \frac{f(st) - f(t)}{a(t)} = \frac{s^\gamma - 1}{\gamma}, \tag{2.2}$$

where the RHS is interpreted as $\log s$ when $\gamma = 0$. The auxiliary function $a(\cdot)$ is often chosen to be

$$a(t) = \begin{cases} \gamma f(t), & \gamma > 0, \\ f(t) - t^{-1} \int_0^t f(u)\mathrm{d}u, & \gamma = 0, \\ -\gamma(f(\infty) - f(t)), & \gamma < 0. \end{cases} \tag{2.3}$$

Note that, for $\gamma = 0$, as $t \to \infty$, we have $a(t) = o(f(t))$ provided $f(\infty) = \infty$ while $a(t) = o(f(\infty) - f(t))$ provided $f(\infty) < \infty$. See Appendix B of de Haan and Ferreira (2006) for more discussions on extended regular variation.

The following lemma is a restatement of Theorem B.2.18 of de Haan and Ferreira (2006), originally attributed to Drees (1998):

Lemma 2.3 *Let $f(\cdot) \in \mathrm{ERV}_\gamma$ for $\gamma \in \mathbb{R}$, namely, relation (2.2) holds for all $s > 0$ and for an auxiliary function $a(\cdot)$ given in (2.3). It holds for every small $\varepsilon, \delta > 0$, some $t_0 = t_0(\varepsilon, \delta) > 0$ and all s, t with $t > t_0$, $st > t_0$ that*

$$\left| \frac{f(st) - f(t)}{a(t)} - \frac{s^\gamma - 1}{\gamma} \right| \leq \varepsilon \left(s^{\gamma+\delta} \vee s^{\gamma-\delta} \right). \tag{2.4}$$

Taking supremum of both sides of inequality (2.4) with respect to s over a small open interval containing 1, and then letting the interval boil down to the point 1, we can easily prove that

$$\lim_{t\to\infty} \frac{f(t \pm 0) - f(t)}{a(t)} = 0.$$

It follows from this and (2.2) that, for all $s > 0$,

$$\lim_{t\to\infty} \frac{f(st) - f(t \pm 0)}{a(t)} = \frac{s^\gamma - 1}{\gamma}. \tag{2.5}$$

2.4 Max-domain of attraction

A distribution function F is said to belong to the max-domain of attraction of a non-degenerate distribution function G, denoted by $F \in \mathrm{MDA}(G)$, if for a simple sample of size n from F, its normalized maximum has a distribution weakly converging to G. The classical Fisher–Tippett theorem, attributed to Fisher and Tippett (1928) and Gnedenko

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(1943), states that G has to be the generalized extreme value distribution whose standard version is given by

$$G_\gamma(t) = \exp\left\{-(1+\gamma t)^{-1/\gamma}\right\}, \qquad \gamma \in \mathbb{R}, 1+\gamma t > 0,$$

where the RHS is interpreted as $\exp\{-e^{-t}\}$ when $\gamma = 0$. The regions $\gamma > 0$, $\gamma = 0$ and $\gamma < 0$ correspond to the Fréchet, Gumbel and Weibull cases, respectively.

It is well known that $F \in \text{MDA}(G_\gamma)$ with $\gamma > 0$ if and only if $\overline{F}(\cdot) \in \text{RV}_{-1/\gamma}(\infty)$, while $F \in \text{MDA}(G_\gamma)$ with $\gamma < 0$ if and only if $\hat{x} < \infty$ and $\overline{F}(\hat{x} - \cdot) \in \text{RV}_{-1/\gamma}(0+)$. Moreover, for $F \in \text{MDA}(G_0)$, we have $\overline{F}(\cdot) \in \text{RV}_{-\infty}(\infty)$ provided $\hat{x} = \infty$ or $\overline{F}(\hat{x} - \cdot) \in \text{RV}_\infty(0+)$ provided $\hat{x} < \infty$. Hence, if $\gamma \leq 0$ then $\text{E}[X_+^p] < \infty$ for all $p > 0$, while if $\gamma > 0$ then $\text{E}[X_+^p] < \infty$ for all $0 < p < 1/\gamma$. See, e.g. Chapter 3 of Embrechts et al. (1997) for these statements.

The MDA of the generalized extreme value distribution is related to extended regular variation through the function

$$U(t) = \left(\frac{1}{\overline{F}}\right)^{\leftarrow}(t) = F^{\leftarrow}\left(1 - \frac{1}{t}\right), \qquad t > 1. \tag{2.6}$$

We have $F \in \text{MDA}(G_\gamma)$ if and only if $U(\cdot) \in \text{ERV}_\gamma$ with the auxiliary function $a(\cdot)$ given in (2.3) in terms of $U(\cdot)$; see Theorem 1.1.6 of de Haan and Ferreira (2006).

3   A General Treatment

Recall that $h = h(x, q)$ is the unique solution to equation (1.1) if $x < \hat{x}$. In what follows, we often write $h(x) = h(x, q)$ if q is fixed or write $h(q) = h(x, q)$ if x is fixed, as long as doing so causes no confusion. As usual, we say that $\varphi(\cdot)$ is differentiable over $\mathbb{R}_+$ if $\varphi(\cdot)$ is differentiable over $(0, \infty)$ and $\varphi'_+(0)$ exists.

Theorem 3.1 *Let $\varphi(\cdot)$ be strictly convex and continuously differentiable over $\mathbb{R}_+$ with $\varphi'_+(0) = 0$, and let $X \in L_0^\varphi$. Then the HG risk measure at level $q \in (0, 1)$ is equal to $H_q[X] = x_* + h_*$, where the minimizer (x_*, h_*) solves (1.1), that is,*

$$\text{E}\left[\varphi\left(\frac{(X-x)_+}{h}\right)\right] = 1 - q, \tag{1.1}$$

and

$$\text{E}\left[\varphi'\left(\frac{(X-x)_+}{h}\right)\right] = \text{E}\left[\varphi'\left(\frac{(X-x)_+}{h}\right)\frac{(X-x)_+}{h}\right]. \tag{3.1}$$

To prove Theorem 3.1, we first prepare several lemmas, some of which are interesting in their own right. The first lemma below can easily be retrieved from the proofs of Proposition 1(c) and Corollary 3 of Bellini et al. (2014); a special case is found in Lemma 2.1(a) of Tang and Yang (2012).

Lemma 3.1 *Let $X \in L_0^{\varphi}$ and define $g(x) = \mathrm{E}\left[\varphi\left((X - x)_+\right)\right]$. Then, for every $x \in \mathbb{R}$,*

$$g'_+(x) = -\mathrm{E}\left[\varphi'_-(X - x)1_{(X>x)}\right] \qquad and \qquad g'_-(x) = -\mathrm{E}\left[\varphi'_+(X - x)1_{(X \geq x)}\right].$$

In particular, if $\varphi(\cdot)$ is continuously differentiable over $\mathbb{R}_+$, and either $\varphi'_+(0) = 0$ or F is continuous at x, then $g'(x) = -\mathrm{E}\left[\varphi'(X - x)1_{(X>x)}\right]$.

The next lemma has a similar flavor and can be proved similarly. We show its proof here for completeness.

Lemma 3.2 *Let $X \in L_0^{\varphi}$ and define $g(s) = \mathrm{E}\left[\varphi\left(sX_+\right)\right]$. Then, for every $s > 0$,*

$$g'_+(s) = \mathrm{E}\left[\varphi'_+\left(sX_+\right)X_+\right] \qquad and \qquad g'_-(s) = \mathrm{E}\left[\varphi'_-\left(sX_+\right)X_+\right].$$

Proof. If $\Pr(X \leq 0) = 1$, then both results are trivial. Therefore, we assume that X has a non-trivial positive part. In this case, denote by $\tilde{X}$ the part of X restricted to $X > 0$; that is, $\tilde{X} = X | X > 0$. We have

$$g(s) = \mathrm{E}\left[\varphi\left(sX\right)1_{(X>0)}\right] = \overline{F}(0)\mathrm{E}\left[\varphi\left(s\tilde{X}\right)\right].$$

By the convexity of $\varphi(\cdot)$ over $\mathbb{R}_+$, it holds for both $0 < \Delta s \leq 1$ and $-s < \Delta s < 0$ that

$$0 \leq \frac{g(s + \Delta s) - g(s)}{\Delta s} \leq \overline{F}(0)\left(\mathrm{E}\left[\varphi\left((s+1)\tilde{X}\right)\right] - \mathrm{E}\left[\varphi\left(s\tilde{X}\right)\right]\right) < \infty.$$

Thus, when taking $\Delta s \downarrow 0$ or $\Delta s \uparrow 0$ to the relation

$$\frac{g(s + \Delta s) - g(s)}{\Delta s} = \overline{F}(0)\mathrm{E}\left[\frac{\varphi\left((s + \Delta s)\tilde{X}\right) - \varphi\left(s\tilde{X}\right)}{\Delta s}\right],$$

we can apply the dominated convergence theorem to interchange the order of the limit and expectation on the right. Hence, we obtain the two expressions for $g'_+(s)$ and $g'_-(s)$. This proof also shows their finiteness. ∎

Lemma 3.3 *Let $\varphi(\cdot)$ be strictly convex and continuously differentiable over $\mathbb{R}_+$, and let $X \in L_0^{\varphi}$. Denote by $h = h(x)$ the implicit function of $x \in (-\infty, \hat{x})$ determined by equation (1.1) with $q \in (0, 1)$ fixed. Then $h(x)$ is continuously differentiable over $(-\infty, \hat{x})$ if either $\varphi'_+(0) = 0$ or F is continuous.*

Proof. We apply the well-known implicit function theorem. Denote by $g(x, h)$ the LHS of (1.1); that is,

$$g(x, h) = \mathrm{E}\left[\varphi\left(\frac{(X - x)_+}{h}\right)\right].$$

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It suffices to check that, over $(-\infty, \hat{x}) \times (0, \infty)$, the bivariate function $g(x, h)$ is continuously differentiable with respect to both x and h and that $\partial g / \partial h \neq 0$. By Lemmas 3.1 and 3.2, for $(x, h) \in (-\infty, \hat{x}) \times (0, \infty)$ we have

$$\frac{\partial}{\partial x} g(x, h) = -\frac{1}{h} \mathrm{E}\left[\varphi'\left(\frac{X - x}{h}\right) 1_{(X > x)}\right] \tag{3.2}$$

and

$$\frac{\partial}{\partial h} g(x, h) = -\mathrm{E}\left[\varphi'\left(\frac{X - x}{h}\right) \frac{X - x}{h^2} 1_{(X > x)}\right]. \tag{3.3}$$

Since $\varphi'(\cdot)$ is continuous and positive over $(0, \infty)$, it is easy to see that both $\partial g / \partial x$ and $\partial g / \partial h$ obtained above are continuous in both x and h and that $\partial g / \partial h$ is negative over $(-\infty, \hat{x}) \times (0, \infty)$. ∎

Proof of Theorem 3.1. By Lemma 3.3, $h(x)$ is continuously differentiable over $(-\infty, \hat{x})$. Thus, for the solution (x, h) to the minimization problem (1.2) subject to (1.1), we have

$$h'(x) = -1. \tag{3.4}$$

Differentiating both sides of (1.1) with respect to x, applying (3.2), (3.3) and (3.4), and noticing that $1 - q$ on the RHS of (1.1) is independent of x, we obtain

$$-\frac{1}{h} \mathrm{E}\left[\varphi'\left(\frac{X - x}{h}\right) 1_{(X > x)}\right] + \mathrm{E}\left[\varphi'\left(\frac{X - x}{h}\right) \frac{X - x}{h^2} 1_{(X > x)}\right] = 0.$$

This gives (3.1) since $\varphi'_+(0) = 0$. ∎

Theorem 3.1 is the starting point of our derivations. Slightly different versions of this result can be found in Remark 7 of Bellini and Rosazza Gianin (2012) and the proof of Corollary 1 of Ahn and Shyamalkumar (2014). By the way, the finiteness of the expectations appearing in Theorem 3.1 is implied by the proofs of Lemmas 3.1 and 3.2.

4 The Limit of the Orlicz Quantile

Through the system of equations (1.1) and (3.1), both minimizers x_* and h_* appearing in (1.2) are implicit functions of q. The next lemma reveals a general limit behavior for the pair (x_*, h_*) as $q \uparrow 1$:

Lemma 4.1 *Let $\varphi(\cdot)$ be convex over $\mathbb{R}_+$ and let $X \in L_0^\varphi$. Then, as $q \uparrow 1$, it holds for every $\delta > 0$ that $x_* + \delta h_* \to \hat{x}$.*

Proof. Let $0 < \delta < 1$ be arbitrarily fixed (hence, $0 < \varphi(\delta) < 1$). Define $\tilde{\varphi}(s) = \varphi(\delta s)/\varphi(\delta)$, which is still a normalized Young function. Since (x_*, h_*) solves (1.1), we have

$$\mathrm{E}\left[\tilde{\varphi}\left(\frac{(X - x_*)_+}{\delta h_*}\right)\right] = \frac{1 - q}{\varphi(\delta)}.$$

Let q be close to 1 such that $0 < (1 - q)/\varphi(\delta) < 1$. Therefore, by Lemma 3.1 and Theorem 3.1 of Goovaerts et al. (2004), we have

$$F^{\leftarrow}\left(1 - \frac{1 - q}{\varphi(\delta)}\right) \leq x_* + \delta h_* \leq x_* + h_* \leq \hat{x}.$$

The conclusion follows immediately. ∎

An implication of Lemma 4.1 is that, when $\hat{x} < \infty$, as $q \uparrow 1$ we have $x_* \to \hat{x}$ and $h_* \downarrow 0$. Therefore, in the theorem below we consider $\hat{x} = \infty$ only.

Theorem 4.1 *In addition to the conditions of Theorem 3.1, assume that $\varphi(\cdot) \in \mathrm{RV}_\alpha(0+) \cap \mathrm{RV}_\beta(\infty)$ for some $1 < \alpha, \beta < \infty$ and that X with $\hat{x} = \infty$ satisfies $\mathrm{E}\left[X_+^p\right] < \infty$ for some $p > \alpha \vee \beta$. Then $x_* \to \infty$ as $q \uparrow 1$.*

Proof. We prove by contradiction that x_* does not diverge to ∞ as $q \uparrow 1$. Then there is a sequence $\{q_n, n = 1, 2, \ldots\}$ increasing to 1 along which $x_* \to x_0 \in [-\infty, \infty)$. In this case, by Lemma 4.1 we have $h_* \to \infty$ as $q_n \uparrow 1$. Our proof is based on equation (3.1).

First consider the case that $x_* \to x_0 \in (-\infty, \infty)$ as $q_n \uparrow 1$. Since $\varphi(\cdot) \in \mathrm{RV}_\alpha(0+) \cap \mathrm{RV}_\beta(\infty)$ is continuously differentiable, by Theorems 1.7.2 and 1.7.2b of Bingham et al. (1987) we have $\varphi'(\cdot) \in \mathrm{RV}_{\alpha-1}(0+) \cap \mathrm{RV}_{\beta-1}(\infty)$. Therefore by Lemma 2.1, for arbitrarily small $\varepsilon > 0$, there is some $\delta > 0$ such that, for all $0 < t, st < \delta$,

$$(1 - \varepsilon)\left(s^{\alpha-1+\varepsilon} \wedge s^{\alpha-1-\varepsilon}\right) \leq \frac{\varphi'(st)}{\varphi'(t)} \leq (1 + \varepsilon)\left(s^{\alpha-1+\varepsilon} \vee s^{\alpha-1-\varepsilon}\right). \qquad (4.1)$$

Let the inequalities in (4.1) and Lemma 4.1 share the same δ. Split the LHS of (3.1) into two parts as

$$\begin{aligned}
\text{LHS of (3.1)} &= \mathrm{E}\left[\varphi'\left(\frac{(X - x_*)_+}{h_*}\right)\left(1_{\left(\frac{(X-x_*)_+}{h_*} \leq \delta\right)} + 1_{\left(\frac{(X-x_*)_+}{h_*} > \delta\right)}\right)\right] \\
&= I_1(x_*, h_*) + I_2(x_*, h_*).
\end{aligned} \qquad (4.2)$$

By the second inequality in (4.1), it holds for all q_n close to 1 that

$$\frac{I_1(x_*, h_*)}{\varphi'\left(\frac{1}{h_*}\right)} \leq (1 + \varepsilon)\mathrm{E}\left[(X - x_*)_+^{\alpha-1+\varepsilon} \vee (X - x_*)_+^{\alpha-1-\varepsilon}1_{(X \leq x_* + \delta h_*)}\right].$$

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Thus, by the dominated convergence theorem and Lemma 4.1, we have

$$\lim_{q_n\uparrow 1}\frac{I_1(x_*,h_*)}{\varphi'\left(\frac{1}{h_*}\right)}=\mathrm{E}\left[\lim_{q_n\uparrow 1}\frac{\varphi'\left(\frac{(X-x_*)_+}{h_*}\right)}{\varphi'\left(\frac{1}{h_*}\right)}1_{\left(\frac{(X-x_*)_+}{h_*}\le\delta\right)}\right]=\mathrm{E}\left[(X-x_0)_+^{\alpha-1}\right].$$

Now we turn to $I_2(x_*,h_*)$. By inequalities (2.1) we have

$$\begin{cases} \varphi'(s)\ge Cs^{\alpha-1+\varepsilon}, & \text{for }0<s\le\delta,\\ \varphi'(s)\le Cs^{\beta-1+\varepsilon}\le Cs^{(\alpha\vee\beta)-1+\varepsilon}, & \text{for }\delta\le s<\infty.\end{cases} \tag{4.3}$$

As $q_n\uparrow 1$, since x_* is bounded, $h_*\uparrow\infty$ and $\mathrm{E}\left[X_+^p\right]<\infty$, by both inequalities in (4.3) and Lemma 4.1 we have

$$\begin{aligned} I_2(x_*,h_*) &\le CE\left[\left(\frac{(X-x_*)_+}{h_*}\right)^{(\alpha\vee\beta)-1+\varepsilon}1_{(X>x_*+\delta h_*)}\right]\\ &= \frac{o(1)}{h_*^{(\alpha\vee\beta)-1+\varepsilon}}=o\left(\varphi'\left(\frac{1}{h_*}\right)\right)=o\left(I_1(x_*,h_*)\right). \end{aligned}$$

It follows from the decomposition in (4.2) that

$$\text{LHS of (3.1)}\sim\varphi'\left(\frac{1}{h_*}\right)\mathrm{E}\left[(X-x_0)_+^{\alpha-1}\right].$$

Since the function $s\varphi'(s)$ for $s\ge 0$ belongs to $\mathrm{RV}_\alpha(0+)\cap\mathrm{RV}_\beta(\infty)$, analogously to the above we can obtain that

$$\text{RHS of (3.1)}\sim\frac{1}{h_*}\varphi'\left(\frac{1}{h_*}\right)\mathrm{E}\left[(X-x_0)_+^{\alpha}\right].$$

A combination of these estimates for both sides of (3.1) yields a contradiction.

Next consider the case that $x_*\to-\infty$ as $q_n\uparrow 1$. By Lemma 4.1, we have $x_*=o(h_*)$ as $q_n\uparrow 1$. For the LHS of (3.1), we still use the decomposition in (4.2). For arbitrarily small $\varepsilon>0$, we have

$$\begin{aligned} I_1(x_*,h_*) &\ge CE\left[\left(\frac{(X-x_*)_+}{h_*}\right)^{\alpha-1+\varepsilon}1_{\left(\frac{(X-x_*)_+}{h_*}\le\delta\right)}\right]\\ &= C\frac{|x_*|^{\alpha-1+\varepsilon}}{h_*^{\alpha-1+\varepsilon}}\mathrm{E}\left[\frac{(X-x_*)_+^{\alpha-1+\varepsilon}}{|x_*|^{\alpha-1+\varepsilon}}1_{(X\le x_*+\delta h_*)}\right]\\ &\sim C\frac{|x_*|^{\alpha-1+\varepsilon}}{h_*^{\alpha-1+\varepsilon}}, \end{aligned}$$

where in the first step we applied the first inequality in (4.3), while in the last step we applied the dominated convergence theorem to prove that the expectation converges to 1 as $x_*\to-\infty$. A similar asymptotic upper bound for $I_1(x_*,h_*)$ can also be established. Thus,

$$C\frac{|x_*|^{\alpha-1+\varepsilon}}{h_*^{\alpha-1+\varepsilon}}\lesssim I_1(x_*,h_*)\lesssim C\frac{|x_*|^{\alpha-1-\varepsilon}}{h_*^{\alpha-1-\varepsilon}}.$$

Now we turn to $I_2(x_*, h_*)$. Applying (4.3), c_r-inequality and Markov's inequality, in turn, we have

$$
\begin{aligned}
I_2(x_*, h_*) &\leq C\mathrm{E}\left[\left(\frac{(X-x_*)_+}{h_*}\right)^{\beta-1+\varepsilon} 1_{(X>x_*+\delta h_*)}\right] \\
&\leq \frac{C}{h_*^{\beta-1+\varepsilon}}\mathrm{E}\left[\left(X_+^{\beta-1+\varepsilon} + |x_*|^{\beta-1+\varepsilon}\right) 1_{(X>x_*+\delta h_*)}\right] \\
&\leq \frac{C}{h_*^{\beta-1+\varepsilon}}\mathrm{E}\left[X_+^{\beta-1+\varepsilon}\frac{X_+^{(\alpha\vee\beta)-\beta+1}}{(x_*+\delta h_*)^{(\alpha\vee\beta)-\beta+1}} + |x_*|^{\beta-1+\varepsilon}\frac{X_+^{(\alpha\vee\beta)+\varepsilon}}{(x_*+\delta h_*)^{(\alpha\vee\beta)+\varepsilon}}\right] \\
&= \frac{C}{h_*^{\beta-1+\varepsilon}}\left(\frac{1}{(x_*+\delta h_*)^{(\alpha\vee\beta)-\beta+1}} + \frac{|x_*|^{\beta-1+\varepsilon}}{(x_*+\delta h_*)^{(\alpha\vee\beta)+\varepsilon}}\right)\mathrm{E}\left[X_+^{(\alpha\vee\beta)+\varepsilon}\right] \\
&= \frac{C}{h_*^{(\alpha\vee\beta)+\varepsilon}},
\end{aligned}
$$

where the last step is due to the fact that $x_* = o(h_*)$ as $q_n \uparrow 1$. Since $x_* \to -\infty$ and $h_* \to \infty$, we have $I_2(x_*, h_*) = o\left(I_1(x_*, h_*)\right)$. It follows from the decomposition in (4.2) that

$$
C\frac{|x_*|^{\alpha-1+\varepsilon}}{h_*^{\alpha-1+\varepsilon}} \lesssim \text{LHS of (3.1)} \lesssim C\frac{|x_*|^{\alpha-1-\varepsilon}}{h_*^{\alpha-1-\varepsilon}}.
$$

Analogously,

$$
C\frac{|x_*|^{\alpha+\varepsilon}}{h_*^{\alpha+\varepsilon}} \lesssim \text{RHS of (3.1)} \lesssim C\frac{|x_*|^{\alpha-\varepsilon}}{h_*^{\alpha-\varepsilon}}.
$$

Since $x_* = o(h_*)$, a combination of these estimates for both sides of (3.1) yields a contradiction. ∎

5   A Deeper Description for the Orlicz Quantile

We restrict our attention to $F \in \mathrm{MDA}(G_\gamma)$ for $\gamma \in \mathbb{R}$. Recall the function $U(\cdot)$ defined in (2.6) and let V be a random variable uniformly distributed over $(0, 1)$. By setting

$$
t_* = \frac{1}{\overline{F}(x_*)},
$$

we have

$$
U(t_*) \leq x_* \leq U(t_* + 0). \tag{5.1}
$$

It is easy to verify that X conditional on $X > x_*$ is equal in distribution to $U(t_*/V)$; namely,

$$
X \,|\, (X > x_*) \stackrel{d}{=} U(t_*/V).
$$

Hence, the LHS of equation (3.1) for the minimizer (x_*, h_*) can be rewritten as

$$
\mathrm{E}\left[\varphi'\left(\frac{(X-x)_+}{h}\right)\right] = \frac{1}{t_*}\mathrm{E}\left[\varphi'\left(\frac{X-x_*}{h_*}\right)\Big| X > x_*\right] = \frac{1}{t_*}\mathrm{E}\left[\varphi'\left(\frac{U(t_*/V)-x_*}{h_*}\right)\right].
$$

We do the same treatment on the RHS of equation (3.1), so that we obtain an equivalent form of (3.1) as

$$\int_0^1 \varphi'\left(\frac{U\left(t_*/v\right)-x_*}{h_*}\right)\mathrm{d}v = \int_0^1 \varphi'\left(\frac{U\left(t_*/v\right)-x_*}{h_*}\right)\frac{U\left(t_*/v\right)-x_*}{h_*}\mathrm{d}v. \qquad (5.2)$$

This relation will be the basis for the proof of the following lemma:

Lemma 5.1 *In addition to the conditions of Theorem 3.1, assume that $\varphi(\cdot) \in \mathrm{RV}_\alpha(0+) \cap \mathrm{RV}_\beta(\infty)$ for some $1 < \alpha, \beta < \infty$ and $F \in \mathrm{MDA}(G_\gamma)$ with $-\infty < \gamma < \alpha^{-1} \wedge \beta^{-1}$. Then the auxiliary function $a(\cdot)$ given by (2.3) with $f(\cdot)$ replaced by $U(\cdot)$ satisfies $a(t_*) \asymp h_*$ as $q \uparrow 1$.*

Proof. We prove this lemma by contradiction based on equation (5.2). First assume that there is a sequence $\{q_n, n = 1, 2, \ldots\}$ increasing to 1 along which $a(t_*)/h_*$ diverges to ∞. Note that $F \in \mathrm{MDA}(G_\gamma)$ if and only if $U(\cdot) \in \mathrm{ERV}_\gamma$. For the LHS of (5.2), by the first inequality in (5.1) and Lemma 2.3, it holds for every small $\varepsilon, \delta > 0$ and all large t_* and all $0 < v < 1$ that

$$\begin{aligned}
\frac{U\left(t_*/v\right)-x_*}{a(t_*)} &\leq \frac{U\left(t_*/v\right)-U(t_*)}{a(t_*)} \\
&\leq \left|\frac{U\left(t_*/v\right)-U(t_*)}{a(t_*)} - \frac{v^{-\gamma}-1}{\gamma}\right| + \frac{v^{-\gamma}-1}{\gamma} \\
&\leq \varepsilon v^{-\gamma-\delta} + \frac{v^{-\gamma}-1}{\gamma}.
\end{aligned}$$

Since $\varphi'(\cdot) \in \mathrm{RV}_{\beta-1}(\infty)$ and $a(t_*)/h_* \to \infty$, by Lemma 2.1, as $q_n \uparrow 1$ we have

$$\begin{aligned}
\frac{\mathrm{LHS\ of\ (5.2)}}{\varphi'\left(\frac{a(t_*)}{h_*}\right)} &\leq \int_0^1 \frac{\varphi'\left(\frac{U(t_*/v)-U(t_*)}{a(t_*)}\frac{a(t_*)}{h_*}\right)}{\varphi'\left(\frac{a(t_*)}{h_*}\right)}\mathrm{d}v \\
&\leq \int_0^1 \frac{\varphi'\left(\left(\varepsilon v^{-\gamma-\delta} + \frac{v^{-\gamma}-1}{\gamma}\right)\frac{a(t_*)}{h_*}\right)}{\varphi'\left(\frac{a(t_*)}{h_*}\right)}\mathrm{d}v \\
&\lesssim (1+\varepsilon)\int_0^1 \left(\varepsilon v^{-\gamma-\delta} + \frac{v^{-\gamma}-1}{\gamma}\right)^{\beta-1+\varepsilon}\mathrm{d}v \\
&< \infty,
\end{aligned}$$

where the first step is due to (5.1) and the last step due to the condition $\gamma < \alpha^{-1} \wedge \beta^{-1}$. An application of dominated convergence theorem yields that, as $q_n \uparrow 1$,

$$\lim_{q_n\uparrow 1}\frac{\mathrm{LHS\ of\ (5.2)}}{\varphi'\left(\frac{a(t_*)}{h_*}\right)} = \lim_{q_n\uparrow 1}\frac{\int_0^1 \varphi'\left(\frac{U(t_*/v)-x_*}{a(t_*)}\frac{a(t_*)}{h_*}\right)\mathrm{d}v}{\varphi'\left(\frac{a(t_*)}{h_*}\right)} = \int_0^1 \left(\frac{v^{-\gamma}-1}{\gamma}\right)^{\beta-1}\mathrm{d}v, \qquad (5.3)$$

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where in the second step we used the two-sided inequality (5.1) and relation (2.5). Analogously, as $q_n \uparrow 1$,

$$\lim_{q_n \uparrow 1} \frac{\text{RHS of (5.2)}}{\varphi'\left(\frac{a(t_*)}{h_*}\right) \frac{a(t_*)}{h_*}} = \int_0^1 \left(\frac{v^{-\gamma} - 1}{\gamma}\right)^\beta dv.$$

Due to the assumption that $a(t_*)/h_* \to \infty$ as $q_n \uparrow 1$, a combination of these two relations contradicts (5.2).

Next assume that there is a sequence $\{q_n, n = 1, 2, \ldots\}$ increasing to 1 along which $a(t_*)/h_*$ converges to 0. We still apply the dominated convergence theorem to obtain a similar limit relation as in (5.3), but we need to verify the requirement for the dominated convergence theorem. Similarly to the above, by Lemma 2.3 we have

$$\text{LHS of (5.2)} \leq \int_0^1 \varphi'\left(\left(\varepsilon v^{-\gamma-\delta} + \frac{v^{-\gamma} - 1}{\gamma}\right)\frac{a(t_*)}{h_*}\right) dv.$$

By Lemma 2.1 and $\varphi'(\cdot) \in \mathrm{RV}_{\alpha-1}(0+)$, for arbitrarily fixed small $0 < \varepsilon < 1$, there is some small $\delta_1 > 0$ such that the inequalities in (4.1) hold for $0 < t, st < \delta_1$. Then define

$$A = \left\{ v \in (0, 1] : \left(\varepsilon v^{-\gamma-\delta} + \frac{v^{-\gamma} - 1}{\gamma}\right)\frac{a(t_*)}{h_*} \geq \delta_1 \right\}, \qquad A^c = (0, 1] - A$$

and do the split

$$\text{LHS of (5.2)} \leq \left(\int_A + \int_{A^c}\right) \varphi'\left(\left(\varepsilon v^{-\gamma-\delta} + \frac{v^{-\gamma} - 1}{\gamma}\right)\frac{a(t_*)}{h_*}\right) dv = I_1 + I_2.$$

By the second inequality in (4.3), we have

$$\begin{aligned}
I_1 &\leq C \int_A \left(\left(\varepsilon v^{-\gamma-\delta} + \frac{v^{-\gamma} - 1}{\gamma}\right)\frac{a(t_*)}{h_*}\right)^{(\alpha \vee \beta)-1+\varepsilon} dv \\
&= o(1)\varphi'\left(\frac{a(t_*)}{h_*}\right) \int_0^{v_0} \left(\varepsilon v^{-\gamma-\delta} + \frac{v^{-\gamma} - 1}{\gamma}\right)^{(\alpha \vee \beta)-1+\varepsilon} dv.
\end{aligned}$$

By the second inequality in (4.1), we have

$$I_2 \leq (1+\varepsilon)\varphi'\left(\frac{a(t_*)}{h_*}\right) \int_{A^c} \left(\varepsilon v^{-\gamma-\delta} + \frac{v^{-\gamma} - 1}{\gamma}\right)^{\alpha-1+\varepsilon} \vee \left(\varepsilon v^{-\gamma-\delta} + \frac{v^{-\gamma} - 1}{\gamma}\right)^{\alpha-1-\varepsilon} dv.$$

Note that both integrals in the bounds for I_1 and I_2 above are finite since $\gamma < \alpha^{-1} \wedge \beta^{-1}$. This validates the application of the dominated convergence theorem in

$$\lim_{q_n \uparrow 1} \frac{\text{LHS of (5.2)}}{\varphi'\left(\frac{a(t_*)}{h_*}\right) \frac{a(t_*)}{h_*}} = \int_0^1 \left(\frac{v^{-\gamma} - 1}{\gamma}\right)^{\alpha-1} dv.$$

Analogously,

$$\lim_{q_n \uparrow 1} \frac{\text{RHS of (5.2)}}{\varphi'\left(\frac{a(t_*)}{h_*}\right) \frac{a(t_*)}{h_*}} = \int_0^1 \left(\frac{v^{-\gamma} - 1}{\gamma}\right)^\alpha dv.$$

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A combination of these two relations contradicts (5.2). ∎

By Theorem 1.1.6 of de Haan and Ferreira (2006), $F \in \mathrm{MDA}(G_\gamma)$ if and only if the relation

$$\lim_{x \uparrow \hat{x}} \frac{\overline{F}(x + ya(1/\overline{F}(x)))}{\overline{F}(x)} = (1 + \gamma y)^{-1/\gamma} \tag{5.4}$$

holds for all $y \in \mathbb{R}$ such that $1 + \gamma y > 0$, where the RHS is understood as e^{-y} when $\gamma = 0$. Define a positive random variable Y distributed by

$$\Pr(Y \le y) = 1 - (1 + \gamma y)^{-1/\gamma} \tag{5.5}$$

for all $y > 0$ such that $1 + \gamma y > 0$. Thus, relation (5.4) implies that

$$\left. \frac{X - x}{a(1/\overline{F}(x))} \right| (X > x) \text{ converges in distribution to } Y, \qquad x \uparrow \hat{x}. \tag{5.6}$$

In the lemma below, the condition $Y \in L_0^{\varphi''}$ can easily be verified to hold if $\varphi''(\cdot) \in \mathrm{RV}_{\alpha-2}(0+)$:

Lemma 5.2 *In addition to the conditions of Theorem 3.1, assume that $\varphi(\cdot) \in \mathrm{RV}_\alpha(0+) \cap \mathrm{RV}_\beta(\infty)$ for some $1 < \alpha, \beta < \infty$. In case $\gamma \le -1$, further assume that $\varphi(\cdot)$ is twice differentiable over $\mathbb{R}_+$ and that Y defined by (5.5) belongs to $L_0^{\varphi''}$. Then the equation*

$$\mathrm{E}\left[\varphi'(\lambda Y)\right] = \mathrm{E}\left[\varphi'(\lambda Y)\lambda Y\right], \qquad \lambda > 0, \tag{5.7}$$

has a unique solution.

Proof. To prove the existence of the solution, denote by $g(\lambda)$ the ratio of both sides of (5.7),

$$g(\lambda) = \frac{\mathrm{E}\left[\varphi'(\lambda Y)\lambda Y\right]}{\mathrm{E}\left[\varphi'(\lambda Y)\right]}, \qquad \lambda > 0.$$

For arbitrarily small $\varepsilon > 0$, by the second inequality in (4.1) and both inequalities in (4.3), there is some small $\delta > 0$ such that, for all $0 < \lambda < \delta$,

$$\begin{aligned}
\frac{\varphi'(\lambda Y)}{\varphi'(\lambda)} &= \frac{\varphi'(\lambda Y)}{\varphi'(\lambda)}\left(1_{(\lambda Y \le \delta)} + 1_{(\lambda Y > \delta)}\right) \\
&\le (1 + \varepsilon)\left(Y^{\alpha-1+\varepsilon} \vee Y^{\alpha-1-\varepsilon}\right) + C\frac{(\lambda Y)^{(\alpha \vee \beta)-1+\varepsilon}}{\lambda^{\alpha-1+\varepsilon}} \\
&\le (1 + \varepsilon)\left(Y^{\alpha-1+\varepsilon} \vee Y^{\alpha-1-\varepsilon}\right) + CY^{(\alpha \vee \beta)-1+\varepsilon},
\end{aligned}$$

which is integrable since $\gamma < \alpha^{-1} \wedge \beta^{-1}$. An application of the dominated convergence theorem gives that $\mathrm{E}\left[\varphi'(\lambda Y)\right] \sim \varphi'(\lambda)\mathrm{E}\left[Y^{\alpha-1}\right]$ as $\lambda \downarrow 0$. Since the function $\lambda\varphi'(\lambda)$ for $\lambda \ge 0$

belongs to $\mathrm{RV}_\alpha(0+) \cap \mathrm{RV}_\beta(\infty)$, we can also obtain $\mathrm{E}\left[\varphi'(\lambda Y)\lambda Y\right] \sim \lambda\varphi'(\lambda)\mathrm{E}\left[Y^\alpha\right]$ as $\lambda \downarrow 0$. It follows that

$$g(\lambda) \sim \lambda \frac{\mathrm{E}\left[Y^\alpha\right]}{\mathrm{E}\left[Y^{\alpha-1}\right]}, \qquad \lambda \downarrow 0.$$

Hence, $\lim_{\lambda\downarrow 0} g(\lambda) = 0$. Similarly, $\lim_{\lambda\uparrow\infty} g(\lambda) = \infty$. Since $g(\lambda)$ is continuous over $\lambda > 0$, we prove the existence of a solution to equation (5.7).

To prove the uniqueness of the solution, it suffices to show that $g'(\lambda) > 0$. Denote by $\hat{y}$ the upper endpoint of Y, so $\hat{y} = \infty$ if $\gamma \geq 0$ while $\hat{y} = -1/\gamma$ if $\gamma < 0$. First assume $\gamma > -1$. Note that

$$\mathrm{E}\left[\varphi'(\lambda Y)\right] = -\int_0^{\hat{y}} \varphi'(\lambda y)\,\mathrm{d}(1+\gamma y)^{-1/\gamma} = \int_0^{\lambda\hat{y}} \left(1 + \frac{\gamma}{\lambda}w\right)^{-1/\gamma-1}\mathrm{d}\varphi(w),$$

Similarly,

$$\mathrm{E}\left[\varphi'(\lambda Y)\lambda Y\right] = \int_0^{\lambda\hat{y}} w\left(1 + \frac{\gamma}{\lambda}w\right)^{-1/\gamma-1}\mathrm{d}\varphi(w).$$

Then we have

$$\begin{aligned}
g'(\lambda) &= \frac{(1+\gamma)\lambda^{-2}}{(\mathrm{E}\left[\varphi'(\lambda Y)\right])^2}\left[\int_0^{\lambda\hat{y}} w^2\left(1+\frac{\gamma}{\lambda}w\right)^{-1/\gamma-2}\mathrm{d}\varphi(w)\int_0^{\lambda\hat{y}}\left(1+\frac{\gamma}{\lambda}w\right)^{-1/\gamma-1}\mathrm{d}\varphi(w)\right.\\
&\quad\left. -\int_0^{\lambda\hat{y}} w\left(1+\frac{\gamma}{\lambda}w\right)^{-1/\gamma-1}\mathrm{d}\varphi(w)\int_0^{\lambda\hat{y}} w\left(1+\frac{\gamma}{\lambda}w\right)^{-1/\gamma-2}\mathrm{d}\varphi(w)\right].
\end{aligned}$$

Every integral in the above is finite for $\gamma > -1$. It remains to verify that the part in the square brackets is positive. Introduce a random variable W with distribution

$$\Pr(W \in \mathrm{d}w) = \frac{\left(1+\frac{\gamma}{\lambda}w\right)^{-1/\gamma-1}\mathrm{d}\varphi(w)}{\int_0^{\lambda\hat{y}}\left(1+\frac{\gamma}{\lambda}w\right)^{-1/\gamma-1}\mathrm{d}\varphi(w)}, \qquad 0 \leq w \leq \lambda\hat{y}.$$

What we need to verify becomes

$$\mathrm{E}\left[\frac{W^2}{1+\frac{\gamma}{\lambda}W}\right] > \mathrm{E}\left[W\right]\mathrm{E}\left[\frac{W}{1+\frac{\gamma}{\lambda}W}\right],$$

or, equivalently,

$$\mathrm{Cov}\left(W, \frac{W}{1+\frac{\gamma}{\lambda}W}\right) > 0. \tag{5.8}$$

By Höffding's formula for covariance, inequality (5.8) holds true since the two random variables are comonotonic and not degenerate. See Dhaene et al. (2002) for the concept of comonotonicity.

Next assume $\gamma \leq -1$. We derive an alternative expression for the derivative of $g(\lambda)$ as

$$g'(\lambda) = \frac{\mathrm{E}\left[\varphi'(\lambda Y)\right]\left(\mathrm{E}\left[\varphi''(\lambda Y)\lambda Y^2\right] + \mathrm{E}\left[\varphi'(\lambda Y)Y\right]\right) - \mathrm{E}\left[\varphi'(\lambda Y)\lambda Y\right]\mathrm{E}\left[\varphi''(\lambda Y)Y\right]}{(\mathrm{E}\left[\varphi'(\lambda Y)\right])^2}.$$

It is easy to verify that every expectation in the above is well defined and finite. We need to verify the positivity of the numerator on the RHS. Introduce two random variables W_1 and W_2 with distributions

$$\begin{cases} \Pr\left(W_1 \in \mathrm{d}w\right) = \frac{\varphi'(\lambda w)}{\mathrm{E}[\varphi'(\lambda Y)]} \Pr\left(Y \in \mathrm{d}w\right), \\ \Pr\left(W_2 \in \mathrm{d}w\right) = \frac{\varphi''(\lambda w)}{\mathrm{E}[\varphi''(\lambda Y)]} \Pr\left(Y \in \mathrm{d}w\right), \end{cases} \qquad 0 \le w \le \hat{y}.$$

It remains to verify that

$$\lambda \mathrm{E}\left[W_2^2\right] + \frac{\mathrm{E}\left[\varphi'(\lambda Y) Y\right]}{\mathrm{E}\left[\varphi''(\lambda Y)\right]} > \lambda \mathrm{E}\left[W_1\right] \mathrm{E}\left[W_2\right]. \tag{5.9}$$

Note that

$$\begin{aligned} \frac{\mathrm{E}\left[\varphi'(\lambda Y) Y\right]}{\mathrm{E}\left[\varphi''(\lambda Y)\right]} &= \frac{\mathrm{E}\left[\varphi'(\lambda Y) Y\right]}{\mathrm{E}\left[\varphi'(\lambda Y)\right]} \frac{\mathrm{E}\left[\varphi'(\lambda Y)\right]}{\mathrm{E}\left[\varphi''(\lambda Y)\right]} \\ &= \mathrm{E}\left[W_1\right] \frac{\lambda \int_0^{\hat{y}} \varphi''(\lambda y)(1+\gamma y)^{-1/\gamma}\,\mathrm{d}y}{\mathrm{E}\left[\varphi''(\lambda Y)\right]} \\ &= \lambda \mathrm{E}\left[W_1\right] \mathrm{E}\left[1+\gamma W_2\right]. \end{aligned}$$

Plugging this into (5.9) and applying Jensen's inequality, we obtain

$$\text{LHS of } (5.9) \ge \lambda \left(\mathrm{E}\left[W_2\right]\right)^2 + \lambda \mathrm{E}\left[W_1\right] \mathrm{E}\left[1+\gamma W_2\right].$$

Then inequality (5.9) follows if we can show that

$$\left(\mathrm{E}\left[W_2\right]\right)^2 + \mathrm{E}\left[W_1\right] \mathrm{E}\left[1+\gamma W_2\right] > \mathrm{E}[W_1]\mathrm{E}[W_2],$$

or, equivalently,

$$\mathrm{E}[W_1] + \gamma \mathrm{E}[W_1]\mathrm{E}[W_2] > \mathrm{E}[W_2]\left(\mathrm{E}[W_1] - \mathrm{E}[W_2]\right). \tag{5.10}$$

Note that both W_1 and W_2 are positive random variables bounded by $\hat{y} = -1/\gamma \le 1$ and that the LHS of (5.10) is positive. Thus, if $\mathrm{E}[W_1] - \mathrm{E}[W_2] \le 0$, then inequality (5.10) naturally holds. On the other hand, if $\mathrm{E}[W_1] - \mathrm{E}[W_2] > 0$, we lower $\gamma \mathrm{E}[W_1]$ to -1 on the LHS while raise the first $\mathrm{E}[W_2]$ to 1 on the RHS and then we see that inequality (5.10) still holds. This completes the proof of Lemma 5.2. ∎

Now we are ready to show the main result of this section:

Theorem 5.1 *Under the conditions of Lemmas 5.1 and 5.2, we have $\lim_{q\uparrow 1} a(t_*)/h_* = \lambda$, where λ is the unique solution to equation (5.7).*

Proof. Let $\{q_n, n = 1, 2, \ldots\}$ be a sequence increasing to 1 along which $a(t_*)/h_* \to \lambda$ for some $\lambda \in (0, \infty)$. As in the proof of Lemma 5.1, for every small $\varepsilon, \delta > 0$, as $q_n \uparrow 1$ it holds

that

$$\int_0^1 \varphi'\left(\frac{U\left(t_*/v\right) - x_*}{h_*}\right) dv \;\leq\; \int_0^1 \varphi'\left(\frac{U\left(t_*/v\right) - U(t_*)}{a(t_*)}\frac{a(t_*)}{h_*}\right) dv$$

$$\leq \int_0^1 \varphi'\left(\left(\varepsilon v^{-\gamma-\delta} + \frac{v^{-\gamma}-1}{\gamma}\right)(\lambda+\varepsilon)\right) dv < \infty,$$

where the last step is due to the conditions $\varphi'(\cdot) \in \mathrm{RV}_{\beta-1}(\infty)$ and $\gamma < \alpha^{-1} \wedge \beta^{-1}$. Hence, by the dominated convergence theorem and relation (5.6),

$$\text{LHS of (3.1)} = \overline{F}(x_*)\mathrm{E}\left[\varphi'\left(\frac{X-x_*}{a(t_*)}\frac{a(t_*)}{h_*}\right)\Bigg|\, X > x_*\right] \sim \overline{F}(x_*)\mathrm{E}\left[\varphi'\left(\lambda Y\right)\right].$$

Similarly,

$$\text{RHS of (3.1)} \sim \overline{F}(x_*)\mathrm{E}\left[\varphi'\left(\lambda Y\right)\lambda Y\right].$$

By Lemma 5.2, λ must be identical to the unique positive solution of equation (5.7). ∎

6   The Main Result

Now we present the main result of this paper:

Theorem 6.1 *Let $\varphi(\cdot)$ be strictly convex and continuously differentiable over $\mathbb{R}_+$ with $\varphi'_+(0) = 0$ and $\varphi(\cdot) \in \mathrm{RV}_\alpha(0+) \cap \mathrm{RV}_\beta(\infty)$ for some $1 < \alpha, \beta < \infty$, and let $F \in \mathrm{MDA}(G_\gamma)$ for $-\infty < \gamma < \alpha^{-1} \wedge \beta^{-1}$. In case $\gamma \leq -1$, further assume that $\varphi(\cdot)$ is twice differentiable over $\mathbb{R}_+$ and that Y defined by (5.5) belongs to the Orlicz heart $L_0^{\varphi''}$. Then the minimizer (x_*, h_*) appearing in (1.2) satisfies*

$$\overline{F}(x_*) \sim \frac{1-q}{\mathrm{E}\left[\varphi\left(\lambda Y\right)\right]} \qquad \text{and} \qquad h_* \sim \frac{a(1/\overline{F}(x_*))}{\lambda}, \qquad q \uparrow 1,$$

where λ is the unique positive solution to equation (5.7) and $a(\cdot)$ is given by (2.3) with $f(\cdot)$ replaced by $U(\cdot) = \left(1/\overline{F}\right)^{\leftarrow}(\cdot)$.

Proof. The second relation for h_* is proved by Theorem 5.1. By relation (5.6) and Theorem 5.1, we have

$$1 - q \;=\; \mathrm{E}\left[\varphi\left(\frac{(X - x_*)_+}{h_*}\right)\right]$$

$$= \overline{F}(x_*)\mathrm{E}\left[\varphi\left(\frac{X-x_*}{a(1/\overline{F}(x_*))}\frac{a(1/\overline{F}(x_*))}{h_*}\right)\Bigg|\, X > x_*\right]$$

$$\sim \overline{F}(x_*)\mathrm{E}\left[\varphi\left(\lambda Y\right)\right],$$

where in the last step we applied the dominated convergence theorem, as justified in the proof of Theorem 5.1. This gives the first relation for $\overline{F}(x_*)$. ∎

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Corollary 6.1 *Under the conditions of Theorem 6.1, as $q \uparrow 1$,*

(a) (the Fréchet case) for $\gamma > 0$,

$$H_q[X] \sim \left(1 + \frac{\gamma}{\lambda}\right) \left(\int_0^\infty \left(1 + \frac{\gamma}{\lambda}z\right)^{-1/\gamma} \mathrm{d}\varphi\,(z)\right)^\gamma F^\leftarrow(q);$$

(b) (the Gumbel case) for $\gamma = 0$, if $\hat{x} = \infty$ then

$$H_q[X] \sim F^\leftarrow\left(1 - \frac{1-q}{\int_0^\infty \mathrm{e}^{-z/\lambda}\mathrm{d}\varphi\,(z)}\right),$$

while if $\hat{x} < \infty$ then

$$\hat{x} - H_q[X] \sim \hat{x} - F^\leftarrow\left(1 - \frac{1-q}{\int_0^\infty \mathrm{e}^{-z/\lambda}\mathrm{d}\varphi\,(z)}\right);$$

(c) (the Weibull case) for $\gamma < 0$,

$$\hat{x} - H_q[X] \sim \left(1 + \frac{\gamma}{\lambda}\right) \left(\int_0^{-\lambda/\gamma} \left(1 + \frac{\gamma}{\lambda}z\right)^{-1/\gamma} \mathrm{d}\varphi\,(z)\right)^\gamma (\hat{x} - F^\leftarrow(q)).$$

Proof. By Theorem 6.1, it remains to identify the quantities $\mathrm{E}\left[\varphi\,(\lambda Y)\right]$ and $a(1/\overline{F}(x_*))$ for the three cases, respectively. Set $t_* = 1/\overline{F}(x_*)$ for notational convenience.

(a) For $\gamma > 0$, we have

$$\mathrm{E}\left[\varphi\,(\lambda Y)\right] = -\int_0^\infty \varphi\,(\lambda y)\,\mathrm{d}\,(1 + \gamma y)^{-1/\gamma} = \int_0^\infty \left(1 + \frac{\gamma}{\lambda}z\right)^{-1/\gamma} \mathrm{d}\varphi\,(z).$$

This together with $\overline{F}(\cdot) \in \mathrm{RV}_{-1/\gamma}(\infty)$ gives

$$x_* \sim \left(\mathrm{E}\left[\varphi\,(\lambda Y)\right]\right)^\gamma F^\leftarrow(q) = \left(\int_0^\infty \left(1 + \frac{\gamma}{\lambda}z\right)^{-1/\gamma} \mathrm{d}\varphi\,(z)\right)^\gamma F^\leftarrow(q).$$

In this case, we have $a(t_*) = \gamma U(t_*)$ by (2.3). Hence,

$$h_* \sim \frac{\gamma}{\lambda}U(t_*) = \frac{\gamma}{\lambda}\left(\frac{1}{\overline{F}}\right)^\leftarrow \left(\frac{1}{\overline{F}(x_*)}\right) \sim \frac{\gamma}{\lambda}x_*.$$

Combining these two asymptotics for x_* and h_* yields the desired result for $H_q[X]$.

(b) For $\gamma = 0$, we have

$$\mathrm{E}\left[\varphi\,(\lambda Y)\right] = -\int_0^\infty \varphi\,(\lambda y)\,\mathrm{d}\mathrm{e}^{-y} = \int_0^\infty \mathrm{e}^{-z/\lambda}\mathrm{d}\varphi\,(z).$$

For $\hat{x} = \infty$, since $a(t_*) = o\,(U(t_*))$ and $U(t_*) \sim x_*$, it holds that $H_q[X] \sim x_*$. Similarly, for $\hat{x} < \infty$, since $a(t_*) = o\,(U\,(\infty) - U(t_*))$, $U\,(\infty) = \hat{x}$ and $U\,(\infty) - U(t_*) \sim \hat{x} - x_*$, it holds that $\hat{x} - H_q[X] \sim \hat{x} - x_*$.

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(c) For $\gamma < 0$, we have

$$\mathrm{E}\left[\varphi\left(\lambda Y\right)\right] = - \int_0^{-1/\gamma} \varphi\left(\lambda y\right) \mathrm{d}\left(1 + \gamma y\right)^{-1/\gamma} = \int_0^{-\lambda/\gamma} \left(1 + \frac{\gamma}{\lambda} z\right)^{-1/\gamma} \mathrm{d}\varphi\left(z\right).$$

This together with $\overline{F}(\hat{x} - \cdot) \in \mathrm{RV}_{-1/\gamma}(0+)$ gives

$$\hat{x} - x_* \sim \left(\mathrm{E}\left[\varphi\left(\lambda Y\right)\right]\right)^\gamma \left(\hat{x} - F^\leftarrow\left(q\right)\right) = \left(\int_0^{-\lambda/\gamma} \left(1 + \frac{\gamma}{\lambda} z\right)^{-1/\gamma} \mathrm{d}\varphi\left(z\right)\right)^\gamma \left(\hat{x} - F^\leftarrow\left(q\right)\right).$$

In addition, since $a(t_*) = -\gamma\left(U\left(\infty\right) - U(t_*)\right)$ by (2.3), we have

$$h_* \sim \frac{a(t_*)}{\lambda} = \frac{-\gamma\left(U\left(\infty\right) - U(t_*)\right)}{\lambda} \sim \frac{-\gamma}{\lambda}\left(\hat{x} - x_*\right).$$

Combining these two asymptotics for x_* and h_* yields the desired result for $H_q[X]$. ∎

7   Numerical Examples

In this section, we use `R` to numerically examine the accuracy of the asymptotic formulas obtained in Corollary 6.1. In order to compare the asymptotic results with the exact value of $H_q[X]$, we choose the Young function to be

$$\varphi(s) = \frac{1}{2}\left(s^{1.1} + s^{2.2}\right), \qquad s \geq 0,$$

so that $\varphi(\cdot) \in \mathrm{RV}_\alpha(0+) \cap \mathrm{RV}_\beta(\infty)$ with $\alpha = 1.1$ and $\beta = 2.2$. Then equation (1.1) becomes

$$\frac{1}{2}\mathrm{E}\left[\left(\frac{(X - x)_+}{h}\right)^{2.2} + \left(\frac{(X - x)_+}{h}\right)^{1.1}\right] = 1 - q,$$

which gives

$$h(x) = \left(\frac{\mathrm{E}\left[(X - x)_+^{1.1}\right] + \sqrt{\left(\mathrm{E}\left[(X - x)_+^{1.1}\right]\right)^2 + 8(1 - q)\mathrm{E}\left[(X - x)_+^{2.2}\right]}}{4(1 - q)}\right)^{1/1.1}.$$

Then the equation $h'(x) = -1$ gives the Orlicz quantile x_*. Plugging this x_* into $h(x)$ gives the value of h_*. Hence, the exact value of $H_q[X] = x_* + h_*$ is obtained. For each of the Fréchet, Gumbel and Weibull cases, we will show a figure in which we compare the asymptotic value of $H_q[X]$ with its exact value on the left and draw their ratio on the right.

Example 7.1 (the Fréchet case) Assume that F is a Pareto distribution given by

$$F(x) = 1 - \left(\frac{\theta}{x + \theta}\right)^\alpha, \qquad x, \alpha, \theta > 0.$$

Thus, $F \in \mathrm{MDA}(G_\gamma)$ with $\gamma = 1/\alpha$ and $U(t) = \theta(t^\gamma - 1)$. In Figure 7.1, we set $\alpha = 2.4, 2.7$ and $\theta = 1$. It shows that the ratio converges to 1 as $q \uparrow 1$ but the accuracy is not as good as in the power Young function case, so the second-order asymptotics becomes desirable.

Example 7.2 (the Gumbel case) Assume that F is a lognormal distribution given by

$$F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right), \qquad x > 0, -\infty < \mu < \infty, \sigma > 0,$$

where $\Phi(\cdot)$ denotes the standard normal distribution function. Thus, $F \in \text{MDA}(G_\gamma)$ with $\gamma = 0$ and the auxiliary function $a(\cdot)$ given by

$$a(x) = \frac{\overline{\Phi}(\sigma^{-1}(\ln x - \mu))\sigma x}{\Phi'(\sigma^{-1}(\ln x - \mu))};$$

see, e.g. page 150 of Embrechts et al. (1997). In Figure 7.2, we set $\mu = 2$ and $\sigma = 0.5$. Overall, the accuracy is very good.

Example 7.3 (the Weibull case) Assume that F is a beta distribution with probability density function given by

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{\text{B}(a,b)}, \qquad 0 < x < 1, a, b > 0.$$

Thus, $F \in \text{MDA}(G_\gamma)$ with $\gamma = -1/b$. In Figure 7.3, we set $a = 2$ and $b = 6, 8$. It shows that the ratio converges to 1 as $q \uparrow 1$ and the accuracy is very good.

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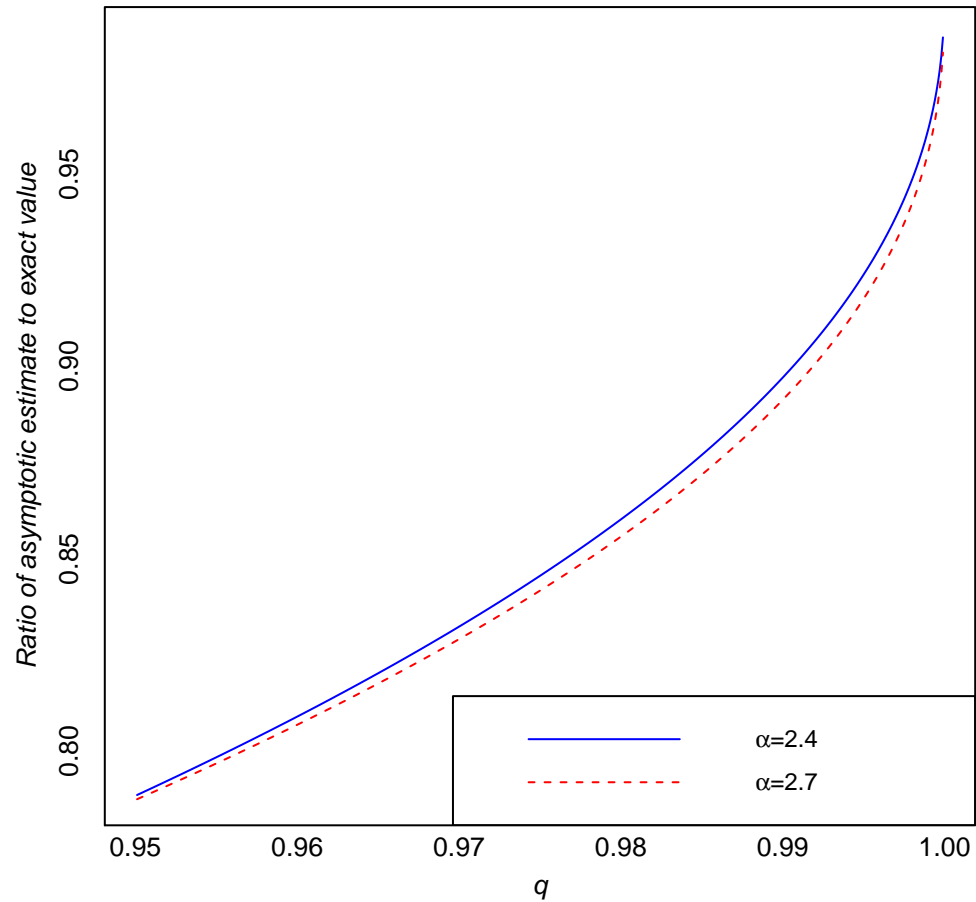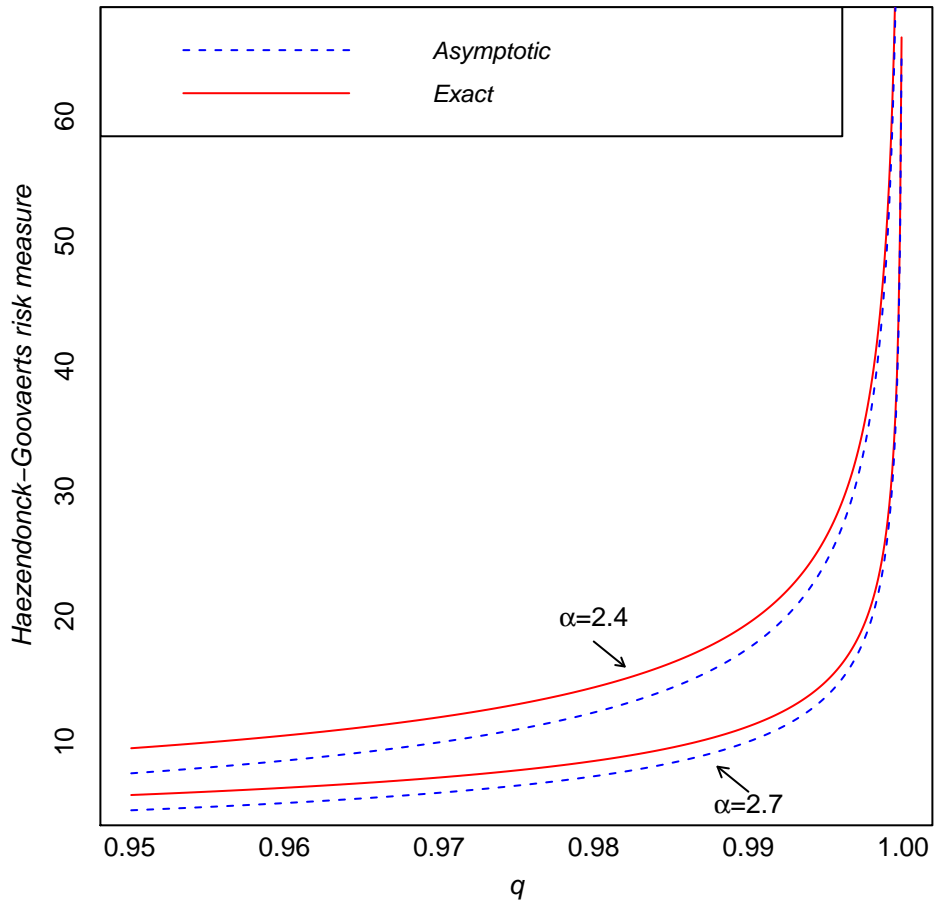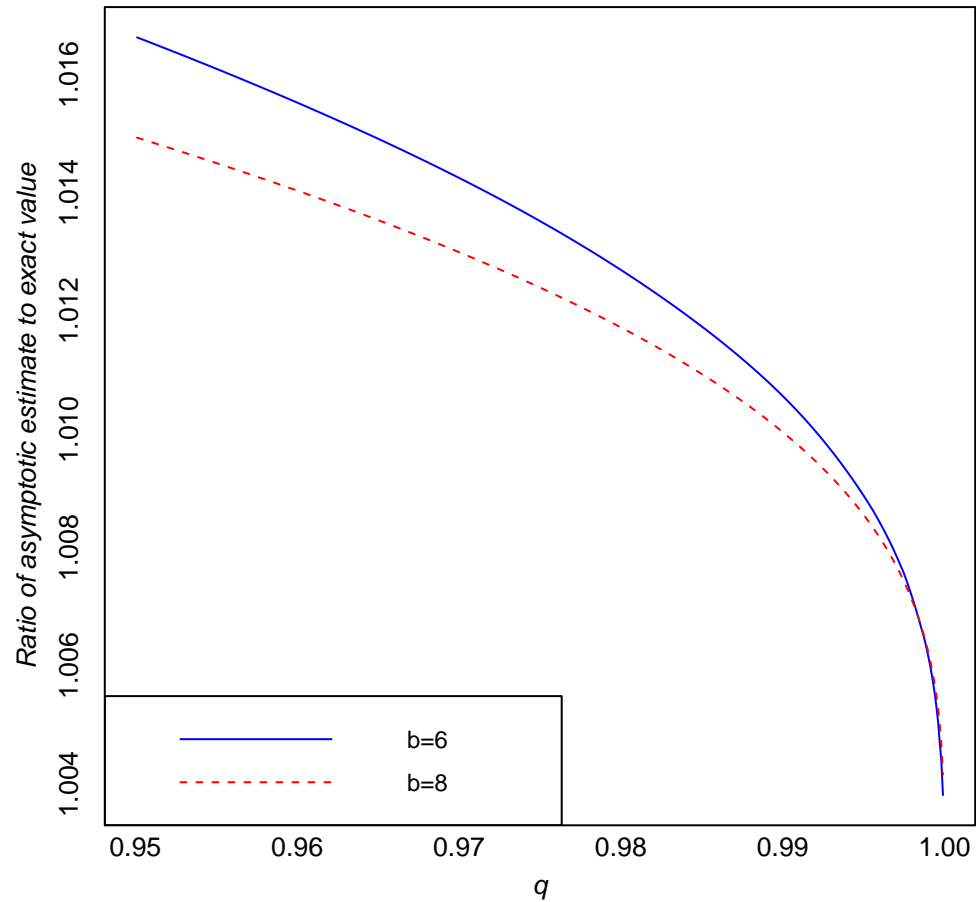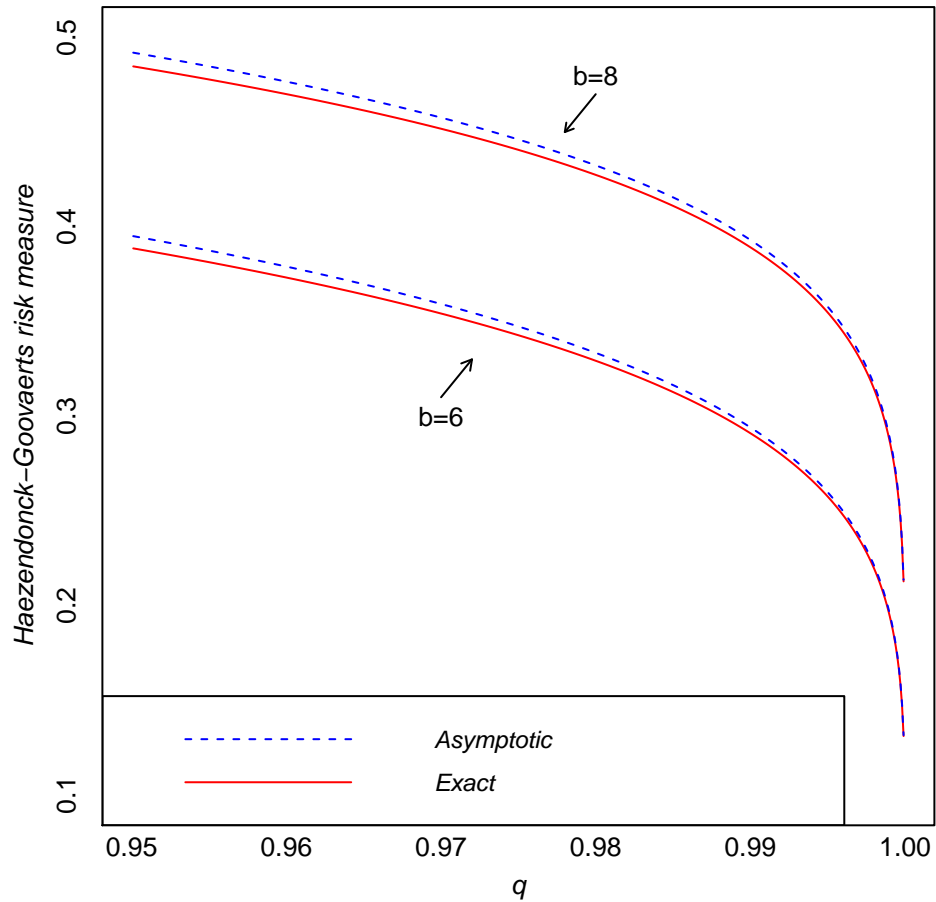
Graph 7.1 Pareto distribution
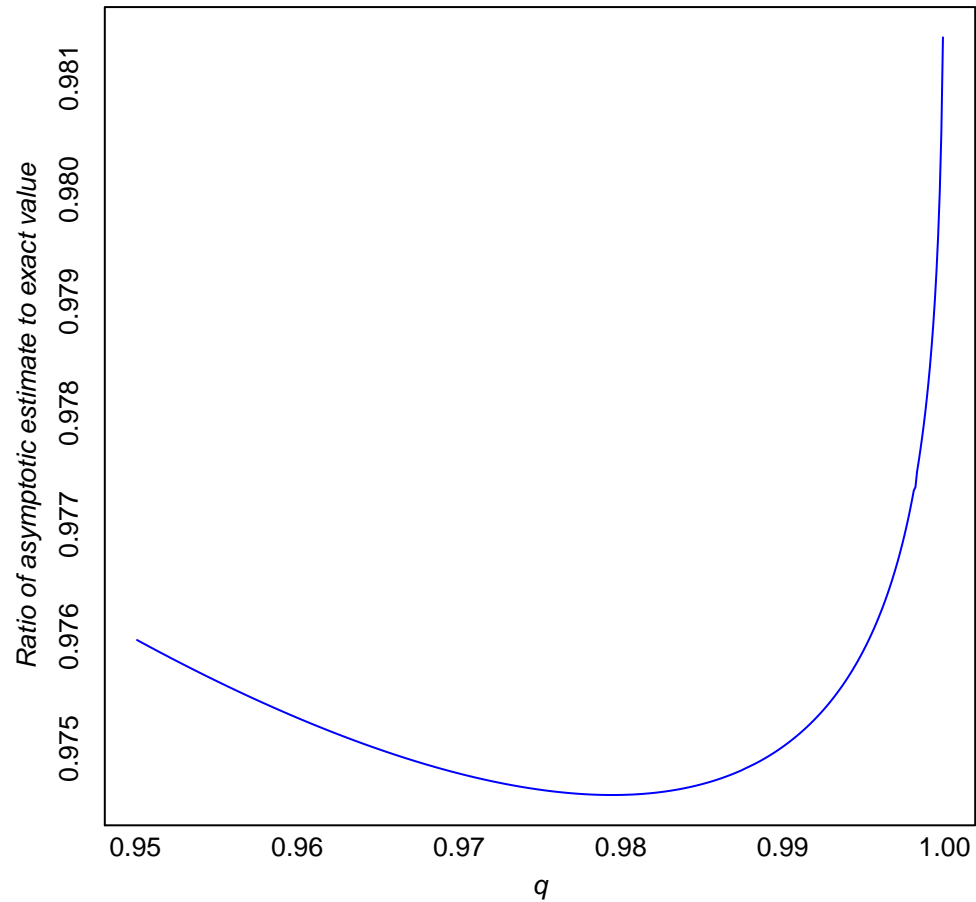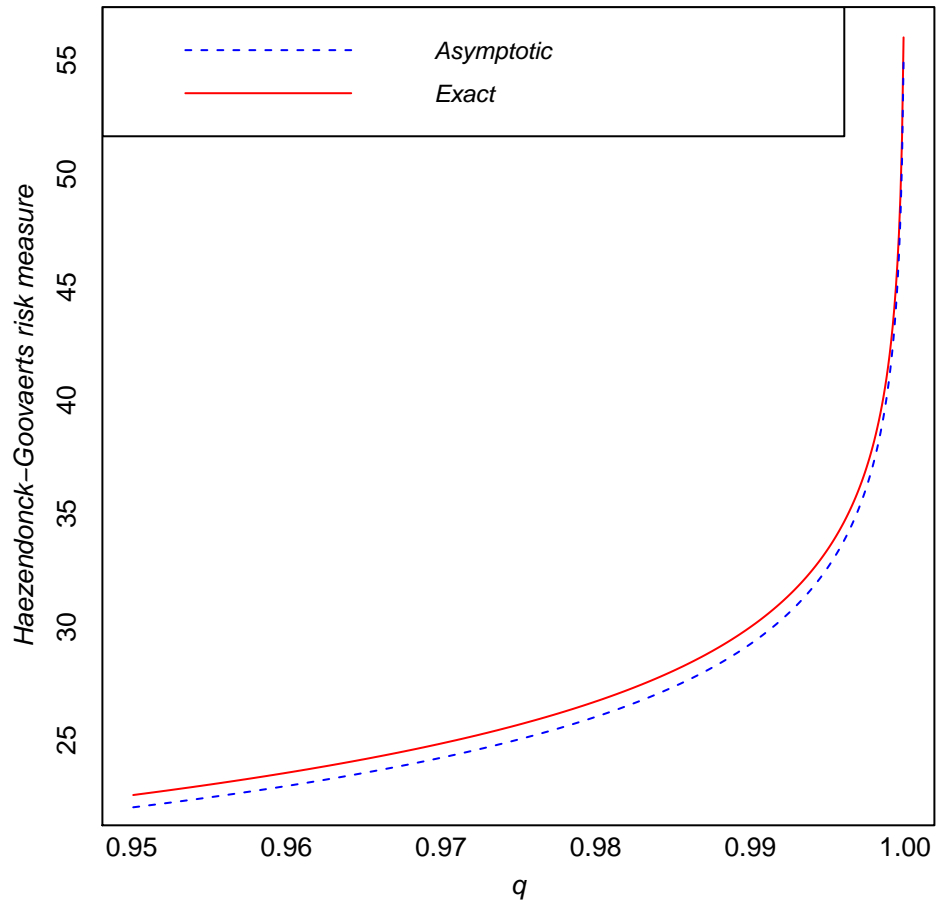
Graph 7.2 Lognormal distribution

Graph 7.3 Beta distribution