A Limit Distribution of Credit Portfolio Losses with Low Default Probabilities

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Abstract

This paper employs a multivariate extreme value theory (EVT) approach to study the limit distribution of the loss of a general credit portfolio with low default probabilities. A latent variable model is employed to quantify the credit portfolio loss, where both heavy tails and tail dependence of the latent variables are realized via a multivariate regular variation (MRV) structure. An approximation formula to implement our main result numerically is obtained. Intensive simulation experiments are conducted, showing that this approximation formula is accurate for relatively small default probabilities, and that our approach is superior to a copula-based approach in reducing model risk.

JEL classification: G210; G320

Keywords: Credit portfolio loss; Extreme risk; Limit distribution; Loss given default; Model risk; Multivariate regular variation; Tail dependence

1 Introduction

Credit risk management, although long residing in the finance literature, has attracted much research attention in the insurance/actuarial community; some recent papers include Vandendorpe et al. (2008), Donnelly and Embrechts (2010), Tang and Yuan (2013), Bernardi et

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al. (2015), Denuit et al. (2015), Hao and Li (2015), Scott and Metzler (2015), Wang et al. (2015), and Wei and Yuan (2016), among many others. This research area is of particular relevance to insurers because credit portfolios in the form of government or corporate bonds take up a large portion of insurers' balance sheets. According to National Association of Insurance Commissioners (NAIC), during 2010–2014 the insurance industry allocated at least 67% of its assets on bonds. Analysis of credit portfolio losses is therefore of enormous importance for both asset management and risk capital determination in insurance. Moreover, the pricing of related insurance, such as mortgage insurance and bond insurance, also requires careful investigation of the loss of the underlying credit portfolio.

Nevertheless, measuring credit portfolio losses has been a challenging task, and despite the progresses made during the past decades, some critical gaps remain. The culprit behind credit risk is extreme risks, which in general result from individual obligors' tail risks or their tail dependence, and both can shift adversely and dramatically in stress times. A credit portfolio with obligors healthy and little correlated in normal times may still suffer from concurrent defaults and losses when subject to common shocks in economic downturns. An increasing amount of evidences demonstrating this phenomena has been presented in the literature. For example, Pesola (2011) applies a reduced-form model estimated from pooled data from nine European countries during 1982–2004 to show that macroeconomic shocks exacerbate the extremely high levels of loan losses in different financial crises. Recent studies on mortgage loss given default (LGD) also report that the severity of loss during the subprime mortgage crisis of 2007–2009 was much higher than before; see e.g. Andersson and Mayock (2014). Clearly, portfolio loss models that neglect extreme risks may significantly underestimate the loss severity in the presence of common shocks, and this underestimation could ruin the effectiveness of prudential capital regulations in banking and insurance.

Generally, there are two approaches employed for LGD estimation in the literature. The first approach is to fit the distribution of LGD to data. This approach finds a number of regularities of the LGD distribution such as bimodality, concentration at both total loss and total recovery, and high variability; see e.g. Renault and Scaillet (2004), Schuermann (2004), and Calabrese and Zenga (2010). These regularities are what motivate our choice of loss settlement functions in conducting numerical studies in Section 5. The second approach to LGD estimation seeks to specify the determinants of LGD to capture it structurally. Altman et al. (2005) examine the recovery rates on corporate bond defaults over the period 1982–2002 and find that the aggregate recovery rates are a function of supply and demand for the securities. During their studies of residential mortgages, Qi and Yang (2009) and Park and Bang (2014) find that the loan-to-value (LTV) is the single most important determinant. Chava et al. (2011) summarize that contract characteristics, firm-specific variables, and

macroeconomic variables are important factors that affect the recovery rate, although none is consistently statistically significant. The determinants found in the literature can serve as guidance for latent variable specification in our model, although we do not purport, also it is not possible, that the latent variable can summarize all important default determinants; see also Duffie et al. (2009) for related discussions.

To address the challenge in the presence of extreme risks, the first distribution fitting approach is no longer as effective due to the sparsity of data in the tails; after all, the occurrence of extreme events such as financial crises is rare. Nevertheless, this literature lends us insights in modeling and simulation, and our experiments closely follow this literature to feature bimodality, concentration, and variability. Regarding the second approach, although no determinants are found universally valid, it can lend us a framework to model and understand the dynamics of shifting correlation of defaults.

The aim of this paper is to provide a methodological framework for modeling credit portfolio losses with extreme risks taken into account. Specifically, we employ a static latent variable model recently proposed by Tang and Yuan (2013) for credit portfolio losses, in which the LGD of an obligor is linked to its severity of default through a loss settlement function that increases from 0 to 1 as the severity of default increases. Such a loss settlement function is general in nature, so that the salient features of the LGD distribution found in the empirical studies can easily be incorporated. Note however that we refrain from empirically identifying the latent variables or the loss settlement functions.

Then we apply the multivariate extreme value theory (EVT) via the multivariate regular variation (MRV) framework to model the latent variables driving the obligors' default. This structure provides us with a rather flexible modeling framework; see Section 3 for details. It is well known that Gaussian copula models cannot describe the strong credit contagion possibly arising from common economic shocks, because of their failure to capture tail dependence; see e.g. Donnelly and Embrechts (2010) and Packham et al. (2016) for discussions on Gaussian copulas in modeling correlated defaults and the strong impact of tail dependence. By contrast, the MRV structure adopted in our paper is flexible enough to allow for tail dependence.

Our main result gives a limit distribution of the loss of a general credit portfolio with low default probabilities, which is relevant to a large array of financial institutions, especially insurance companies. This limit distribution captures the tail dependence information of the latent variables. An immediate application of the result is to generate stress testing scenarios for such institutions. Also, this result offers a solution to risk measure calculation for low default portfolios (LDPs), which is usually a challenging task for both financial institutions and regulators. In addition, the explicit expression of the limit distribution

enables us to conduct sensitivity analysis of the portfolio loss with respect to certain risk characteristics such as the tail dependence of the latent variables.

We run a large volume of numerical simulations to test our main result and find the following. When the individual default probabilities are reasonably small (say, below 5%), the limit distribution of the portfolio loss usually gives a good approximation for the true distribution. In particular, the underestimation of the portfolio loss can be substantial if tail dependence is not accounted for. Moreover, the limit distribution of the portfolio loss is not sensitive to the regularities of LGDs, which sheds new light that portfolio losses depend more on the underlying extreme risks than on other risk characteristics recognized in normal times.

The rest of this paper is organized as follows. Section 2 introduces our static latent variable model for credit portfolio losses, Section 3 describes the MRV structure for the latent variables, Section 4 derives the limit distribution of the loss of a low default portfolio, Section 5 presents our numerical results, Section 6 concludes the paper with some remarks, and finally, Section 7 collects related technical discussions, a corollary, and the proof of the main result.

2   A Static Latent Variable Model

In this section, we follow Tang and Yuan (2013) to use a static structural model to model credit portfolio losses. We assume that for each obligor there is a latent variable X that summarizes the determinants governing the process of the obligor's default. Such a latent variable model underlies CreditMetrics and KMV, and has been widely used in the literature. Depending on the portfolio under consideration, the latent variable may or may not be observable. For example, for corporate bond portfolios, the asset value used by CreditMetrics and KMV is an observable latent variable, and so is the running minimum of the asset value suggested by Giesecke (2004a). For mortgage portfolios, one may use the LTV as a latent variable, as justified by Qi and Yang (2009), which is also observable. By contrast, in a restricted version of CreditMetrics, Gordy (2000) considers the case where the latent variable is unobservable, equal to a linear combination of other input variables plus an independent and normally distributed error term. Note that under this framework, since the input variables themselves are typically observable, statistical inference on the latent variable is still possible. See also Merton (1974), Frey and McNeil (2003), Giesecke (2004b), Glasserman and Li (2005), and Duffie et al. (2009) for more discussions on latent variables in credit risk modeling.

Suppose that an obligor with its default driven by a latent variable X has a default

probability $p \in (0,1)$. Default occurs when X exceeds the default threshold, which is specified to be the $(1-p)$th quantile of the distribution function F of X,

$$F^{\leftarrow}(1-p) = \inf\left\{x \in \mathbb{R} : F(x) \geq 1-p\right\},$$

with $\inf \varnothing = \infty$ by convention. In a similar spirit to Merton-type modeling, we define the severity of default as

$$S = \frac{X}{F^{\leftarrow}(1-p)} - 1, \tag{2.1}$$

which is the percentage exceedance of the latent variable over the threshold, indicating how severe the obligor's default is. Previous literature in both theoretical and empirical regards indicates that loss ratio and default risk are linked; see e.g. Altman et al. (2005) and Frye and Jacobs Jr (2012). Accordingly, we use a settlement function

$$G(s) : \mathbb{R} \to [0,1], \text{ non-decreasing with } G(s) = 0 \text{ for } s \leq 0 \text{ and } G(\infty) = 1,$$

to relate the LGD to the default risk through S defined in (2.1). The settlement process after the obligor's default determines the amount of the defaulted debt that can be recovered, which we use $1 - G$ to quantify. The LGD of an obligor with exposure e is then $eG(S)$.

Consequently, the credit portfolio loss of a portfolio of d obligors is modeled by

$$L(\mathbf{p}) = \sum_{i=1}^{d} e_i G_i\left(\frac{X_i}{F_i^{\leftarrow}(1-p_i)} - 1\right), \tag{2.2}$$

where, for each obligor i, X_i is its latent variable distributed by F_i, G_i is its settlement function, $e_i > 0$ is the exposure to the obligor, $p_i \in (0,1)$ is a given small probability of default, and $F_i^{\leftarrow}(1-p_i)$ is the corresponding default threshold. Without loss of generality, the d exposures $e_1, \ldots, e_d$ are scaled so that $\sum_{i=1}^{d} e_i = 1$. Note that the LGD extensively discussed in the literature is essentially the portfolio loss given its positiveness, and therefore our study of $L(\mathbf{p})$ can be readily used to characterize the LGD.

Several advantages of using model (2.2) for the credit portfolio loss are worth highlighting. First, it departs from the traditional way of using the historical average LGD, which is a constant, to calculate the expected loss of a credit portfolio. Ours considers the LGD per se as a randomized parameter that accords with empirical evidence. Second, our modeling of the portfolio loss incorporates Merton-type structural modeling of default risk in a more general way, which enables us to easily extend to include an array of default determinants in a coherent manner. Third, the general expression of loss settlement functions lends us advantages to incorporate the aforementioned salient features of LGD. The linkage of the LGD to the default probabilities is explicitly expressed. Departing from the literature where a mixture of specified distributions is often constructed to model and estimate the LGD, our expression is quite general.

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3 Multivariate Regular Variation (MRV)

Typically, the multivariate EVT method can be applied in two steps. The first step applies the univariate EVT to fit the marginal distribution functions, and the second fits a dependence structure (usually through a copula) to join the marginals. Credit risk management poses at least two challenges to this way of applying EVT. First, credit defaults are rare events and data are usually sparse, truncated or censored, which cannot guarantee a copula to be fitted globally. Second, as discussed in the literature, modeling credit portfolio losses in this way may be subject to substantial model risk. The copula chosen for the dependence structure can be misspecified, and due to the scarcity of credit loss data it is hard to calibrate. On the other hand, the portfolio loss distribution can be largely affected by even small changes to the parameters of the copula, and the effect is usually remarkable in the tail. See e.g. Frey and McNeil (2002, 2003) for related discussions.

The concept of MRV stems from multivariate EVT and has become a promising modeling framework that models multivariate heavy-tailed risks and their tail dependence structure in a unified manner. In modeling extreme risks, the MRV method is advantageous over the usual one that assembles the univariate EVT approach and copula approach, in that the MRV method offers a nonparametric modeling of the tail dependence, and thus reduces the model risk. Moreover, since an MRV structure only focuses on the (right) tail area of risks, it is no longer a concern to have left truncated or censored data. Just like an EVT method having an advantage in fitting tails over e.g. a least square method, by focusing only on the tail area fitting an MRV structure also has an advantage in fitting tail dependence over fitting a copula globally. In summary, the two challenges of credit risk modeling mentioned above can be avoided to a certain extent by using an MRV model.

To understand MRV, we start with the concept of regular variation. A positive function $h(\cdot)$ on $\mathbb{R}_+ = [0, \infty)$ is said to be regularly varying at ∞ with regularity index $\gamma \in \mathbb{R}$, written as $h(\cdot) \in \mathrm{RV}_\gamma$, if

$$\lim_{x \to \infty} \frac{h(xy)}{h(x)} = y^\gamma, \qquad y > 0.$$

We say that $h(\cdot)$ is slowly varying when $\gamma = 0$. It is known that $h(\cdot) \in \mathrm{RV}_\gamma$ if and only if

$$h(x) = x^\gamma l(x) \tag{3.1}$$

for some slowly varying function $l(\cdot)$. See Bingham et al. (1987) and Resnick (1987) for textbook treatments of regular variation.

For a random variable X distributed by F on $\mathbb{R}_+$ with tail $\overline{F} = 1 - F \in \mathrm{RV}_{-\alpha}$, $\alpha > 0$, by relation (3.1) $\overline{F}(x)$ decays roughly at a power rate as $x \to \infty$. Hence, the corresponding random variable X and its distribution function F are said to be of Pareto type. Commonly

used distributions such as Pareto, Student's t, Snedecor's F, Burr, loggamma, Fréchet, and inverse gamma distributions are all of Pareto type. Note that a Pareto-type distribution is heavy tailed in the sense that its right tail is heavier than that of any exponential distribution. Empirical evidence shows that many economic risk factors are heavy tailed. See e.g. Embrechts et al. (1997) and Gabaix (2009) for general discussions on the applications of Pareto-type distributions in modeling economic factors.

The concept of MRV of a random vector is defined through regular variation of its joint tail. Precisely, a random vector $\mathbf{X} = (X_1, \ldots, X_d)$ on $\mathbb{E} = [0, \infty]^d \backslash \{\mathbf{0}\}$, or its distribution, is said to possess a multivariate regularly varying tail if there exists a distribution function F and a Radon measure ν, not identically 0 over $[0, \infty)^d \backslash \{\mathbf{0}\}$, such that

$$\lim_{x \to \infty} \frac{1}{\overline{F}(x)} P\left(\bigcup_{i=1}^{d} (X_i > xt_i)\right) = \nu\left[\mathbf{0}, \mathbf{t}\right]^c, \qquad \mathbf{t} > \mathbf{0}. \tag{3.2}$$

In (3.2) and throughout the paper, for a Borel subset A of $\mathbb{E}$, its measure $\nu(A)$ is often abbreviated as νA as long as no confusion arises, and A^c denotes the complement set $\mathbb{E} \backslash A$. Note that a measure on $\mathbb{E}$ is called Radon if its value is finite on every compact Borel subset of $\mathbb{E}$. The definition of MRV in (3.2) implies that $\overline{F} \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 0$, and hence, we write $\mathbf{X} \in \mathrm{MRV}_{-\alpha}$. The requirement of an MRV structure for a random vector $\mathbf{X}$ is essentially a regular-variation condition for the probability of at least one component of $\mathbf{X}$ being large. This also means that at least one component of $\mathbf{X}$ is of Pareto type. Roughly, any multivariate distribution with regularly varying marginals joined by a practically useful copula possesses an MRV structure. Although different choices of F may result in different limit measures, these limit measures can only differ by a constant factor.

The information of tail dependence in the upper-right tail of $\mathbf{X}$ is contained in the limit measure ν. Under (3.2), if $\nu\left(\mathbf{1}, \boldsymbol{\infty}\right] > 0$ then

$$\lim_{x \to \infty} \frac{1}{\overline{F}(x)} P\left(\bigcap_{i=1}^{d} (X_i > x)\right) = \nu\left(\mathbf{1}, \boldsymbol{\infty}\right] > 0,$$

which means that $\mathbf{X}$ exhibits large joint movements and can thus be used to model the latent variables subject to common shocks. In the case that the limit measure ν is concentrated on a straight line $\{\mathbf{x} > \mathbf{0} : l_1 x_1 = \cdots = l_d x_d\}$ for some $\mathbf{l} > \mathbf{0}$, the components of $\mathbf{X}$ are so-called fully tail dependent, of which comonotonicity is a special case. In contrast, if $\nu\left(\mathbf{1}, \boldsymbol{\infty}\right] = 0$, then $\mathbf{X}$ does not exhibit large joint movements.

Since its introduction by de Haan and Resnick (1981), the concept of MRV has found applications in many areas of insurance, finance, and risk management that involve extreme risks. For example, Schmidt (2003) applies MRV in credit risk modeling by making a

connection between MRV and various copulas, and Böcker and Klüppelberg (2010) find its application in operational risk management. Additional applications can be found in Mainik and Rüschendorf (2010), Part IV of Rüschendorf (2013), and Tang and Yuan (2013), among others. For thorough theoretical discussions on MRV, we refer the reader to Resnick (1987, 2007).

4   A Limit Distribution of the Portfolio Loss

4.1   Modeling assumptions

Consider the credit portfolio loss (2.2) in which the latent vector $\mathbf{X}$ has marginal distributions F_1, ..., F_d. Assume that $\mathbf{X}^+ = (\max\{X_1, 0\}, \dots, \max\{X_d, 0\})$ follows an MRV structure, or more precisely, relation (3.2) holds for $\mathbf{X}^+$ for some auxiliary distribution function F with $\overline{F} \in \mathrm{RV}_{-\alpha}$, $\alpha > 0$, and some limit measure ν. Further assume that, for each $i = 1, \dots, d$, the limit measure ν satisfies

$$a_i = \nu\left([0, \infty] \times \cdots \times (1, \infty] \times \cdots \times [0, \infty]\right) > 0,$$

which implies that

$$\lim_{x \to \infty} \frac{\overline{F_i}(x)}{\overline{F}(x)} = a_i. \tag{4.1}$$

Hence, all latent variables X_1, ..., X_d are comparable in the right tail.

Furthermore, we assume that these default probabilities p_1, ..., p_d decay to 0 at rates of the same order, or more precisely, there are positive constants b_1, ..., b_d such that

$$\lim_{p \downarrow 0} \frac{p_i}{p} = b_i, \qquad i = 1, \dots, d. \tag{4.2}$$

This is to assume that the assets in the credit portfolio are currently of investment grade, and hence, the default probabilities of all individual assets are small and the credit portfolio is an LDP. Note that regulated financial institutions may need such portfolios to meet capital requirements. According to NAIC, during 2010–2014 at least 94% bond holdings of the insurance industry are of investment grade (i.e., rated Baa or better under Moody's rating tiers). Moreover, according to Moody's (2015), U.S. municipal bonds with a Moody's rating of A or better, Baa, or B have a historical one-year default rate of 0, 0.01%, or 2.92%, respectively. Global corporate bonds with a Moody's rating of A, Baa, or Ba have a historical one-year default rate of 0.06%, 0.17%, or 1.07%, respectively. This motivates us to consider LDPs with default probabilities p_1, ..., p_d close to 0. Note that the portfolio loss of an LDP is more challenging to estimate than that of a below-investment-grade portfolio, because for the latter default occurrences are more frequent and hence can be simulated relatively more

easily. The asymptotic approach used here provides a possible avenue to overcoming the estimation challenge.

Under the modeling assumptions above, we shall study the limit distribution of $L(\mathbf{p})$ as $p \downarrow 0$. We conduct the study through the survival function of $L(\mathbf{p})$ to reflect our focus on the tail risk of the portfolio.

4.2 An exploratory numerical study

For arbitrarily fixed $l \in [0,1]$, we first give a heuristic analysis of the rare-event probability $P(L(\mathbf{p}) > l)$. For $L(\mathbf{p})$ to be larger than l, at least one of the obligors needs to default, i.e., its latent variable X_i needs to exceed the default threshold $F_i^{\leftarrow}(1 - p_i)$, which by (4.2) has a probability of order p. Depending on the value of l, it may require multiple obligors to default, which under the MRV structure has a probability at most of order p. Thus, we expect that as $p \downarrow 0$ the probability of $L(\mathbf{p})$ exceeding l decays to 0 at rate p. An exploratory numerical study is conducted below to back this surmise.

Suppose that the portfolio consists of $d = 5$ obligors with equal exposures $e_1 = \cdots = e_5 = 20\%$. The latent variable of each obligor i is distributed by a Pareto distribution with shape parameter α and scale parameter θ_i,

$$F_i(x) = 1 - \left(\frac{\theta_i}{x + \theta_i} \right)^{\alpha}, \qquad x > 0. \tag{4.3}$$

The dependence structure of $\mathbf{X}$ is given by a Gumbel copula

$$C(\mathbf{u}) = \exp \left\{ - \left(\sum_{i=1}^{d} (-\ln u_i)^r \right)^{1/r} \right\}, \qquad \mathbf{u} \in (0,1)^d, 1 < r < \infty \tag{4.4}$$

which yields tail dependence in the upper-right tail and tail independence in the lower-left tail; see Section 7.4 of McNeil et al. (2015). We set the parameters to $\alpha = 1$, $\theta_i = i$, and $r = 5$, respectively. Meanwhile, the loss settlement functions for the 5 obligors are assumed to be the same and chosen to be either a mixed beta distribution G^I or a uniform distribution G^{II}, respectively having probability density functions

$$\begin{cases} g^I(x) = s \times \mathrm{beta}(x; \beta_{11}, \beta_{12}) + (1-s) \times \mathrm{beta}(x; \beta_{21}, \beta_{22}), & 0 < s, x < 1, \\ g^{II}(x) = \frac{x}{y_G}, & 0 < x < y_G, \end{cases} \tag{4.5}$$

where $\mathrm{beta}(x; \beta_1, \beta_2)$ denotes the beta probability density function

$$\mathrm{beta}(x; \beta_1, \beta_2) = \frac{\Gamma(\beta_1 + \beta_2)}{\Gamma(\beta_2)\Gamma(\beta_2)} x^{\beta_1 - 1}(1 - x)^{\beta_2 - 1}, \qquad 0 < x < 1,$$

with $\Gamma(\cdot)$ being the gamma function and β_1, β_2 being two positive shape parameters. The choice G^I is motivated by the bimodal feature of LGD, and the choice G^{II} is provided as a robustness check. In our simulation, the parameters appearing in (4.5) are set to

$$s = 0.7, \quad \beta_{11} = 2, \quad \beta_{12} = 5, \quad \beta_{21} = 5, \quad \beta_{22} = 2, \quad \text{and} \quad y_G = 2,$$

the last specification meaning that the percentage loss will attain its maximum 100% when the latent variable exceeds three times its $(1 - p)$th quantile.

For $p_1 = \cdots = p_5 = p$ ranging from 0.1% to 5%, l chosen to be 0.1 or 0.3, we estimate the probability $P\left(L(p) > l\right)$ empirically using 10^6 simulation samples of $\mathbf{X}$. The following graphs demonstrate the decaying rate of $P\left(L(p) > l\right)$ as p approaches 0.

$$\boxed{\text{Figure 4.1 is here.}}$$

We observe that in each graph the points all appear roughly on a straight line. This suggests that the probability $P\left(L(p) > l\right)$ decays at rate p as $p \downarrow 0$; that is, when p is small,

$$P\left(L(\mathbf{p}) > l\right) \approx c(l)p$$

for some positive coefficient $c(l)$. In our theoretical contribution below, we make this statement precise.

4.3   The main result

As the main contribution of this paper, below we derive the limit distribution of $L(\mathbf{p})$ (2.2) as $p \downarrow 0$.

Theorem 4.1 *Under the modeling assumptions in Subsection 4.1, it holds for every $l \in [0,1]$ that*

$$\lim_{p \downarrow 0} \frac{P\left(L(\mathbf{p}) > l\right)}{p} = \nu\left(A_l\right), \tag{4.6}$$

where

$$A_l = \left\{ \mathbf{x} \in [0, \infty]^d : \sum_{i=1}^d e_i G_i \left((a_i/b_i)^{-1/\alpha} x_i - 1 \right) > l \right\}.$$

Pedantically, it is possible that $\nu\left(A_l\right) = 0$, for which case relation (4.6), while still valid, can no longer serve as an exact approximation for $P\left(L(\mathbf{p}) > l\right)$. Nevertheless, this happens only when $\mathbf{X}$ is tail independent and only for relatively large l. For example, in the two-dimensional case it is easy to see that, for $l > \max\{e_1, e_2\}$, the set A_l does not contain any point on the axes, and thus, $\nu\left(A_l\right) = 0$ if X_1 and X_2 are tail independent. In conclusion,

for most practical cases we have $\nu(A_l) > 0$ and relation (4.6) gives an exact asymptotic formula for $P(L(\mathbf{p}) > l)$.

A substantial simplification of relation (4.6) can be made in the case that $\mathbf{X}$ is fully tail dependent, i.e., the limit measure ν is concentrated on a straight line; see Corollary 7.1 in the Appendix.

5   Numerical Studies

This section contains three parts. First, we use numerical examples to show why the use of an MRV structure can help us arrive at a better estimate for the portfolio loss distribution when there is uncertainty in tail dependence. Second, we check the accuracy of the asymptotic formula (4.6) as an approximation for the survival function of $L(\mathbf{p})$. And last, we present some examples to demonstrate the limit distribution approximated by our theorem.

In all three subsections below, we consider a credit portfolio consisting of $d = 5$ obligors with equal exposures. Demonstrations for larger portfolios would require more computation time but we expect the conclusions to be the same. For the loss settlement function, we consider the two choices G^I and G^{II} in Subsection 4.2, whose probability density functions are given by (4.5) with parameters

$$s = 0.7, \quad \beta_{11} = 2, \quad \beta_{12} = 5, \quad \beta_{21} = 5, \quad \beta_{22} = 2, \quad \text{and} \quad y_G = 2.$$

The individual default probabilities $p_1, \ldots, p_d$ are assumed to be all equal to p. Hence, $b_1 = \cdots = b_d = 1$ and $L(\mathbf{p})$ can be rewritten as $L(p)$.

We would like to point out that, although the numerical results are presented under a simplified setup, according to our extensive tests with more realistic parameters (e.g., nonidentical exposures, nonidentical loss settlement functions, and nonidentical individual default probabilities), the setup here can well represent the general cases for illustrating the key insights.

5.1   Uncertainty in tail dependence

Using a copula-based approach to describe the tail dependence of a credit portfolio requires an appropriate choice of the copula, and the possibility of choosing a wrong copula poses substantial model misspecification risk. In this subsection we consider examples where the tail dependence structure of the latent vector $\mathbf{X}$ is misspecified, and then investigate the impact of the misspecification on the portfolio loss distribution. Specifically, we use a Gumbel copula to generate the benchmark tail dependence of a credit portfolio, and use a

Gaussian copula and a t copula to fit the generated data to see how the fitted distributions diverge from the benchmark.

Now that our concern in this subsection is mainly about tail dependence and that univariate distribution functions can be easily estimated by standard statistical approaches, we simply assume that the marginal distributions of $\mathbf{X}$ are known to be the Pareto distribution (4.3) with $\alpha = 1$ and $\theta_i = i$. Moreover, we consider two cases with the default probability p equal to 0.5% or 1%.

Our study is based on a synthetic dataset containing 10^6 samples of $\mathbf{X}$, simulated using a Gumbel copula (4.4) with $r = 2$ or 5. Suppose that a Gaussian copula or a t copula, both with an equicorrelation matrix, is (mis)specified for the data. For both copulas, we use the function `fitCopula` in the `R` software to obtain maximum likelihood estimates for the parameters; see Yan (2007) for details about the function `fitCopula`. The estimation results are summarized in Table 5.1.

$$\boxed{\text{Table 5.1 is here.}}$$

For $r = 2$ and $r = 5$, the off-diagonal elements of the correlation matrix of the Gaussian copula are estimated to be 0.70 and 0.95, respectively, those of the t copula are 0.71 and 0.95, respectively, and the degree of freedom of the t copula is 8.3 and 4.3, respectively.

Next, for each value of l between 10% and 90%, with step-size 2.5%, we estimate the probability $P\left(L(p) > l\right)$ empirically, by generating 10^6 samples of $\mathbf{X}$, using the (true) Gumbel copula with $r = 2$ or 5, the fitted Gaussian copula, and the fitted t copula. The obtained portfolio distribution is compared for the three cases to show the impact of misspecification. We compare the portfolio loss distribution, again in terms of its survival function $P\left(L(p) > l\right)$, and demonstrate the results in Figures 5.1 and 5.2.

$$\boxed{\text{Figures 5.1 and 5.2 are here.}}$$

One may observe severe discrepancies among the estimated survival functions for the three cases, especially in the tail area. For example, the estimation for the probability of the loss exceeding 90% of the total exposure under the true Gumbel-copula specification can be 25 times as much as under the Gaussian-copula specification, and can be 7 times as much as under the t-copula specification. Therefore, a misspecified dependence structure may lead to a substantial over- or under-estimation of the portfolio loss and its risk measures.

We demonstrate in the next subsection that the limit distribution obtained by Theorem 4.1, free of copula specifications, can provide a very accurate approximation for the portfolio loss distribution.

5.2 Accuracy of the approximation

In this subsection we check the accuracy of the approximation for the survival probability $P(L(p) > l)$ provided by relation (4.6). Again, we assume that each latent variable X_i is distributed by the Pareto distribution (4.3) with $\alpha = 1$ and $\theta_i = i$ for $i = 1, \ldots, d$, and that their dependence structure is described by the Gumbel copula (4.4) with $r = 5$. This leads to an MRV structure for the vector $\mathbf{X}$; see Tang and Yuan (2013) for related discussions.

For each of p ranging from 0.1% to 5% and each of l equal to 10%, 30%, or 50%, on the one hand we simulate 10^6 (and for comparison purpose 10^7) samples of $\mathbf{X}$ and estimate the probability $P(L(p) > l)$ empirically, and on the other hand we use relation (4.6) in Theorem 4.1 to approximate the probability by $\nu(A_l)\,p$. The value of $\nu(A_l)$ is estimated according to the method given by Section 9.2 of Resnick (2007), also with a sample size of $N = 10^6$. Precisely, with a proper choice of k such that both k and N/k are large, we estimate $\nu(A_l)$ by

$$\frac{1}{k} \sum_{i=1}^{N} \epsilon_{\frac{\mathbf{x}_i}{F_1^{\leftarrow}(1-k/N)}} (A_l),$$

where ϵ is the Dirac measure and the random vectors $\mathbf{X}_1, \ldots, \mathbf{X}_N$ are a sample from $\mathbf{X}$. The value of k is usually determined by some exploratory methods; see Stărică (1999) and Subsection 9.2.4 of Resnick (2007) for more details, and see Einmahl and Segers (2009) and Kiriliouk et al. (2015) for more discussions on the estimation of such a limit measure. We remark that, although it still requires a simulation to estimate $\nu(A_l)$, such a simulation usually works more efficiently than using rare event simulation to directly estimate the survival probability empirically.

The empirical estimates and approximations are compared in Figures 5.3–5.6.

$$\boxed{\text{Figures 5.3–5.6 are here.}}$$

The left graphs of Figures 5.3 and 5.5 show that the absolute differences between the empirical estimates and approximations are always small for p reasonably close to 0. The right graphs show that the ratio of the empirical estimates to approximations is close to 1, confirming the quality of approximation. The fluctuations of the ratio for p around 0 are due to the fluctuations of the empirical estimations, which become less stable when the event of $L(p)$ exceeding l becomes rarer as p gets closer to 0. The asymptotic approximations, by nature, are not subject to such a limitation. In fact, as is seen from Figures 5.4 and 5.6, increasing the sample size to 10^7 leads to much improved convergence around 0. This comparison with different sample sizes is also indicative of the high simulation cost in the

case of very low default probabilities and the benefit of using the asymptotic method. Although Figures 5.3 and 5.5 show that the convergence of (4.6) is fast under the setup above, we need to point out that different choices of parameters may lead to different convergence rates. For example, our tests show that a larger value of α may lead to a slower convergence, while changes of the value of r generally do not affect the convergence rate.

5.3   Demonstration of the limit distribution

Using the approximation provided by relation (4.6) and the same parameters as in Subsection 5.2, Figure 5.7 demonstrates the distribution of the portfolio loss $L(p)$ in terms of its survival function, for $p = 1\%$, 3% and $r = 2$, 3, and 5.

$$\boxed{\text{Figure 5.7 is here.}}$$

Obviously, since our approximation for $P(L(p) > l)$ is proportional to p, the survival probabilities in the right graphs, which are for $p = 3\%$, are just three times of those in the left graphs, which are for $p = 1\%$. This feature may be useful when there is a need for comparing different credit portfolios, of the same obligors, consisting of debts with different seniority.

In addition, notice that a larger value of r leads to a stronger tail dependence. We can clearly see that, for a value of l that requires multiple defaults for $L(p)$ to exceed l, a stronger tail dependence leads to a larger tail probability of $L(p)$. For example, for $l = 30\%$, since each exposure is 20%, it requires at least two obligors to simultaneously default. A stronger tail dependence leads to a larger probability of simultaneous defaults and, hence, a larger tail probability of $L(p)$. This also partly explains the intriguing reverse of the relations observed in Figure 5.7 for values of l that can be exceeded by one single default (i.e., $l < 20\%$).

6   Concluding Remarks

In this paper, we have studied the distribution of the credit portfolio loss of an LDP in the presence of extreme risks. Our study is conducted in a static latent variable model in which the percentage LGD of each obligor is assumed to be random and linked to the severity of the obligor's default through a loss settlement function. Such loss settlement functions can be general enough to accommodate empirical features of LGD distributions. Furthermore, the obligors' defaults are driven by corresponding latent variables. The extreme risks are then represented by the obligors' latent variables that are heavy tailed and tail dependent,

modeled by an MRV structure. We demonstrate through numerical examples that the nonparametric feature of the MRV structure can help reduce model risk.

As our main result, we obtain a limit distribution of the credit portfolio loss as individual default probabilities tend to 0. For many practical cases, our formula can serve as an accurate approximation for the portfolio loss distribution for reasonably small default probabilities, say, not greater than 5%; by its nature the asymptotic method works even better for smaller default probabilities. Our study also reveals that in the presence of extreme risks, as individual default probabilities decrease to 0 the tail probability of the portfolio loss with tail dependent latent variables decreases at the same rate.

Note that our limit procedure is taken over $p \to 0$ with the portfolio size d being fixed. It would be interesting to also study large credit portfolios where d tends to infinity with p either fixed or tending to 0 at a similar rate.

7 Appendix

In order to gain a rigorous understanding of the concept of MRV, let us first introduce the concept of vague convergence of Radon measures. Consider a d-dimensional cone $\mathbb{E} = [0, \infty]^d \backslash \{\mathbf{0}\}$ equipped with a Borel sigma-field $\mathcal{B}$. Denote the space of nonnegative Radon measures on $\mathbb{E}$ by $M_+(\mathbb{E})$. For a sequence of Radon measures $\{\mu, \mu_n, n \in \mathbb{N}\}$ in $M_+(\mathbb{E})$, we say that $\mu_n \to \mu$ vaguely if the relation

$$\lim_{n \to \infty} \int_{\mathbb{E}} h(z) \mu_n(\mathrm{d}z) = \int_{\mathbb{E}} h(z) \mu(\mathrm{d}z)$$

holds for every nonnegative continuous function $h(\cdot)$ with compact support. It is known that $\mu_n \to \mu$ vaguely in $M_+(\mathbb{E})$ if and only if the convergence

$$\lim_{n \to \infty} \mu_n \left[\mathbf{0}, \mathbf{t}\right]^c = \mu \left[\mathbf{0}, \mathbf{t}\right]^c$$

holds for every continuity point $\mathbf{t} \in \mathbb{E}$ of the limit $\mu \left[\mathbf{0}, \cdot\right]^c$. See Lemma 6.1 and Subsection 3.3.5 of Resnick (2007) for this assertion and related discussions.

Note that relation (3.2) essentially means that

$$\frac{1}{\overline{F}(x)} P\left(\frac{\mathbf{X}}{x} \in \cdot\right) \to \nu(\cdot) \qquad \text{vaguely in } M_+(\mathbb{E}).$$

The very definition of MRV by relation (3.2) implies that the limit measure ν is homogeneous; that is,

$$\nu(tB) = t^{-\alpha} \nu(B) \tag{7.1}$$

15

holds for some $\alpha > 0$ and all Borel subset B. See Page 178 of Resnick (2007) for related proofs.

In order to apply vague convergence on a set, one often needs to verify that ν assigns no mass on the boundary of the set. The following lemma describes an elementary property of the limit measure ν, which is helpful for such a verification:

Lemma 7.1 *Let ν be a Radon measure on $\mathbb{E}$ satisfying the homogeneity property (7.1) with $\alpha > 0$, and let Δ be a Borel set bounded away from $\mathbf{0}$. If $t\Delta \cap \Delta = \varnothing$ for every $t > 1$, then $\nu(\Delta) = 0$.*

Proof. Since Δ is bounded away from $\mathbf{0}$, so is the union U of $t\Delta$ over $1 \leq t \leq 2$, which implies that $\nu(U) < \infty$. If $\nu(\Delta) > 0$, then, with $\mathbb{Q}$ denoting the set of rational numbers,

$$\nu(U) \geq \nu\left(\bigcup_{t \in \mathbb{Q} \cap [1,2]} t\Delta\right) = \sum_{t \in \mathbb{Q} \cap [1,2]} t^{-\alpha}\nu(\Delta) \geq 2^{-\alpha} \sum_{t \in \mathbb{Q} \cap [1,2]} \nu(\Delta) = \infty,$$

which is a contradiction. Hence, $\nu(\Delta) = 0$ must hold. ∎

Corollary 7.1 *If in Theorem 4.1 the latent vector $\mathbf{X}$ is fully tail dependent, then relation (4.6) is simplified to relation (7.2) below.*

Proof. It is easy to see that the limit measure ν is concentrated on the straight line

$$a_1^{-1/\alpha} x_1 = \cdots = a_d^{-1/\alpha} x_d, \qquad \mathbf{x} \in [0, \infty]^d.$$

On this straight line we have

$$\sum_{i=1}^{d} e_i G_i\left((a_i/b_i)^{-1/\alpha} x_i - 1\right) = \sum_{i=1}^{d} e_i G_i\left(b_i^{1/\alpha} a_1^{-1/\alpha} x_1 - 1\right).$$

Denote by $h(x_1)$ the right-hand side above, which is a non-decreasing function in x_1. It follows from Theorem 4.1 that

$$
\begin{aligned}
\lim_{p \downarrow 0} \frac{P(L(\mathbf{p}) > l)}{p} &= \nu\{\mathbf{x} : h(x_1) > l\} \\
&= \nu\{\mathbf{x} : x_1 > h^{\rightarrow}(l)\} \\
&= a_1 (h^{\rightarrow}(l))^{-\alpha},
\end{aligned}
\tag{7.2}
$$

where $h^{\rightarrow}(l) = \sup\{x \in \mathbb{R} : h(x) \leq l\}$ with $\sup \varnothing = -\infty$ by convention, and the second step holds because ν assigns no mass on the plain $\{\mathbf{x} : x_1 = h^{\rightarrow}(l)\}$ by Lemma 7.1. ∎

Proof of Theorem 4.1. In this proof, for two positive functions $f_1(\cdot)$ and $f_2(\cdot)$, as usual we write $f_1(\cdot) \sim f_2(\cdot)$ if $\lim f_1(\cdot)/f_2(\cdot) = 1$ and write $f_1(\cdot) \lesssim f_2(\cdot)$ if $\limsup f_1(\cdot)/f_2(\cdot) \le 1$.

It is clear that

$$P(L(\mathbf{p}) > l) = P\left(\left(\frac{X_1}{F_1^{\leftarrow}(1-p)}, \ldots, \frac{X_d}{F_d^{\leftarrow}(1-p)}\right) \in B_l\right), \tag{7.3}$$

where $B_l = \left\{\mathbf{x} \in [0, \infty]^d : \sum_{i=1}^d e_i G_i(x_i - 1) > l\right\}$. For each $i = 1, \ldots, d$, by Proposition 0.8(V) of Resnick (1987), relations (4.1)–(4.2) imply that, as $p \downarrow 0$,

$$F_i^{\leftarrow}(1-p) \sim (a_i/b_i)^{1/\alpha} F^{\leftarrow}(1-p).$$

Thus, it holds for arbitrarily chosen $\varepsilon \in (0,1)$ and all small $p > 0$ that

$$(1-\varepsilon)(a_i/b_i)^{1/\alpha} F^{\leftarrow}(1-p) \le F_i^{\leftarrow}(1-p) \le (1+\varepsilon)(a_i/b_i)^{1/\alpha} F^{\leftarrow}(1-p).$$

From this two-sided inequality and identity (7.3), it follows that, for all small $p > 0$,

$$P\left(\frac{\mathbf{X}}{F^{\leftarrow}(1-p)} \in (1+\varepsilon)A_l\right) \le P(L(\mathbf{p}) > l) \le P\left(\frac{\mathbf{X}}{F^{\leftarrow}(1-p)} \in (1-\varepsilon)A_l\right).$$

Here we tacitly applied a property of the set B_l that, for $\mathbf{x}_1 \le \mathbf{x}_2$, if $\mathbf{x}_1 \in B_l$ then $\mathbf{x}_2 \in B_l$. Also by this property, it is easy to see that $t(\partial B_l) \cap (\partial B_l) = \varnothing$ for every $t > 1$. Hence, $\nu(\partial B_l) = 0$ by Lemma 7.1. Similarly, ν assigns no mass to the boundaries of the sets $(1 \pm \varepsilon)A_l$ either. Thus, it follows from relation (3.2) that

$$(1+\varepsilon)^{-\alpha}\nu(A_l) \lesssim \frac{P(L(\mathbf{p}) > l)}{p} \lesssim (1-\varepsilon)^{-\alpha}\nu(A_l).$$

By the arbitrariness of ε, this concludes the proof of Theorem 4.1. ∎

References

[1] Altman, E. I.; Brady, B.; Resti, A.; Sironi, A. The link between default and recovery rates: theory, empirical evidence, and implications. Journal of Business 78 (2005), 2203–2228.

[2] Andersson, F.; Mayock, T. Loss severities on residential real estate debt during the Great Recession. Journal of Banking & Finance 46 (2014), 266–284.

[3] Bernardi, E.; Falangi, F.; Romagnoli, S. A hierarchical copula-based worldwide valuation of sovereign risk. Insurance: Mathematics and Economics 61 (2015), 155–169.

[4] Bingham, N. H.; Goldie, C. M.; Teugels, J. L. Regular Variation. Cambridge University Press, Cambridge, 1987.

[5] Böcker, K.; Klüppelberg, C. Multivariate models for operational risk. Quantitative Finance 10 (2010), 855–869.

[6] Calabrese, R.; Zenga, M. Bank loan recovery rates: measuring and nonparametric density estimation. Journal of Banking & Finance 34 (2010), 903–911.

[7] Chava, S.; Stefanescu, C.; Turnbull, S. Modeling the loss distribution. Management Science 57 (2011), 1267–1287.

[8] de Haan, L.; Resnick, S. I. On the observation closest to the origin. Stochastic Processes and Their Applications 11 (1981), no.3, 301–308.

[9] Denuit, M.; Kiriliouk, A; Segers, J. Max-factor individual risk models with application to credit portfolios. Insurance: Mathematics and Economics 62 (2015), 162–172.

[10] Donnelly, C.; Embrechts, P. The devil is in the tails: actuarial mathematics and the subprime mortgage crisis. ASTIN Bulletin 40 (2010), 1–33.

[11] Duffie, D.; Eckner, A.; Horel, G.; Saita, L. Frailty correlated default. The Journal of Finance 64 (2009), 2089–2123.

[12] Einmahl, J. H. J.; Segers, J. Maximum empirical likelihood estimation of the spectral measure of an extreme-value distribution. The Annals of Statistics 37 (2009), 2953–2989.

[13] Embrechts, P.; Klüppelberg, C.; Mikosch, T. Modelling Extremal Events for Insurance and Finance. Springer-Verlag, Berlin, 1997.

[14] Frey, R.; McNeil, A. J. VaR and expected shortfall in portfolios of dependent credit risks: conceptual and practical insights. Journal of Banking & Finance 26 (2002), 1317–1344.

[15] Frey, R.; McNeil, A. J. Dependent defaults in models of portfolio credit risk. Journal of Risk 6 (2003), 59–92.

[16] Frye, J.; Jacobs Jr, M. Credit loss and systematic loss given default. The Journal of Credit Risk 8 (2012), 109–140.

[17] Gabaix, X. Power laws in economics and finance. Annual Review of Economics 1 (2009), 255–293.

[18] Giesecke, K. Correlated default with incomplete information. Journal of Banking & Finance 28 (2004a), 1521–1545.

[19] Giesecke, K. Credit risk modeling and valuation: an introduction. Credit Risk: Models and Management (vol. 2, edited by D. Shimko), Risk Books, London, 2004b.

[20] Glasserman, P.; Li, J. Importance sampling for portfolio credit risk. Management science 51 (2005), 1643–1656.

[21] Gordy, M. B. A comparative anatomy of credit risk models. Journal of Banking & Finance 24 (2000), 119–149.

[22] Hao, X.; Li, X. Pricing credit default swaps with a random recovery rate by a double inverse Fourier transform. Insurance: Mathematics and Economics 65 (2015), 103–110.

[23] Kiriliouk, A.; Segers, J.; Warchol, M. Nonparametric estimation of extremal dependence. Extreme Value Modeling and Risk Analysis: Methods and Applications (edited by D. K. Dey and J. Yan), Chapman & Hall/CRC, 2015.

[24] Mainik, G.; Rüschendorf, L. On optimal portfolio diversification with respect to extreme risks. Finance and Stochastics 14 (2010), 593–623.

[25] McNeil, A. J.; Frey, R.; Embrechts, P. Quantitative Risk Management: Concepts, Techniques, and Tools. Princeton University Press, Princeton, NJ, 2015.

[26] Merton, R. C. On the pricing of corporate debt: the risk structure of interest rates. Journal of Finance 29 (1974), 449–470.

[27] Moody's. US Municipal Bond Defaults and Recoveries, 1970–2014. Technical report, 2015.

[28] Packham, N.; Kalkbrener, M.; Overbeck, L. Asymptotic behaviour of multivariate default probabilities and default correlations under stress. Journal of Applied Probability 53 (2016), 71–81.

[29] Park, Y. W.; Bang, D. W. Loss given default of residential mortgages in a low LTV regime: role of foreclosure auction process and housing market cycles. Journal of Banking & Finance 39 (2014), 192–210.

[30] Pesola, J. Joint effect of financial fragility and macroeconomic shocks on bank loan losses: evidence from Europe. Journal of Banking & Finance 35 (2011), 3134–3144.

[31] Qi, M.; Yang, X. Loss given default of high loan-to-value residential mortgages. Journal of Banking & Finance 33 (2009), 788–799.

[32] Renault, O.; Scaillet, O. On the way to recovery: a nonparametric bias free estimation of recovery rate densities. Journal of Banking & Finance 28 (2004), 2915–2931.

[33] Resnick, S. I. Extreme Values, Regular Variation, and Point Processes. Springer-Verlag, New York, 1987.

[34] Resnick, S. I. Heavy-tail Phenomena. Probabilistic and Statistical Modeling. Springer, New York, 2007.

[35] Rüschendorf, L. Mathematical Risk Analysis. Dependence, Risk Bounds, Optimal Allocations and Portfolios. Springer, Heidelberg, 2013.

[36] Schmidt, R. Credit risk modelling and estimation via elliptical copulae. Credit Risk – Measurement, Evaluation and Management (edited by G. Bohl, G. Nakhaeizadeh, S. T. Rachev, T. Ridder and K. H. Vollmer). Physica-Verlag, Heidelberg, 2003.

[37] Schuermann, T. What do we know about loss given default? In Credit Risk: Models and Management, 2nd Edition (edited by D. Shimko), Risk Books, London, 2004.

[38] Scott, A.; Metzler, A. A general importance sampling algorithm for estimating portfolio loss probabilities in linear factor models. Insurance: Mathematics and Economics 64 (2015), 279–293.

[39] Stărică, C. Multivariate extremes for models with constant conditional correlations. Journal of Empirical Finance 6 (1999), no. 5, 515–553.

[40] Tang, Q.; Yuan, Z. Asymptotic analysis of the loss given default in the presence of multivariate regular variation. North American Actuarial Journal 17 (2013), 253–271.

[41] Vandendorpe, A.; Ho, N.; Vanduffel, S.; Van Dooren, P. On the parameterization of the CreditRisk+ model for estimating credit portfolio risk. Insurance: Mathematics and Economics 42 (2008), 736–745.

[42] Wang, R.; Peng, L.; Yang, J. CreditRisk+ model with dependent risk factors. North American Actuarial Journal 19 (2015), no. 1, 24–40.

[43] Wei, L.; Yuan, Z. The loss given default of a low-default portfolio with weak contagion. Insurance: Mathematics and Economics 66 (2016), 113–123.

[44] Yan, J. Enjoy the joy of copulas: with a package copula. Journal of Statistical Software 21 (2007), 1–21.
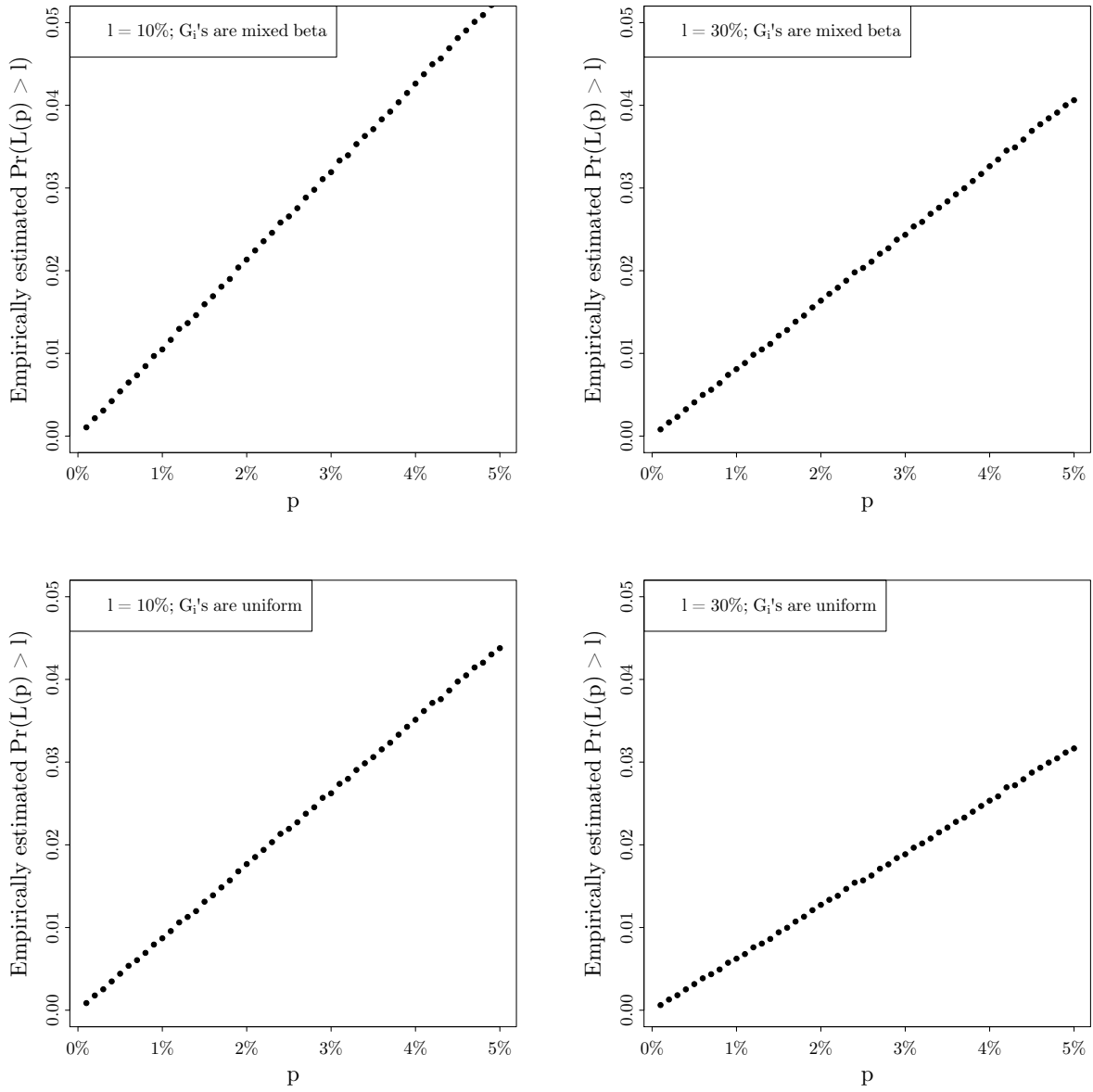
Figure 4.1. The simulated probability $P\left(L(p) > l\right)$ for small values of p, $l = 10\%$ or 30%, and $G_i = G^I$ or G^{II}, $i = 1, \ldots, d$.

	Gaussian copula	t copula	
	off-diagonal elements of correlation matrix (standard deviation)	off-diagonal elements of correlation matrix (standard deviation)	degree of freedom (standard deviation)
$r = 2$	0.70 (4.1×10^{-4})	0.71 (4.6×10^{-4})	8.3 (0.03)
$r = 5$	0.95 (7.6×10^{-5})	0.95 (9.6×10^{-5})	4.3 (0.01)

**Table 5.1.** Estimation results for the Gaussian copula and t copula.
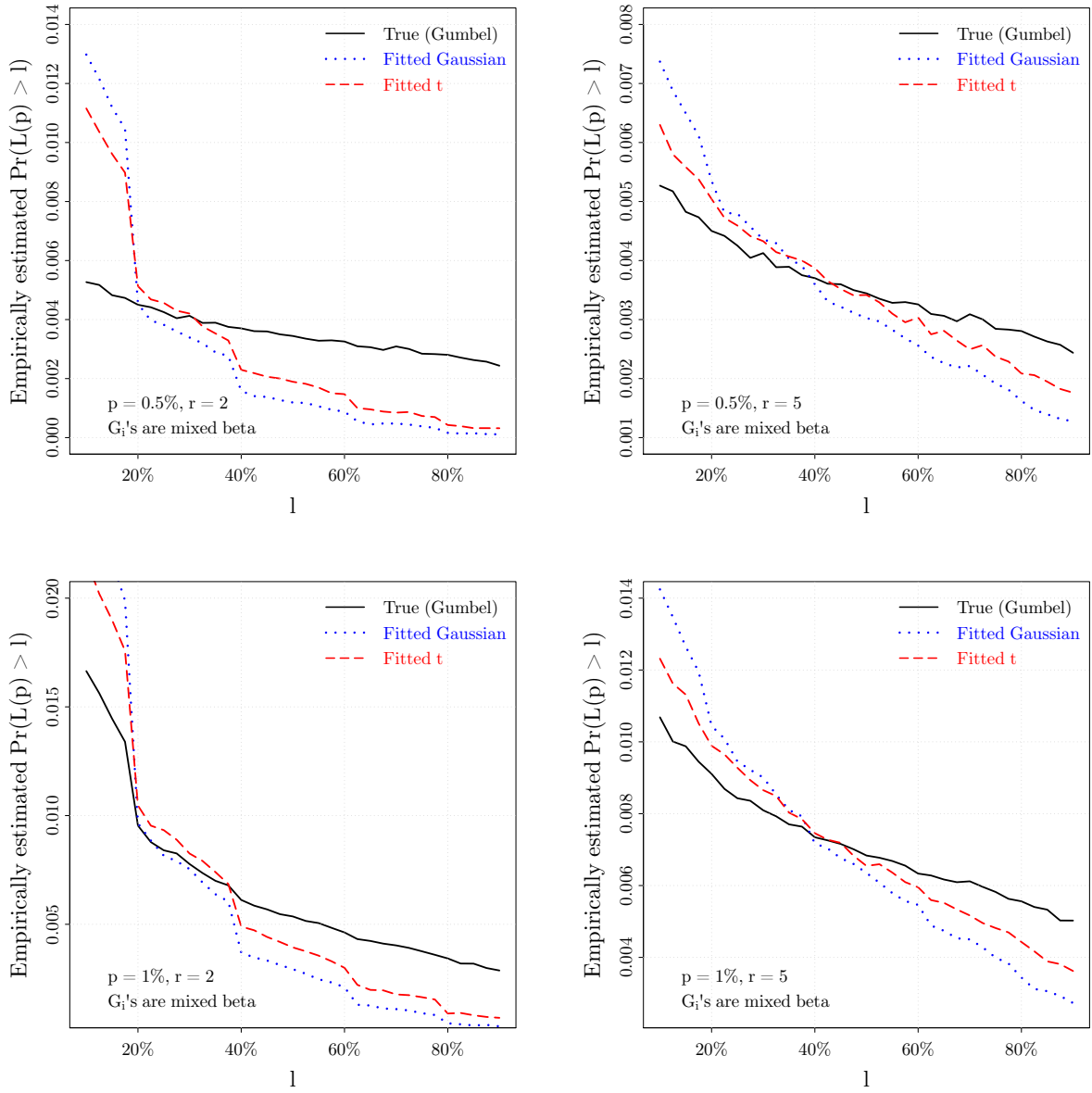
Figure 5.1. The empirically estimated survival function of $L(p)$ based on the true copula (Gumbel), the fitted Gaussian copula, and the fitted t copula, for $p = 0.5\%$ or 1%, $r = 2$ or 5, and $G_i = G^I$, $i = 1, \ldots, d$.
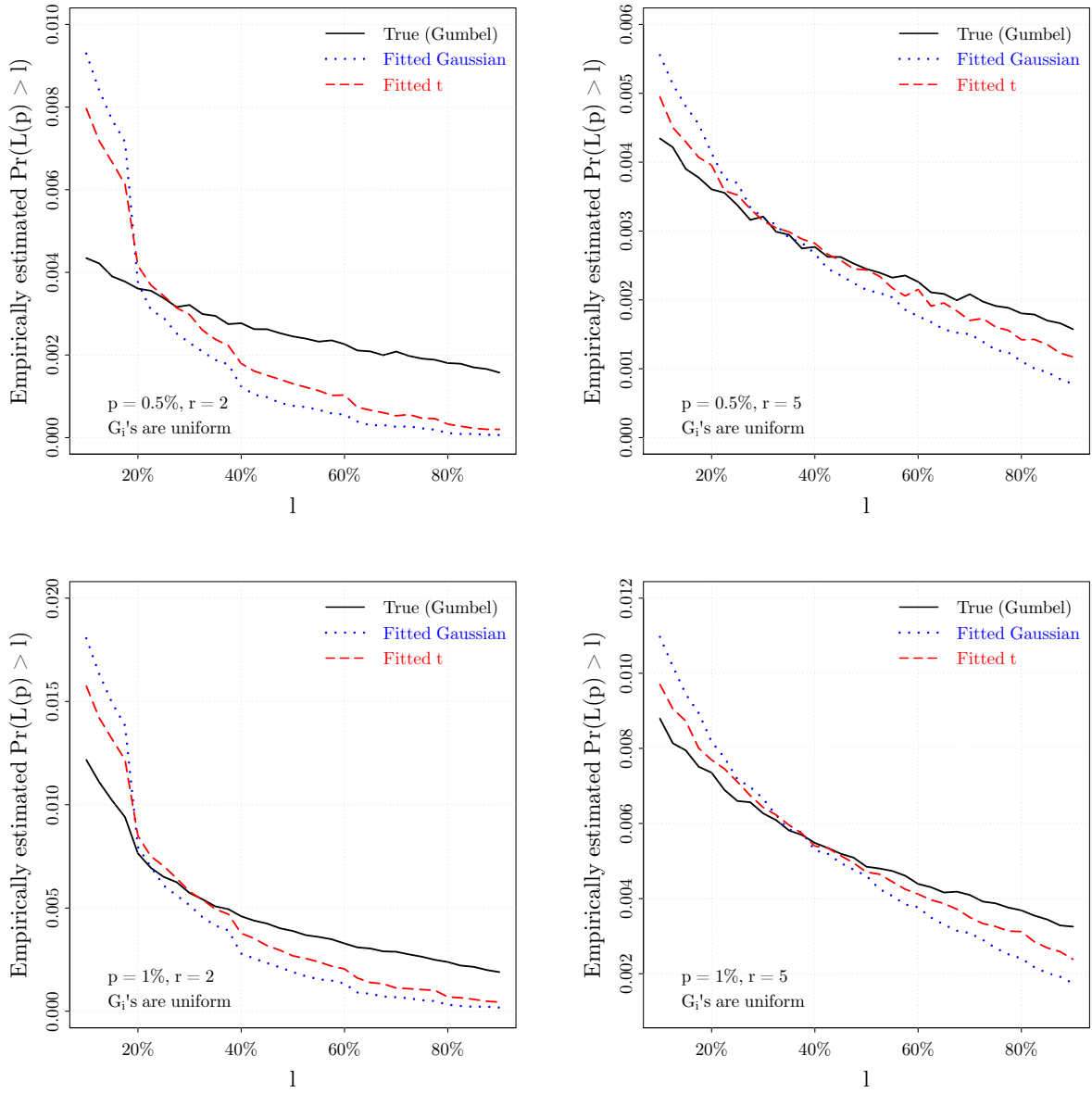
**Figure 5.2.** The empirically estimated survival function of $L(p)$ based on the true copula (Gumbel), the fitted Gaussian copula, and the fitted t copula, for $p = 0.5\%$ or 1%, $r = 2$ or 5, and $G_i = G^{II}$, $i = 1, \ldots, d$.
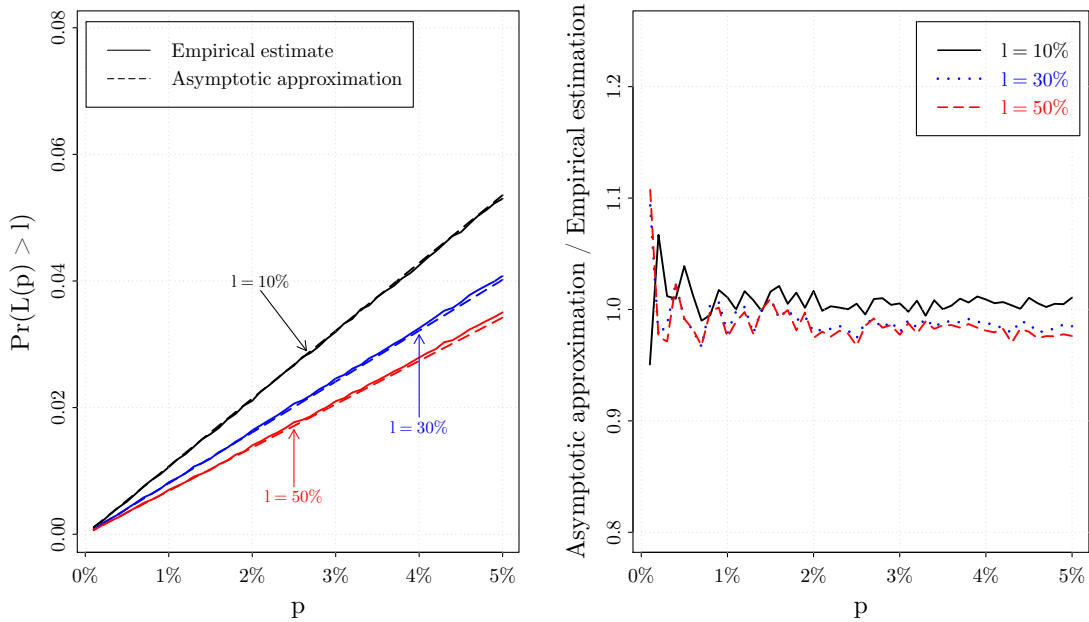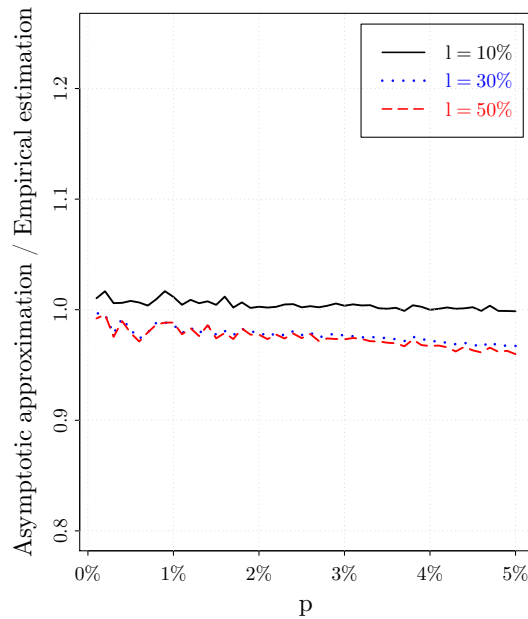
Figure 5.3. Comparison between the asymptotic approximations of $P(L(p) > l)$ and its empirical estimates (obtained based on a sample size of 10^6), for $l = 10\%$, 30%, or 50%, and $G_i = G^I$, $i = 1, \ldots, d$.
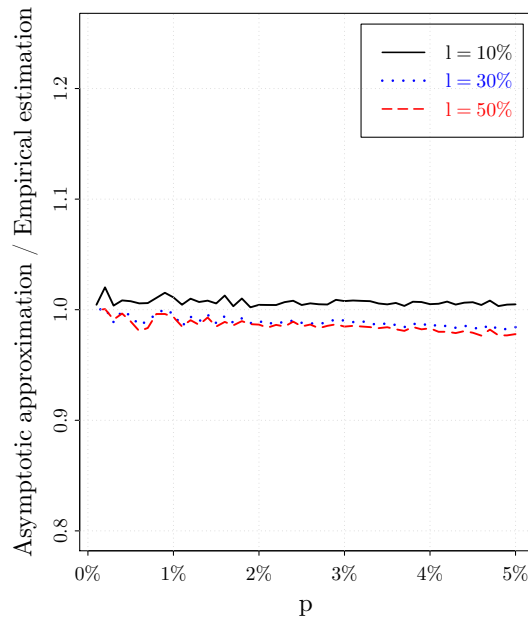

Figure 5.4. Ratio of the asymptotic approximations of $P(L(p) > l)$ to its empirical estimates (obtained based on a sample size of 10^7) for $l = 10\%$, 30%, or 50%, and $G_i = G^I$, $i = 1, \ldots, d$.
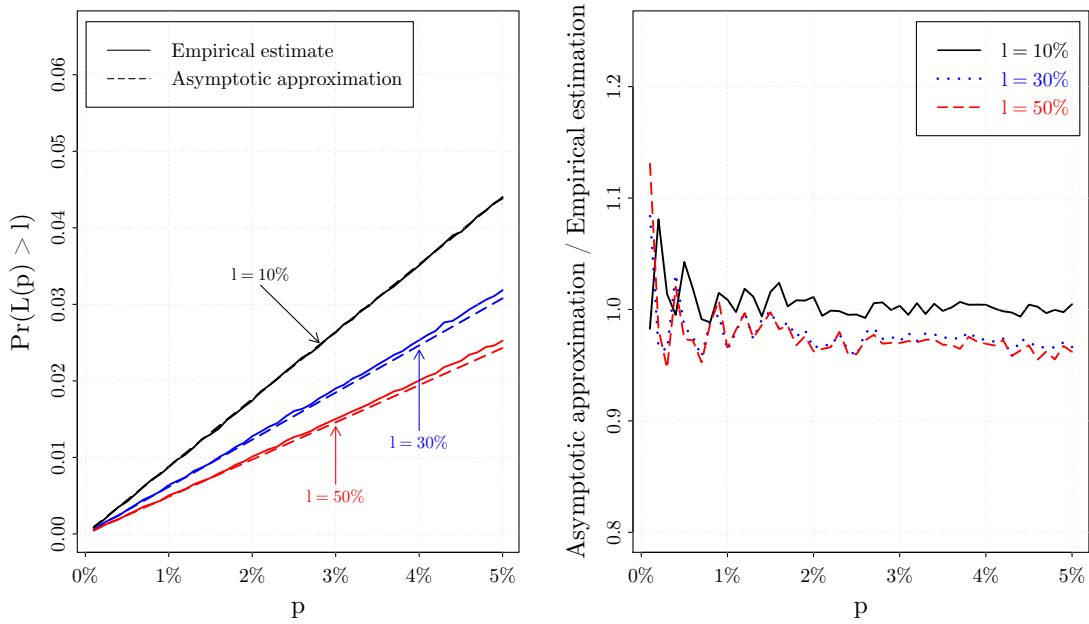
Figure 5.5. Comparison between the asymptotic approximations of $P(L(p) > l)$ and its empirical estimates (obtained based on a sample size of 10^6), for $l = 10\%$, 30%, or 50%, and $G_i = G^{II}$, $i = 1, \ldots, d$.

Figure 5.6. Ratio of the asymptotic approximations of $P(L(p) > l)$ to its empirical estimates (obtained based on a sample size of 10^7) for $l = 10\%$, 30%, or 50%, and $G_i = G^{II}$, $i = 1, \ldots, d$.
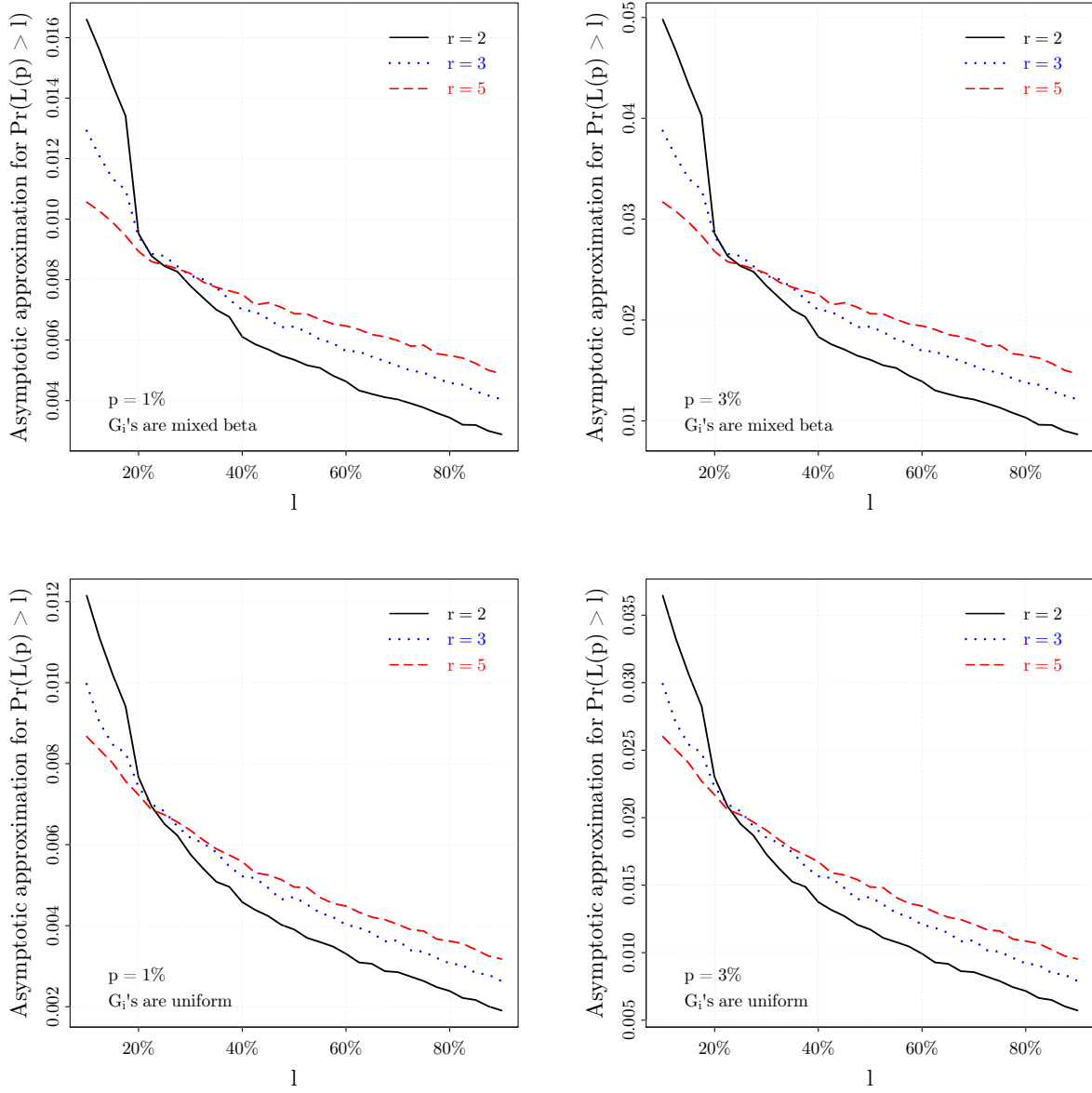
Figure 5.7. Approximation of $P(L(p) > l)$ based on relation (4.6), for $p = 1\%$ or 3%, $r = 2$, 3, or 5, and $G_i = G^I$ or G^{II}, $i = 1, \ldots, d$.