Characterizations of optimal reinsurance treaties: A cost-benefit approach

Ka Chun Cheung[*] and Ambrose Lo[*,†]

[*]*Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong*
[†]*Department of Statistics and Actuarial Science, The University of Iowa, 241 Schaeffer Hall, Iowa City, IA 52242, USA*

Abstract

This article investigates optimal reinsurance treaties minimizing an insurer's risk-adjusted liability, which encompasses a risk margin quantified by distortion risk measures. Via the introduction of a transparent cost-benefit argument, we extend the results in Cui et al. [Cui, W., Yang, J., Wu, L., 2013. Optimal reinsurance minimizing the distortion risk measure under general reinsurance premium principles. Insurance: Math. Econ. 53, 74-85] and provide full characterizations on the set of optimal reinsurance treaties within the class of non-decreasing, 1-Lipschitz functions. Unlike conventional studies, our results address the issue of (non-)uniqueness of optimal solutions and indicate that ceded loss functions beyond the traditional insurance layers can be optimal in some cases. The usefulness of our novel cost-benefit approach is further demonstrated by readily solving the dual problem of minimizing the reinsurance premium while maintaining the risk-adjusted liability below a fixed tolerance level.

Keywords: Risk-adjusted liability; Risk margin; Premium constraint; VaR; TVaR; Wang's premium; Actuarial pricing principle; 1-Lipschitz

*Corresponding author. Email: ambrose-lo@uiowa.edu; Tel.: +319 335 1915.

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1 Introduction

The design of optimal reinsurance treaties is a classical and delicate problem that has aroused sustained interest in academia as well as in industry. Practically, reinsurance is a popular risk management strategy on which insurers can capitalize to optimize their risk exposure in accordance with their risk appetite. Technically, optimal reinsurance problems constitute a class of infinite-dimensional (constrained) optimization problems whose solution involves the search for an optimal function, in lieu of an optimal parameter value, and requires unconventional and ingenious techniques. The study of optimal reinsurance is therefore a research area that is of theoretical and practical importance.

Mathematically, the aggregate ground-up loss faced by an insurer in a certain reference period is modelled by a non-negative integrable random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Under a reinsurance arrangement, the reinsurer pays $f(x)$ to the insurer, when x is the realization of X, with the insurer retaining a loss of $x - f(x)$. The function $f : [0, \operatorname{ess\,sup}(X)) \to \mathbb{R}^+$, where $\operatorname{ess\,sup}(X)$ is the essential supremum[i] of X, is known as the *ceded loss function*. In return, the insurer is charged the reinsurance premium $\mu(f(X))$ payable to the reinsurer. Therefore, the total retained loss of the insurer corresponding to a ceded loss function f is

$$T_f(X) := X - f(X) + \mu(f(X)). \tag{1.1}$$

Conceivably, ceding more loss to the reinsurer results in the insurer bearing a smaller retained loss, but a higher reinsurance premium. The problem of optimal reinsurance then lies in the design of the ceded loss function f within a class of feasible functions to achieve an optimal level of balance.

A complete description of any optimal reinsurance model requires the specification of four components: (1) The optimization criterion; (2) The class of feasible ceded loss functions; (3) The reinsurance premium principle; (4) Constraints, if any, on the reinsurance premium. Different choices may lead to completely different solutions. In the pioneering work of Borch (1960), the minimization of the variance of the total retained loss was considered, and the optimality of stop-loss reinsurance in the class of ceded loss functions having the same expected reinsurance premium was established. The seminal paper by Arrow (1963), also under the expectation premium principle, demonstrated that stop-loss reinsurance maximized the expected utility of an insurer's terminal wealth when the insurer had a concave utility function. More recently, Young (1999) and Kaluszka and Okolewski (2008) generalized Arrow's result to the cases of Wang's premium principle and maximum possible claims principle respectively.

Over the last decade, there was a proliferation of research work centering on risk-measure-

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[i]We restrict the domain of f to $[0, \operatorname{ess\,sup}(X))$ rather than $\mathbb{R}^+$ because ceding losses beyond the maximum size of loss makes no practical sense and hinders the discussion of unique optimal solutions in this paper.

based reinsurance models beyond the classical criteria of variance and expected utility, driven in part by the axiomatic approach to risk measures formally laid in Artzner et al. (1999) and by the prevalence of risk measures used in insurance companies and banks for risk assessment. Studies in this direction predominantly revolved around Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR) as prominent choices of risk measures. Using these two risk measures, Cai and Tan (2007) derived the optimal retention of a stop-loss reinsurance analytically under the expectation premium principle. By minimizing the VaR and TVaR of the total retained loss, Cai et al. (2008) deduced the optimal ceded loss functions being insurance layers within the class of non-decreasing convex functions via some sophisticated approximation and convergence arguments. Since then, VaR-based and TVaR-based reinsurance models continued to receive substantial attention and were extended in different directions (see Bernard and Tian (2009), Cheung (2010), Chi (2012), Lu et al. (2013) and Cai and Weng (2014) among others).

In spite of the vast amount of work in the optimal reinsurance literature, the majority of reinsurance models studied could be justifiably critiqued in the following three substantive aspects:

1. The optimal reinsurance problems were studied separately under specific risk measures and reinsurance premium principles. The analysis, which heavily relies on the risk measure that is chosen as the measurement of risk as well as the premium principle that is adopted to price the reinsurance contract, may not carry over to other reinsurance models. Such a disparate treatment also conceals the general mathematical structure of and inter-relationships between different classes of reinsurance models. For these reasons, a unified analysis of optimal reinsurance problems covering a wide class of risk measures and reinsurance premium principles is warranted.

2. Most reinsurance models imposed no constraint on the amount of reinsurance premium. In reality, profitability concerns may dictate that only a limited budget can be allocated to reinsurance activities. This realistic constraint effectively undermines the extent to which an insurer can engage in reinsurance and must be taken into account when devising the optimal reinsurance strategy.

3. Last but not least, the issue of uniqueness of optimal solutions remains unanswered in virtually all reinsurance models. Many papers suggest reinsurance layers as the optimal ceded loss function. However, are reinsurance layers the only possible design to achieve optimality? For mathematical and practical reasons, it is highly desirable to identify all optimal reinsurance strategies in a given reinsurance model.

The first and second shortcomings were remedied in a recent paper by Cui et al. (2013), where the total retained risk of an insurer was measured by distortion risk measures introduced by Yaari (1987) and Wang (1996), a general premium principle was in force, and a reinsurance premium constraint was also considered. Distortion risk measures not only

include VaR, TVaR and other well-known risk measures as special cases, but also have far-reaching applications to decision making under uncertainty. In particular, the perception of risk they reflect is more consistent with human behavior than what the traditional expected utility paradigm provides (see Chapter 4 of Quiggin (1993) for some anomalies associated with expected utility theory). The premium principle considered in Cui et al. (2013) was also general enough to incorporate a wide class of premium principles, including the expectation premium principle. By virtue of some complicated partition arguments, Cui et al. (2013) developed the optimality of reinsurance layers in the class of non-decreasing, 1-Lipschitz ceded loss functions. Based on the techniques in Cui et al. (2013), Zheng and Cui (2014) and Zheng et al. (2015) tackled the same optimization problem within the class of non-decreasing and left-continuous retained loss functions, but with the specification of the expectation premium principle. Because of their high generality, the results in Cui et al. (2013) arguably represent groundbreaking contributions to the literature of optimal reinsurance and merit further studies.

In the same spirit as Cui et al. (2013), this paper revisits one-period distortion-risk-measure-based optimal reinsurance models with reinsurance premium constraints and with ceded loss functions residing in the class of non-decreasing, 1-Lipschitz functions. This class of reinsurance treaties not only is more general than that studied in Cai et al. (2008), Cheung (2010) and Lu et al. (2013), but also effectively eliminates the issue of moral hazard. Therefore, it appears to be the most natural feasible set in theory and in practice. Instead of minimizing the distortion risk measure of the insurer's total retained risk, we suggest examining the minimization of his/her *risk-adjusted liability* (also known as market consistent value of liability; see The Chief Risk Officer Forum (2006)), which is determined as the sum of the actuarial reserve on the insurer's total retained risk and a risk margin quantified by distortion risk measures. In addition to encompassing the models studied in Cui et al. (2013), the reinsurance model in this paper also enjoys a natural economic interpretation. We refer to the comprehensive introduction in Tsanakas and Christofides (2006) for the economic versatility of distortion risk measures and to Dowd and Blake (2006) for specific applications of distortion risk measures to insurance risk problems. By rewriting the objective function and constraint function in some equivalent integral forms and introducing an intuitive benefit-to-cost ratio, we dispense with the complicated partition arguments in Cui et al. (2013) and, more importantly, provide complete solutions to the distortion-risk-measure-based optimal reinsurance problem. It is shown that the derivations of optimal solutions amount to comparing the benefit and cost of reinsurance. In particular, characterizations of the optimal ceded loss functions are provided and all of the three above-mentioned shortcomings of classical optimal reinsurance models are addressed in full.

Given the extensive literature on optimal reinsurance, it is instructive to pinpoint the following highlights of the current article:

1. In many previous contributions such as Cai and Tan (2007), Cai et al. (2008), Bernard

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and Tian (2009), Cai et al. (2013) and Lu et al. (2013), it was routinely assumed that the ground-up loss X was a continuous random variable with a strictly increasing cumulative distribution function on the positive real line and a possible jump at 0. Such a distributional assumption, often invoked to simplify the presentation of optimal solutions, eliminates a variety of loss random variables. In contrast, we assume in this paper that X is a general random variable, not necessarily continuous or discrete. Our results, which can be displayed in concise forms, hold in much greater generality.

2. As opposed to Cui et al. (2013), in which the distortion function and premium function are assumed to be left-continuous for technical reasons, in this paper we overcome the technical challenges therein and do not require any continuity assumption on these functions.

3. In the literature, the standard way to tackle optimal reinsurance problems, as adopted, for instance, by Cai et al. (2008), Cheung (2010) and Lu et al. (2013), is to reduce the infinite-dimensional optimization problem to a finite-dimensional one, which can then be solved by calculus methods. We pursue a radically different approach in this paper, where we prefer to work in an infinite-dimensional setting. It is in this high level of generality that the mathematical structure of the reinsurance model, particularly the benefits and costs associated with reinsurance, is made distinctively clear. Furthermore, it will be shown in this paper (Theorems 3.1 and 3.3) that the set of optimal solutions can be infinite-dimensional. Reducing the feasible set of ceded loss functions to a finite-dimensional set will inevitably result in a loss of optimal solutions.

4. The cost-benefit approach illustrated in this paper has the following merits:

   - By sidestepping the technicalities in Cai et al. (2008), Cui et al. (2013) and Zheng and Cui (2014), our approach provides a tremendously intuitive and transparent method to deduce the optimal reinsurance strategies.

   - It is flexible in the sense of being applicable to both the constrained and unconstrained optimal reinsurance problems.

   - It is founded upon the cost-benefit considerations that form the very essence of optimal reinsurance problems. In this connection, it proves to be a natural method of solution which aligns considerably better with the character of optimal reinsurance problems than other solution schemes in the literature.

   - It allows us to provide full characterizations of the set of optimal solutions, which other solution methods may not offer.

The rest of the paper is organized as follows. Section 2 recalls fundamental notions such as distortion risk measures and generalized inverse functions that are essential to the later development of this paper, and lays the mathematical formulation of the distortion-risk-measure-based optimal reinsurance model. Economic interpretations of various quantities

in the model are also given. In Section 3, we solve the optimal reinsurance problems with and without a reinsurance premium constraint. The cost-benefit perspective that underlies the derivations of the optimal solutions is elucidated. Concrete examples involving well-known risk measures are studied to illustrate the construction of optimal reinsurance strategies by graphical inspection. Section 4 demonstrates an application of the cost-benefit technique by solving a dual optimal reinsurance problem. Finally, Section 5 concludes the paper.

2   Preliminaries

For any non-decreasing right-continuous function $\xi\colon \mathbb{R} \to \mathbb{R}$, we define its generalized left-continuous inverse function and generalized right-continuous inverse function respectively by

$$\xi^{-1}(p) := \inf\{x \in \mathbb{R} \mid \xi(x) \geq p\}$$

and

$$\xi^{-1+}(p) := \inf\{x \in \mathbb{R} \mid \xi(x) > p\},$$

with the convention $\inf \emptyset = \infty$. It is known that the following equivalence holds:

$$\xi^{-1}(p) \leq x \quad \Leftrightarrow \quad p \leq \xi(x). \tag{2.1}$$

If F_X is the distribution function of a random variable X, and f is a non-decreasing and continuous function, then

$$F_{f(X)}^{-1}(p) = f\left(F_X^{-1}(p)\right) \quad \text{and} \quad F_{f(X)}^{-1+}(p) = f\left(F_X^{-1+}(p)\right), \quad 0 < p < 1. \tag{2.2}$$

For more information about generalized inverse functions, see Section 3 of Dhaene et al. (2002). In the sequel, we denote the indicator function of a given set A by 1_A, i.e. $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$, and the identity function by Id, i.e. $\mathrm{Id}(x) = x$.

2.1   Mathematical formulation of distortion-risk-measure-based optimal reinsurance models

Following Chi (2012) and Cai and Weng (2014), in this paper we study the minimization of an insurer's risk-adjusted liability, which is comprised of: (1) The actuarial reserve on the insurer's total retained risk, namely, $\mathbb{E}[T_f(X)]$, where $T_f(X)$ is given in Equation (1.1) and f is a ceded loss function; (2) A risk margin quantified by means of distortion risk measures, which in turn are defined in terms of distortion functions. By definition, a *distortion function* $g\colon [0,1] \to [0,1]$ is a non-decreasing function such that $g(0) = 0$ and $g(1) = 1$. We do not assume a priori that g is convex, concave or differentiable (see

Example 3.5 for a distortion function that is neither convex nor concave). Corresponding to a distortion function g, the *distortion risk measure* of a random variable X is defined as

$$\rho_g(X) := \int_0^\infty g\left(S_X(t)\right) \mathrm{d}t - \int_{-\infty}^0 \left[1 - g\left(S_X(t)\right)\right] \mathrm{d}t, \tag{2.3}$$

where $S_X(t) := \mathbb{P}(X > t)$ is the survival function of X, and it is assumed that at least one of the two integrals in Equation (2.3) is finite. If X is non-negative, then the second integral vanishes, leaving

$$\rho_g(X) = \int_0^\infty g\left(S_X(t)\right) \mathrm{d}t.$$

By Fubini's theorem, Equation (2.3) can be rewritten as

$$\rho_g(X) = \int_0^1 F_X^{-1}(1-p) \, \mathrm{d}g(p) \tag{2.4}$$

if g is left-continuous, and as

$$\rho_g(X) = \int_0^1 F_X^{-1+}(1-p) \, \mathrm{d}g(p) \tag{2.5}$$

if g is right-continuous (see Theorems 4 and 6 of Dhaene et al. (2012)), with both integrals understood in a Lebesgue-Stieltjes sense. The risk margin is prescribed mathematically by a scalar multiple of the distortion risk measure of the discrepancy between the total retained risk and its actuarial reserve:

$$\text{Risk margin} = \delta\rho_g\left(T_f(X) - \mathbb{E}[T_f(X)]\right),$$

where δ is a positive constant known as the *cost-of-capital rate*. The insurer's risk-adjusted liability, being the minimization objective of this paper, is then given by

$$L_f(X) := \mathbb{E}[T_f(X)] + \delta\rho_g\left(T_f(X) - \mathbb{E}[T_f(X)]\right).$$

This form of $L_f(X)$ exhibits the risk margin as a buffer added to the actuarial reserve, being the best estimate of the total retained loss, to cover adverse deviation of risk from its expected value. The risk margin is also indispensable, taking into consideration the fact that insurance liabilities are not actively traded in deep and liquid markets, and therefore non-hedgeable. Allowance for prudence can be flexibly heightened by raising the cost-of-capital rate δ or varying the distortion function g[ii]. For more information about risk-adjusted liability, we refer to The Chief Risk Officer Forum (2006) and Risk Margin Working Group (2009).

Another perspective on $L_f(X)$ is furnished by the translation invariance of distortion risk measures, i.e. $\rho_g(X + c) = \rho_g(X) + c$ for any real constant c (see Section 5 of Dhaene et al. (2006)). This leads to

$$L_f(X) = (1 - \delta)\mathbb{E}[T_f(X)] + \delta\rho_g\left(T_f(X)\right), \tag{2.6}$$

[ii]If $g(x) \geq x$ for all $x \in [0, 1]$, then the risk margin must be non-negative.

which can be viewed conveniently as a weighted average of the actuarial reserve on and distortion risk measure of the total retained risk. Moreover, when $\delta = 1$, $L_f(X)$ reduces to the objective function studied in Cui et al. (2013), Zheng and Cui (2014) and Zheng et al. (2015).

To reduce its risk-adjusted liability, the insurer can purchase reinsurance treaties characterized by ceded loss functions belonging to the set $\mathcal{F}$ of non-decreasing and 1-Lipschitz functions, i.e.

$$\mathcal{F} = \left\{ f\colon [0, \operatorname{ess\,sup}(X)) \to [0, \operatorname{ess\,sup}(X)) \left| \begin{array}{l} 0 \le f(x) \le x \text{ for all } x \ge 0, \\ 0 \le f(x_1) - f(x_2) \le x_1 - x_2 \text{ if } 0 \le x_2 \le x_1 \end{array} \right. \right\}.$$

With all ceded loss functions restricted to the set $\mathcal{F}$, both the insurer and reinsurer will incur higher payments for heavier ground-up losses, thereby having no incentive to over-report claims. A technical by-product is that every $f \in \mathcal{F}$ is absolutely continuous with $f(0) = 0$.

Premium-wise, we assume that the reinsurance premium is calibrated, for a given ceded loss function $f \in \mathcal{F}$, by

$$\mu_r\left(f(X)\right) := \int_0^\infty r\left(S_{f(X)}(t)\right) \mathrm{d}t, \tag{2.7}$$

where $r\colon [0, 1] \to \mathbb{R}^+$ is a given non-decreasing function with $r(0) = 0$. To avoid trivialities, we assume that r is not zero almost everywhere. It is important to note that the reinsurance premium defined in Equation (2.7) not only possesses a concrete representation, but also can be easily configured to satisfy a number of desirable properties by imposing appropriate conditions on the function r, as noted in Proposition 2.3 of Cui et al. (2013). In addition, with different specifications of r, various premium principles, including the well-known expectation premium principle, Wang's premium principle, generalized percentile premium and the maximum possible claims principle, can be retrieved from (see Cui et al. (2013) and Remark 2.2).

With every ingredient of the optimal reinsurance model at hand, we are now in a position to formally state the optimization problems to be studied in this paper:

1. *Free-premium problem:*

$$\inf_{f \in \mathcal{F}} L_f(X). \tag{2.8}$$

2. *Budget-constrained problem:*

$$\begin{cases} \inf_{f \in \mathcal{F}} & L_f(X) \\ \text{s.t.} & \mu_r\left(f(X)\right) \le \pi, \end{cases} \tag{2.9}$$

where π is an exogenously given strictly positive quantity which can be interpreted as the budget allocated to reinsurance.

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Note that Problems (2.8) and (2.9) are always well-posed with non-empty feasible sets because the ceded loss function $f \equiv 0$ is admissible. Throughout this paper, we assume that all integrals underlying Problems (2.8) and (2.9) exist and are finite.

2.2 First-step analysis

To solve Problems (2.8) and (2.9), we first recast their objective and constraint functions into some equivalent integral forms in the following lemma, which extends Proposition 2.2 of Cui et al. (2013) and will be crucial to further analysis. The proof, which is deferred to the Appendix, is long and technical because of the general discontinuity of the functions g and r, as opposed to Cui et al. (2013). For notational convenience, we write $\int_0^\infty$ for $\int_0^{\mathrm{ess\,sup}(X)}$, keeping in mind that all ceded loss functions in this paper are defined on $[0, \mathrm{ess\,sup}(X))$.

Lemma 2.1. *(Integral representations of objective and constraint functions) For any $f \in \mathcal{F}$, we have*

$$\mathbb{E}[f(X)] = \int_0^\infty S_X(t)\,\mathrm{d}f(t). \tag{2.10}$$

Moreover, the objective function of Problems (2.8) and (2.9) and the reinsurance premium can be expressed respectively as

$$L_f(X) = \delta\rho_g(X) + (1-\delta)\mathbb{E}[X] + \int_0^\infty G_X(t)\,\mathrm{d}f(t), \tag{2.11}$$

$$= \delta\rho_g(X) + (1-\delta)\mathbb{E}[X] + \int_0^\infty G_X(t)f'(t)\,\mathrm{d}t, \tag{2.12}$$

where

$$G_X(t) = r\left(S_X(t)\right) - \left[\delta g\left(S_X(t)\right) + (1-\delta)S_X(t)\right], \tag{2.13}$$

and

$$\mu_r\left(f(X)\right) = \int_0^\infty r\left(S_X(t)\right)\mathrm{d}f(t) = \int_0^\infty r\left(S_X(t)\right)f'(t)\,\mathrm{d}t. \tag{2.14}$$

Remark 2.2. The integral representations in Equations (2.10) and (2.14) can be used to show that the maximum possible claims principle (see Kaluszka and Okolewski (2008)) can be captured by $\mu_r\left(f(X)\right)$ for an appropriate function r. Specifically, with $r(x) = (1-\beta)x + \beta 1_{\{x>0\}}$, where β is a fixed weight in $(0,1)$, we have

$$\mu_r\left(f(X)\right) = \int_0^\infty \left[(1-\beta)S_X(t) + \beta 1_{\{S_X(t)>0\}}\right]\mathrm{d}f(t)$$

$$= (1-\beta)\int_0^\infty S_X(t)\,\mathrm{d}f(t) + \beta\int_0^\infty 1_{\{F_X(t)<1\}}\mathrm{d}f(t).$$

Applying (2.1) and Equation (2.10) yields

$$
\begin{aligned}
\mu_r\left(f(X)\right) &= (1-\beta)\mathbb{E}[f(X)] + \beta\int_0^\infty 1_{\{0\le t < F_X^{-1}(1)\}}\mathrm{d}f(t)\\
&= (1-\beta)\mathbb{E}[f(X)] + \beta f\left(F_X^{-1}(1)\right)\\
&= (1-\beta)\mathbb{E}[f(X)] + \beta F_{f(X)}^{-1}(1).
\end{aligned}
$$

3 Characterizations of optimal solutions

Armed with the integral expressions of the objective and constraint functions in Section 2, in this section we give complete solutions to Problems (2.8) and (2.9). We first treat the free-premium problem, which offers important insights and forms the basis for tackling the more challenging budget-constrained problem.

3.1 Free-premium problem

As $\delta\rho_g(X) + (1-\delta)\mathbb{E}[X]$ is independent of the ceded loss function f, it follows from Equation (2.11) of Lemma 2.1 that Problem (2.8) boils down to minimizing the integral $\int_0^\infty G_X(t)f'(t)\,\mathrm{d}t$, where G_X is given in Equation (2.13), among all $f \in \mathcal{F}$. This can be easily achieved in the following theorem.

Theorem 3.1. *(Solution of Problem* (2.8)*) Every optimal solution of Problem* (2.8) *is of the form*

$$
f^{*1}(x) = \int_0^x 1_{\{G_X<0\}}(t)\,\mathrm{d}t + \int_0^x 1_{\{G_X=0\}}(t)\,\mathrm{d}h^{*1}(t), \tag{3.1}
$$

where h^{*1} is an arbitrary function in $\mathcal{F}$. In particular, the optimal solution is unique if and only if $\{G_X = 0\}$ is a Lebesgue null set. The minimum risk-adjusted liability is

$$
\inf_{f\in\mathcal{F}} L_f(X) = \int_0^\infty \min\left\{r\left(S_X(t)\right), \delta g\left(S_X(t)\right) + (1-\delta)S_X(t)\right\}\,\mathrm{d}t.
$$

Proof. By construction, $f^{*1} \in \mathcal{F}$. Since all $f \in \mathcal{F}$ is non-decreasing and 1-Lipschitz, the corresponding derivative f', which exists almost everywhere, satisfies $0 \le f' \le 1$. For any

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$f \in \mathcal{F}$, observe that

$$
\begin{aligned}
\int_0^\infty G_X(t) f'(t) \, dt &= \int_{\{G_X<0\}} G_X(t) f'(t) \, dt + \int_{\{G_X=0\}} G_X(t) f'(t) \, dt \\
&\quad + \int_{\{G_X>0\}} G_X(t) f'(t) \, dt \\
&\geq \int_{\{G_X<0\}} G_X(t) f'(t) \, dt & (3.2) \\
&\geq \int_{\{G_X<0\}} G_X(t) \, dt & (3.3) \\
&= \int_0^\infty G_X(t) \, df^{*1}(t).
\end{aligned}
$$

Hence $\inf_{f \in \mathcal{F}} L_f(X) = L_{f^{*1}}(X)$.

To show that all optimal solutions of Problem (2.8) are given by the form in Equation (3.1), we observe that Inequality (3.2) is sharp if and only if $f' = 0$ almost everywhere on $\{G_X > 0\}$. As $f' \leq 1$, the second equality

$$
\int_{\{G_X<0\}} G_X(t) f'(t) \, dt = \int_{\{G_X<0\}} G_X(t) \, dt
$$

prevails if and only if $f' = 1$ almost everywhere on $\{G_X < 0\}$. Overall, $L_f(X) = L_{f^{*1}}(X)$ if and only if

$$
f'(t) = 1_{\{G_X<0\}}(t) + \tilde{h}(t) 1_{\{G_X=0\}}(t) \quad \text{for almost all } t \in [0, \operatorname{ess\,sup}(X)), \qquad (3.4)
$$

where $\tilde{h}$ is any function that takes values between 0 and 1. Because f is absolutely continuous with $f(0) = 0$ and integrals of almost everywhere equal functions are the same, Equation (3.4) is further equivalent to

$$
f(x) = \int_0^x f'(t) \, dt = \int_0^x \left[1_{\{G_X<0\}}(t) + \tilde{h}(t) 1_{\{G_X=0\}}(t) \right] dt,
$$

or

$$
f(x) = \int_0^x 1_{\{G_X<0\}}(t) \, dt + \int_0^x 1_{\{G_X=0\}}(t) \, dh(t),
$$

in which h is an arbitrary function in $\mathcal{F}$. This form of f corresponds precisely to the construction of f^{*1} in Equation (3.1). A direct inspection of Equation (3.1) also shows that f^{*1} simplifies to $f^{*1}(x) = \int_0^x 1_{\{G_X<0\}}(t) \, dt$ as the unique solution of Problem (2.8) if and only if $\{G_X = 0\}$ is a Lebesgue null set.

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The minimum liability can be computed as

$$
\begin{aligned}
L_{f^{*1}}(X) \;&=\; \delta\rho_g(X) + (1-\delta)\mathbb{E}[X] + \int_{\{G_X<0\}} G_X(t)\,\mathrm{d}t \\
&=\; \delta \int_0^\infty g\left(S_X(t)\right)\mathrm{d}t + (1-\delta)\int_0^\infty S_X(t)\,\mathrm{d}t \\
&\quad + \int_{\{G_X<0\}} \left(r\left(S_X(t)\right) - [\delta g\left(S_X(t)\right) + (1-\delta)S_X(t)] \right)\mathrm{d}t \\
&=\; \int_{\{G_X<0\}} r\left(S_X(t)\right)\mathrm{d}t + \int_{\{G_X\geq 0\}} [\delta g\left(S_X(t)\right) + (1-\delta)S_X(t)]\,\mathrm{d}t \\
&=\; \int_0^\infty \min\left\{r\left(S_X(t)\right), \delta g\left(S_X(t)\right) + (1-\delta)S_X(t)\right\}\mathrm{d}t.
\end{aligned}
$$

$\square$

Remark 3.2. The optimal solutions given in Theorem 3.1, despite their seemingly awkward analytic expression, can be appreciated by simple cost-benefit considerations. To this end, notice that each unit of excess loss the insurer cedes leads to a reduction in the risk-adjusted liability captured by the risk function $\delta g\left(S_X(\cdot)\right)+(1-\delta)S_X(\cdot)$ (*benefit*), but an increase in the reinsurance premium quantified by the premium function $r\left(S_X(\cdot)\right)$ (*cost*). Theorem 3.1 suggests that the insurer, with a view to minimizing his/her risk-adjusted liability, should purchase full coverage on an excess layer of loss on which the benefit of reinsurance strictly outweighs the cost of reinsurance. The latter condition translates mathematically into

$$
\delta g\left(S_X(\cdot)\right) + (1-\delta)S_X(\cdot) > r\left(S_X(\cdot)\right) \quad \Leftrightarrow \quad G_X(\cdot) < 0.
$$

This explains the first integral in Equation (3.1). On the other hand, no reinsurance coverage should be purchased when the cost of reinsurance strictly exceeds the cost of reinsurance, or in mathematical terms, $G_X(\cdot) > 0$. On the excess layers of losses where the benefit and cost of reinsurance are equal, i.e. $G_X(\cdot) = 0$, the insurer is indifferent to what form of reinsurance coverage, if any, is purchased; the value of the risk-adjusted liability is not altered in any way. This justifies why the optimal solutions may differ in design on the set $\{G_X = 0\}$, as the second integral in Equation (3.1) shows. These simple cost-benefit observations will be crucial to solving the budget-constrained problem in the next subsection.

3.2 Budget-constrained problem

In this subsection, we turn our attention to the budget-constrained Problem (2.9), the solution of which allows the insurer to select the reinsurance strategy leading to the smallest risk-adjusted liability while staying within the premium budget.

In the presence of a reinsurance premium budget, the interaction between the risk function $\delta g\left(S_X(\cdot)\right)+(1-\delta)S_X(\cdot)$ and premium function $r\left(S_X(\cdot)\right)$ continues to play an instrumental

role in the search of optimal reinsurance strategies. To begin with, we make the following observations:

- If $\int_{\{G_X \leq 0\}} r\left(S_X(t)\right) dt \leq \pi$, then f^{*1} as defined in Theorem 3.1 satisfies the premium constraint and solves Problem (2.9) readily.

- We are more interested in the intricate case where $\int_{\{G_X \leq 0\}} r\left(S_X(t)\right) dt > \pi$, meaning that ceding all the excess layers of loss when the benefit of reinsurance exceeds or is equal to its cost violates the tight reinsurance premium constraint and therefore refutes the feasibility of f^{*1}. The critical question is how to cede the excess losses in the set $\{G_X \leq 0\}$ as "efficiently" as possible so as to make the most of the limited reinsurance premium budget.

To formalize the notion of "efficiency", we define the *benefit-to-cost ratio*[iii] $H_X : [0, \operatorname{ess\,sup}(X))$ $\to \bar{\mathbb{R}}^+$ by

$$H_X(t) := \frac{\delta g\left(S_X(t)\right) + (1 - \delta) S_X(t)}{r\left(S_X(t)\right)}. \tag{3.5}$$

It is clear that $H_X \geq 1$ if and only if $G_X \leq 0$. Observe that the numerator and denominator of H_X capture the benefit and cost of reinsurance respectively. By ceding the excess layers of loss with a higher value of H_X, the insurer can achieve a greater reduction in the risk-adjusted liability at a smaller reinsurance premium. To minimize the total liability given the limited budget, the above consideration suggests that the insurer should cede starting from where H_X is the greatest until all the reinsurance premium is exhausted. Theorem 3.3 below formally confirms that such a heuristic tactic based on the level sets of H_X indeed gives rise to the optimal solutions of Problem (2.9).

Theorem 3.3. *(Solution of Problem* (2.9)*) Let*

$$c^* = \inf \left\{ c > 1 \,\middle|\, \int_{\{H_X \geq c\}} r\left(S_X(t)\right) dt \leq \pi \right\}.$$

(a) *If $\int_{\{G_X \leq 0\}} r\left(S_X(t)\right) dt \leq \pi$, then the set of optimal solutions of Problem* (2.9) *is the same as that of Problem* (2.8) *given in Theorem 3.1.*

(b) *If $\int_{\{G_X < 0\}} r\left(S_X(t)\right) dt \leq \pi < \int_{\{G_X \leq 0\}} r\left(S_X(t)\right) dt$, then the optimal solution of Problem* (2.9) *takes the form*

$$f^{*2}(x) = \int_0^x 1_{\{G_X < 0\}}(t)\, dt + \int_0^x 1_{\{G_X = 0\}}(t)\, dh^{*2}(t),$$

where h^{*2} is any function in $\mathcal{F}$ such that

$$\int_{\{G_X < 0\}} r\left(S_X(t)\right) dt + \int_{\{G_X = 0\}} r\left(S_X(t)\right) dh^{*2}(t) \leq \pi. \tag{3.6}$$

[iii]When $t \geq \operatorname{ess\,sup}(X)$, $r\left(S_X(t)\right) = g\left(S_X(t)\right) = S_X(t) = 0$, in which case $H_X(t)$ will be undefined.

(c) If $\int_{\{G_X<0\}} r\left(S_X(t)\right) \mathrm{d}t > \pi$ and $\int_{\{H_X\geq c^*\}} r\left(S_X(t)\right) \mathrm{d}t = \pi$, then the unique optimal solution of Problem (2.9) is

$$f^{*3}(x) = \int_0^x 1_{\{H_X\geq c^*\}}(t)\,\mathrm{d}t.$$

(d) If $\int_{\{G_X<0\}} r\left(S_X(t)\right) \mathrm{d}t > \pi$ and $\int_{\{H_X>c^*\}} r\left(S_X(t)\right) \mathrm{d}t \leq \pi < \int_{\{H_X\geq c^*\}} r\left(S_X(t)\right) \mathrm{d}t$, then every optimal solution of Problem (2.9) is of the form

$$f^{*4}(x) = \int_0^x 1_{\{H_X>c^*\}}(t)\,\mathrm{d}t + \int_0^x 1_{\{H_X=c^*\}}(t)\,\mathrm{d}h^{*4}(t),$$

where h^{*4} is any function in $\mathcal{F}$ such that

$$\int_{\{H_X>c^*\}} r\left(S_X(t)\right) \mathrm{d}t + \int_{\{H_X=c^*\}} r\left(S_X(t)\right) \mathrm{d}h^{*4}(t) = \pi. \tag{3.7}$$

Proof. Since all of the four cases can be proved in a similar manner, we only prove Cases (c) and (d), which are the most interesting.

(c) By definition, $f^{*3} \in \mathcal{F}$ with $\mu_r\left(f^{*3}(X)\right) = \int_{\{H_X\geq c^*\}} r\left(S_X(t)\right) \mathrm{d}t = \pi$. Now fix any $f \in \mathcal{F}$ satisfying the premium constraint. To verify the optimality of f^{*3}, it suffices to prove

$$\int_{\{G_X<0\}} G_X(t)[(f^{*3})'(t) - f'(t)]\,\mathrm{d}t \leq 0, \tag{3.8}$$

which is equivalent to

$$\int_{\{H_X\geq c^*\}} G_X(t)[1 - f'(t)]\,\mathrm{d}t \ \leq \ \int_{\{G_X<0\}\backslash\{H_X\geq c^*\}} G_X(t)f'(t)\,\mathrm{d}t$$
$$= \ \int_{\{1<H_X<c^*\}} G_X(t)f'(t)\,\mathrm{d}t.$$

Because $\{H_X \geq c^*\} = \{G_X(\cdot) \leq (1-c^*)r\left(S_X(\cdot)\right)\}$,

$$\int_{\{H_X\geq c^*\}} G_X(t)[1 - f'(t)]\,\mathrm{d}t \ \leq \ (1-c^*)\int_{\{H_X\geq c^*\}} r\left(S_X(t)\right)[1 - f'(t)]\,\mathrm{d}t \tag{3.9}$$
$$= \ (1-c^*)\left(\pi - \int_{\{H_X\geq c^*\}} r\left(S_X(t)\right) f'(t)\,\mathrm{d}t\right),$$

in which the last equality follows because $\int_{\{H_X\geq c^*\}} r\left(S_X(t)\right) \mathrm{d}t = \pi$ by hypothesis. As

$$\int_{\{H_X>1\}} r\left(S_X(t)\right) f'(t)\,\mathrm{d}t \leq \int_0^\infty r\left(S_X(t)\right) f'(t)\,\mathrm{d}t \leq \pi$$

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and $c^* > 1$ (if $c^* = 1$, then

$$\int_{\{H_X \geq 1\}} r\left(S_X(t)\right) \mathrm{d}t = \int_{\{G_X \leq 0\}} r\left(S_X(t)\right) \mathrm{d}t \geq \int_{\{G_X < 0\}} r\left(S_X(t)\right) \mathrm{d}t > \pi,$$

a contradiction to the hypothesis), we further have

$$\int_{\{H_X \geq c^*\}} G_X(t)[1 - f'(t)] \, \mathrm{d}t \;\; \leq \;\; (1 - c^*) \left(\int_{\{1 < H_X < c^*\}} r\left(S_X(t)\right) f'(t) \, \mathrm{d}t \right) \tag{3.10}$$

$$\leq \;\; \int_{\{1 < H_X < c^*\}} G_X(t) f'(t) \, \mathrm{d}t, \tag{3.11}$$

where the last inequality is due to $\{1 < H_X < c^*\} \subset \{(1 - c^*)r\left(S_X(\cdot)\right) < G_X(\cdot)\}$. This proves Inequality (3.8) and confirms the optimality of f^{*3}.

It remains to prove that f^{*3} is the only optimal ceded loss function of Problem (2.9). Note that equality prevails in Inequality (3.8) if and only if Inequalities (3.9), (3.10) and (3.11) all hold with equalities. It is easy to see that the latter is achieved by f^{*3} with $(f^{*3})' = 1_{\{H_X \geq c^*\}}$ almost everywhere. Conversely, suppose that equalities hold in Inequalities (3.9), (3.10) and (3.11). To show that $f \equiv f^{*3}$, we observe that the functions $G_X(\cdot)$ and $(1 - c^*)r(S_X(\cdot))$ not only are ordered, but also have the same integral on each of $\{H_X \geq c^*\}$ and $\{1 < H_X < c^*\}$. It follows that

$$[(1 - c^*)r\left(S_X(\cdot)\right) - G_X(\cdot)][1 - f'] = 0 \quad \text{almost everywhere on } \{H_X \geq c^*\}$$

and

$$[(1 - c^*)r\left(S_X(\cdot)\right) - G_X(\cdot)]f' = 0 \quad \text{almost everywhere on } \{1 < H_X < c^*\}.$$

These imply respectively that $f'(t) = 1$ for almost all t in $\{H_X > c^*\}$ and $f'(t) = 0$ for almost all t in $\{1 < H_X < c^*\}$. Together with the equality in (3.10), we have

$$\begin{aligned}
\pi &= \int_{\{H_X > 1\}} r\left(S_X(t)\right) f'(t) \, \mathrm{d}t \\
&= \int_{\{H_X > c^*\}} r\left(S_X(t)\right) \mathrm{d}t + \int_{\{H_X = c^*\}} r\left(S_X(t)\right) f'(t) \, \mathrm{d}t \\
&\leq \int_{\{H_X > c^*\}} r\left(S_X(t)\right) \mathrm{d}t + \int_{\{H_X = c^*\}} r\left(S_X(t)\right) \mathrm{d}t \\
&= \int_{\{H_X \geq c^*\}} r\left(S_X(t)\right) \mathrm{d}t \\
&= \pi,
\end{aligned}$$

so that $f'(t) = 1$ for almost all t in $\{H_X = c^*\}$ by the same argument as well. Consequently, $f(x) = \int_0^x f'(t) \, \mathrm{d}t = \int_0^x 1_{\{H_X \geq c^*\}}(t) \, \mathrm{d}t = f^{*3}(x)$ for all $x \in [0, \mathrm{ess\,sup}(X))$.

(d) By construction, f^{*4} lies in $\mathcal{F}$ and satisfies

$$\int_0^\infty r\left(S_X(t)\right) \mathrm{d}f^{*4}(t) = \int_{\{H_X > c^*\}} r\left(S_X(t)\right) \mathrm{d}t + \int_{\{H_X = c^*\}} r\left(S_X(t)\right) \mathrm{d}h^{*4}(t) = \pi.$$

The optimality of f^{*4} and uniqueness of any optimal solution on $\{H_X > c^*\}$ can be established as in the proof of Case (c) by splitting the integral in Inequality (3.8) over the three sets $\{1 < H_X < c^*\}$, $\{H_X = c^*\}$ and $\{H_X > c^*\}$. $\qquad\square$

Several remarks concerning the different cases presented in Theorem 3.3 and the structures of the various optimal solutions are in order.

Remark 3.4.   (i) To solve Problem (2.8), the function $G_X(\cdot)$, defined as the difference between the premium function $r\left(S_X(\cdot)\right)$ and the risk function $\delta g\left(S_X(\cdot)\right)+(1-\delta)S_X(\cdot)$, plays a vital role. In contrast, H_X, being the quotient of the risk function and the premium function, is instrumental in the solution of Problem (2.9). In essence, H_X considers not only whether the benefit of reinsurance is greater than the cost, but also how much greater. Basing the construction of the optimal ceded loss function on H_X therefore allows the insurer to cede the excess losses most "efficiently" and, as a consequence, obtain the lowest risk-adjusted liability with the limited reinsurance budget.

(ii) Our results in Theorem 3.3 generalize those in Theorems 2.2 and 2.3 of Cui et al. (2013). Our additional contributions are:

- We showcase the tradeoff of benefit and cost of reinsurance explicitly. Such a cost-benefit consideration goes a long way towards a simple and transparent investigation of the optimal solutions of Problem (2.9).

- The cost-benefit consideration is also critical to a complete search of optimal ceded loss functions. In Theorem 3.3, we fully solve Problem (2.9) by successfully identifying all possible optimal reinsurance strategies. In particular, we provide a full characterization of the solution set and determine when the optimal solution is unique.

(iii) It is necessary to distinguish Case (b) from Case (d), albeit their similarity. On the set $\{G_X = 0\}$, the ceded loss function does not contribute to any reduction in the risk-adjusted liability. As long as the form of the ceded loss function on $\{G_X = 0\}$ allows the insurer to stay within budget, optimality can be attained, hence the inequality in (3.6). In contrast, on the set $\{H_X = c^*\}$ with $c^* > 1$ in Case (d), the structure of the ceded loss function does affect the risk-adjusted liability. The more excess loss the insurer cedes, the greater the reduction in the risk-adjusted liability will be. This explains the equality in Equation (3.7), signifying the exhaustion of the reinsurance premium budget by the optimal ceded loss functions.

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(iv) Case (d) occurs only if there exists some interval $[t_1, t_2]$ on which H_X is constant (see Example 3.6 in the next subsection). In this case, ceding all the excess losses in the entire interval violates the reinsurance premium constraint. Theorem 3.3 (d) shows that as a remedy, optimality can be achieved by setting f to be any arbitrary non-decreasing, 1-Lipschitz function on $[t_1, t_2]$ while maintaining its continuity and consuming the full amount of reinsurance premium budget.

(v) The rather surprising conclusion that the form of the optimal solution on $\{H_X = c^*\}$ in Case (d) does not matter, as long the resulting function resides in $\mathcal{F}$ and satisfies the reinsurance premium constraint, can be appreciated by writing the objective function on $\{H_X = c^*\}$ as

$$\int_{\{H_X=c^*\}} G_X(t) f'(t)\, \mathrm{d}t = (1 - c^*) \int_{\{H_X=c^*\}} r\left(S_X(t)\right) f'(t)\, \mathrm{d}t$$

and observing that the same integral $\int_{\{H_X=c^*\}} r\left(S_X(t)\right) f'(t)\, \mathrm{d}t$ contributes to the reinsurance premium on $\{H_X = c^*\}$.

3.3 Examples

In the remainder of this section, we provide two carefully worked-out examples illuminating Theorem 3.3 in some particular reinsurance models. In the course of doing so, we also demonstrate the possibility of Cases (b) and (d) in Theorem 3.3 involving more than one optimal reinsurance strategy. In each example, a complete solution would require one to proceed carefully and consider a number of different cases in order to identify all the optimal strategies, whose forms vary critically with the confidence level of the risk measure and premium budget. Verbal interpretations based on the characteristics of the chosen risk measures are provided to justify why the optimal solutions make intuitive sense.

For any real x and y, we write $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$ and $x_+ = x \vee 0$.

Example 3.5. *(Value-at-Risk)* The *Value-at-Risk* (VaR) of a random variable X with distribution function F_X at confidence level $\alpha \in [0, 1]$ is defined by

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha).$$

The distortion function that gives rise to VaR is $g(x) = 1_{\{x>1-\alpha\}}$ (see Equation (44) of Dhaene et al. (2006)), which is neither convex nor concave.

In this example, we solve Problem (2.9) under the expectation premium principle with relative security loading θ, i.e. $r(x) = (1 + \theta)x$ with $\theta \geq 0$. The optimal solution of Problem (2.8) can then be obtained from that of Problem (2.9) by disregarding the premium constraint. Using (2.1), we have the following equivalences:

$$S_X(t) > 1 - \alpha \quad \Leftrightarrow \quad F_X(t) < \alpha \quad \Leftrightarrow \quad t < F_X^{-1}(\alpha).$$
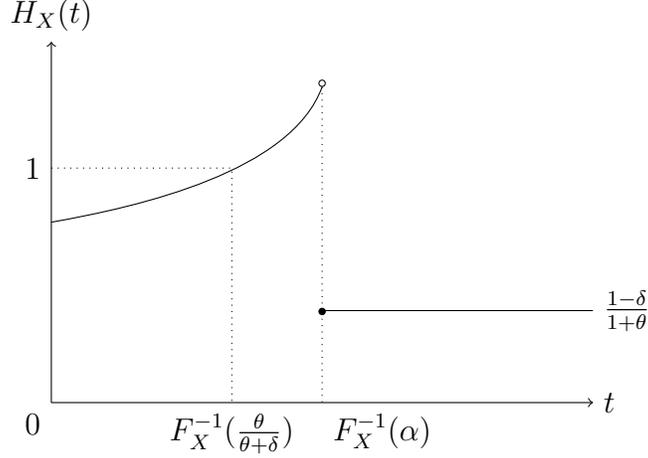
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Figure 1: The graph of H_X in the case of VaR and expectation premium principle.

In this case,

$$H_X(t) = \begin{cases} \frac{\delta}{(1+\theta)S_X(t)} + \frac{1-\delta}{1+\theta}, & \text{if } t < F_X^{-1}(\alpha), \\ \frac{1-\delta}{1+\theta}, & \text{if } t \geq F_X^{-1}(\alpha), \end{cases}$$

which is illustrated in Figure 1.

Using (2.1) again, we note that

$$\frac{\delta}{(1+\theta)S_X(t)} + \frac{1-\delta}{1+\theta} \geq 1 \quad \Leftrightarrow \quad t \geq F_X^{-1}\left(\frac{\theta}{\theta+\delta}\right),$$

so

$$H_X(t) \geq 1 \quad \Leftrightarrow \quad F_X^{-1}\left(\frac{\theta}{\theta+\delta}\right) \leq t < F_X^{-1}(\alpha).$$

Case 1. If $\alpha < \theta/(\theta + \delta)$, then $H_X(t) < 1$ for all $t \in [0, \operatorname{ess sup}(X))$. In this case, no reinsurance should be optimally purchased in Problems (2.8) and (2.9), i.e. the only optimal solution is $f^* \equiv 0$. A possible interpretation is that when the level of confidence is small, the insurer, who is not concerned with extreme losses, decides that the current level of risk exposure is appropriate and that it is not worthwhile at all to purchase any form of reinsurance.

Case 2. Suppose that $\alpha \geq \theta/(\theta + \delta)$.

    Case i. If $\int_{F_X^{-1}(\theta/(\theta+\delta))}^{F_X^{-1}(\alpha)} S_X(t)\,dt \leq \pi/(1+\theta)$, then according to Theorem 3.3 (a), the optimal solution of Problem (2.9) is of the form

$$f^{*1}(x) = \left[x \wedge F_X^{-1}(\alpha) - F_X^{-1+}\left(\frac{\theta}{\theta+\delta}\right)\right]_+ + h^{*1}\left[x \wedge F_X^{-1+}\left(\frac{\theta}{\theta+\delta}\right)\right],$$

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where h^{*1} is an arbitrary non-negative, non-decreasing 1-Lipschitz function with $h^{*1}\left[F_X^{-1}\left(\theta/(\theta+\delta)\right)\right] = 0$. Note that f^{*1} also solves Problem (2.8).

Case ii. If $\int_{F_X^{-1+}(\theta/(\theta+\delta))}^{F_X^{-1}(\alpha)} S_X(t)\,dt \leq \pi/(1+\theta) < \int_{F_X^{-1}(\theta/(\theta+\delta))}^{F_X^{-1}(\alpha)} S_X(t)\,dt$, then in accordance with Theorem 3.3 (b), the optimal solution of Problem (2.9) is

$$f^{*2}(x) = \left[x \wedge F_X^{-1}(\alpha) - F_X^{-1+}\left(\frac{\theta}{\theta+\delta}\right)\right]_+ + h^{*2}\left[x \wedge F_X^{-1+}\left(\frac{\theta}{\theta+\delta}\right)\right],$$

where h^{*2} is an arbitrary non-negative, non-decreasing 1-Lipschitz function with $h^{*2}\left[F_X^{-1}\left(\theta/(\theta+\delta)\right)\right] = 0$ and

$$\int_{F_X^{-1+}(\theta/(\theta+\delta))}^{F_X^{-1}(\alpha)} S_X(t)\,dt + \int_{F_X^{-1}(\theta/(\theta+\delta))}^{F_X^{-1+}(\theta/(\theta+\delta))} S_X(t)\,dh^{*2}(t) \leq \frac{\pi}{1+\theta}.$$

Case iii. If $\int_{F_X^{-1+}(\theta/(\theta+\delta))}^{F_X^{-1}(\alpha)} S_X(t)\,dt > \pi/(1+\theta)$, then we determine the optimal cutoff level $c^* > 1$ as the smallest $c > 1$ such that

$$\int_{\{H_X \geq c\}} S_X(t)\,dt \leq \frac{\pi}{1+\theta}.$$

- If $\int_{\{H_X \geq c^*\}} S_X(t)\,dt = \pi/(1+\theta)$, then from Theorem 3.3 (c), the unique optimal solution is

$$f^{*3}(x) = \left[x \wedge F_X^{-1}(\alpha) - H_X^{-1}(c^*)\right]_+,$$

where $H_X^{-1}(c^*)$ is well-defined because H_X is non-decreasing on $[0, F_X^{-1}(\alpha))$.

- If $\int_{\{H_X > c^*\}} S_X(t)\,dt \leq \pi/(1+\theta) < \int_{\{H_X \geq c^*\}} S_X(t)\,dt$, then it follows from Theorem 3.3 (d) that the optimal solution is

$$f^{*4}(x) = \left[x \wedge F_X^{-1}(\alpha) - H_X^{-1+}(c^*)\right]_+ + h^{*4}\left[x \wedge H_X^{-1+}(c^*)\right],$$

where h^{*4} is an arbitrary non-negative, non-decreasing 1-Lipschitz function such that $h^{*4}\left[H_X^{-1}(c^*)\right] = 0$ and

$$\int_{H_X^{-1+}(c^*)}^{F_X^{-1}(\alpha)} S_X(t)\,dt + \int_{H_X^{-1}(c^*)}^{H_X^{-1+}(c^*)} S_X(t)\,dh^{*4}(t) = \frac{\pi}{1+\theta}.$$

Observe that when $\{G_X = 0\}$ and $\{H_X = c^*\}$ are both Lebesgue null sets, the optimal solution of Problem (2.9) reduces to a limited stop-loss reinsurance contract. Such a ceded loss function is consistent with the character of VaR, adopting which the insurer only cares about losses up to the α-th quantile $F_X^{-1}(\alpha)$ and is not interested in the severity of further extreme losses. $\qquad\square$
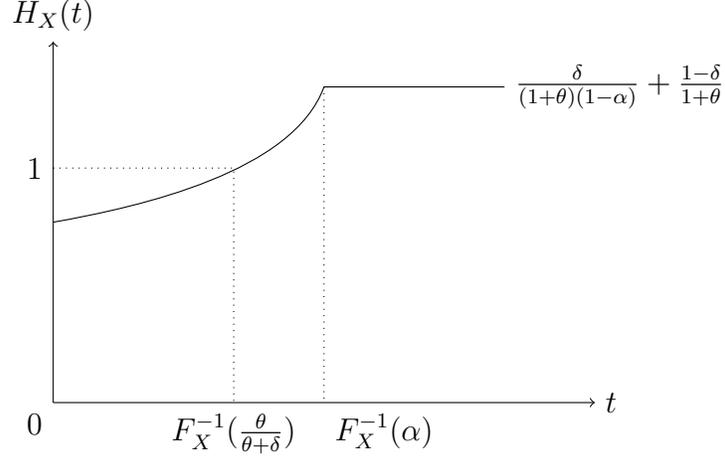
Figure 2: The graph of H_X in the case of TVaR and expectation premium principle.

Example 3.6. *(Tail Value-at-Risk)* The *Tail Value-at-Risk* (TVaR) of an integrable random variable X at confidence level $\alpha \in [0, 1)$ is defined by

$$\text{TVaR}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 \text{VaR}_p(X) \, dp.$$

The distortion function corresponding to TVaR is $g(x) = \frac{x}{1-\alpha} \wedge 1$ (see Equation (45) of Dhaene et al. (2006)).

We now solve Problem (2.9) under the expectation premium principle with relative security loading θ. In this setting, we have

$$H_X(t) = \begin{cases} \frac{\delta}{(1+\theta)S_X(t)} + \frac{1-\delta}{1+\theta}, & \text{if } t < F_X^{-1}(\alpha), \\ \frac{\delta}{(1+\theta)(1-\alpha)} + \frac{1-\delta}{1+\theta}, & \text{if } t \geq F_X^{-1}(\alpha), \end{cases}$$

which is non-decreasing in t (see Figure 2).

Case 1. If $\frac{\delta}{(1+\theta)(1-\alpha)} + \frac{1-\delta}{1+\theta} < 1$, or $\alpha < \theta/(\theta + \delta)$, then $H_X(t) < 1$ for all $t \geq 0$. As in Example 3.5, no reinsurance should be optimally purchased when the level of confidence is sufficiently small.

Case 2. Assume that $\alpha \geq \theta/(\theta + \delta)$. Then $H_X(t) \geq 1$ if and only if $t \geq F_X^{-1}(\theta/(\theta + \delta))$.

Case i. If $\int_{F_X^{-1}(\theta/(\theta+\delta))}^\infty S_X(t) \, dt \leq \pi/(1 + \theta)$, then by Theorem 3.3 (a), the optimal solution of Problem (2.9) (as well as Problem (2.8)) is

$$f^{*1}(x) = \left[x - F_X^{-1+} \left(\frac{\theta}{\theta + \delta} \right) \right]_+ + h^{*1} \left[x \wedge F_X^{-1+} \left(\frac{\theta}{\theta + \delta} \right) \right], \quad (3.12)$$

where h^{*1} is an arbitrary non-negative, non-decreasing 1-Lipschitz function with $h^{*1} \left[F_X^{-1} \left(\theta/(\theta + \delta) \right) \right] = 0$.

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Case ii. If $\int_{F_X^{-1+}(\theta/(\theta+\delta))}^{\infty} S_X(t)\,\mathrm{d}t \le \pi/(1+\theta) < \int_{F_X^{-1}(\theta/(\theta+\delta))}^{\infty} S_X(t)\,\mathrm{d}t$, then by virtue of Theorem 3.3 (b), the optimal solution of Problem (2.9) is

$$f^{*2}(x) = \left[x - F_X^{-1+}\left(\frac{\theta}{\theta+\delta}\right)\right]_+ + h^{*2}\left[x \wedge F_X^{-1+}\left(\frac{\theta}{\theta+\delta}\right)\right],$$

where h^{*2} is an arbitrary non-negative, non-decreasing 1-Lipschitz function with $h^{*2}\left[F_X^{-1}(\theta/(\theta+\delta))\right] = 0$ and

$$\int_{F_X^{-1+}(\theta/(\theta+\delta))}^{\infty} S_X(t)\,\mathrm{d}t + \int_{F_X^{-1}(\theta/(\theta+\delta))}^{F_X^{-1+}(\theta/(\theta+\delta))} S_X(t)\,\mathrm{d}h^{*2}(t) \le \frac{\pi}{1+\theta}.$$

Case iii. If $\int_{F_X^{-1+}(\theta/(\theta+\delta))}^{\infty} S_X(t)\,\mathrm{d}t > \pi/(1+\theta)$, then we determine the optimal cutoff level $c^* > 1$ as the smallest $c > 1$ such that

$$\int_{\{H_X \ge c\}} S_X(t)\,\mathrm{d}t \le \frac{\pi}{1+\theta}.$$

- If $\int_{\{H_X \ge c^*\}} S_X(t)\,\mathrm{d}t = \pi/(1+\theta)$, then by Theorem 3.3 (c), the unique optimal solution is

$$f^{*3}(x) = \left[x - H_X^{-1}(c^*)\right]_+,$$

  where $H_X^{-1}(c^*)$ is well-defined again because H_X is a non-decreasing function.

- If $\int_{\{H_X > c^*\}} S_X(t)\,\mathrm{d}t \le \pi/(1+\theta) < \int_{\{H_X \ge c^*\}} S_X(t)\,\mathrm{d}t$, then by Theorem 3.3 (d), the optimal solution is

$$f^{*4}(x) = \left[x - H_X^{-1+}(c^*)\right]_+ + h^{*4}\left[x \wedge H_X^{-1+}(c^*)\right],$$

  where h^{*4} is an arbitrary non-negative, non-decreasing 1-Lipschitz function such that $h^{*4}\left[H_X^{-1}(c^*)\right] = 0$ and

$$\int_{H_X^{-1+}(c^*)}^{\infty} S_X(t)\,\mathrm{d}t + \int_{H_X^{-1}(c^*)}^{H_X^{-1+}(c^*)} S_X(t)\,\mathrm{d}h^{*4}(t) = \frac{\pi}{1+\theta}.$$

If $\{G_X = 0\}$ and $\{H_X = c^*\}$ are both Lebesgue null sets, then the optimal solution is solely in the form of stop-loss reinsurance. This is consistent with the character of TVaR. When the insurer is TVaR-regulated, he/she cares about not only the probability of incurring a high loss, but also the severity of extreme losses. As a result, the insurer will find it advantageous in terms of minimizing his/her risk-adjusted liability to purchase full coverage on all extreme losses by means of a stop-loss reinsurance treaty.

As a concrete illustration of the wide range of optimal reinsurance treaties in the last case of the above solution, we suppose specifically that the ground-up loss X is exponentially

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distributed with mean 1,000, $\alpha = 0.95$, $\theta = 0.1$, $\delta = 0.6$ and $\pi = 44$. Now $\frac{\delta}{(1-\alpha)(1+\theta)} + \frac{1-\delta}{1+\theta} = 124/11 > 1$, $F_X^{-1}(\alpha) = 1{,}000 \ln 20$ and

$$\int_{F_X^{-1}(\alpha)}^{\infty} S_X(t)\, \mathrm{d}t = \int_{1{,}000 \ln 20}^{\infty} e^{-t/1{,}000}\, \mathrm{d}t = 50 > \frac{\pi}{1+\theta} = 40,$$

which implies that $c^* = 124/11$ in this case of a tight reinsurance budget. Because $H_X^{-1}(c^*) = 1{,}000 \ln 20$ and $H_X^{-1+}(c^*) = \infty$, the optimal solution can be any function $f^* \in \mathcal{F}$ such that $f^* \left(F_X^{-1}(\alpha) \right) = f^*(1{,}000 \ln 20) = 0$ and

$$\int_{1{,}000 \ln 20}^{\infty} S_X(t)\, \mathrm{d}f^*(t) = 40. \tag{3.13}$$

It is straightforward to check that the following functions are all members of $\mathcal{F}$, null on $[0, F_X^{-1}(\alpha)] = [0, 1{,}000 \ln 20]$, satisfy Equation (3.13) and, most importantly, result in the same level of risk-adjusted liability:

$$
\begin{aligned}
\text{(Stop-loss)} \quad f_1(x) &= (x - 1{,}000 \ln 25)_+, \\
\text{(Limited stop-loss)} \quad f_2(x) &= (x \wedge 1{,}000 \ln 100 - 1{,}000 \ln 20)_+, \\
\text{(Quota-share)} \quad f_3(x) &= \frac{4}{5}(x - 1{,}000 \ln 20)_+, \\
\text{(Mixtures of insurance layers)} \quad f_4(x) &= (x \wedge 1{,}000 \ln 40 - 1{,}000 \ln 20)_+ \\
&\quad + \frac{3}{4}(x - 1{,}000 \ln 50)_+, \\
\text{(Irregular)} \quad f_5(x) &= c \int_{1{,}000 \ln 20}^{x \vee 1{,}000 \ln 20} \Phi(t)\, \mathrm{d}t,
\end{aligned}
$$

where Φ is the distribution function of the standard normal distribution, and

$$
\begin{aligned}
c &= \frac{40}{\int_{1{,}000 \ln 20}^{\infty} e^{-x/1{,}000}\Phi(x)\, \mathrm{d}x} \\
&= \frac{40}{50\Phi(1{,}000 \ln 20) + 1{,}000 e^{-1/(2 \times 1{,}000^2)}\left[1 - \Phi(1{,}000 \ln 20 + 1/1{,}000)\right]}.
\end{aligned}
$$

In addition, any non-decreasing convex function f_6 satisfying $0 \le f_6(x) \le x$ for all $x \ge 0$ is 1-Lipschitz (see Section 5 of Cheung et al. (2014)), and will solve Problem (2.9) provided that f_6 is null on $[0, F_X^{-1}(\alpha)] = [0, 1{,}000 \ln 20]$ and satisfies Equation (3.13). $\qquad\square$

Remark 3.7. (i) For the purpose of illustration, we have chosen the expectation premium principle in Examples 3.5 and 3.6 because it is widely used and extensively studied in a number of articles. In the special case when $\delta = 1$, our results in Examples 3.5 and 3.6 are consistent with those in Corollary 3.1 and Example 3.1 of Cui et al. (2013). However, with Theorem 3.3 we manage to exhaust all possible optimal solutions and assert uniqueness in some cases.

(ii) It is also intriguing to explore the effect of δ and θ on the optimal ceded loss functions in Examples 3.5 and 3.6. With the exception of Case 1 (i.e. when the confidence level is too low), it can be seen that the insurer in both examples cedes more excess loss as δ increases or θ decreases, implying a more stringent risk margin or reinsurance becoming less expensive respectively. This is in agreement with observed practice.

(iii) If we replace the expectation premium principle by Wang's premium principle, i.e. r is a non-decreasing concave function, we have

$$H_X^{\text{VaR}}(t) = \begin{cases} \frac{\delta+(1-\delta)S_X(t)}{r(S_X(t))}, & \text{if } t < F_X^{-1}(\alpha), \\ \frac{(1-\delta)S_X(t)}{r(S_X(t))}, & \text{if } t \geq F_X^{-1}(\alpha), \end{cases}$$

and

$$H_X^{\text{TVaR}}(t) = \begin{cases} \frac{\delta+(1-\delta)S_X(t)}{r(S_X(t))}, & \text{if } t < F_X^{-1}(\alpha), \\ \frac{[\delta+(1-\delta)(1-\alpha)]S_X(t)}{(1-\alpha)r(S_X(t))}, & \text{if } t \geq F_X^{-1}(\alpha). \end{cases}$$

respectively in the VaR and TVaR cases. Note that H_X^{VaR} and H_X^{TVaR} can take an irregular shape without further information about the function r. If $\delta = 1$, then H_X^{VaR} and H_X^{TVaR} are both non-decreasing on $[0, F_X^{-1}(\alpha))$, so that in Case (c) of Theorem 3.3, we have

$$f^{*\text{VaR}}(x) = \left[x \wedge F_X^{-1}(\alpha) - H_X^{-1}(c^*) \right]_+ ,$$

for some $c^* > 1$ such that $\int_{H_X^{-1}(c^*)}^{F_X^{-1}(\alpha)} r\left(S_X(t)\right) \mathrm{d}t = \pi$, and

$$f^{*\text{TVaR}}(x) = \left(x \wedge d_2 - d_1^* \right)_+ ,$$

for some d_1^* and d_2^* such that $d_1^* < d_2^*$ and $\int_{d_1^*}^{d_2^*} r\left(S_X(t)\right) \mathrm{d}t = \pi$, as the respective solutions of the VaR-based and TVaR-based Problem (2.9). These results are again consistent with Corollary 3.1 and Example 3.1 of Cui et al. (2013).

4 A dual optimal reinsurance problem

As an application of the cost-benefit technique illustrated in Section 3, in this section we solve a related optimal reinsurance problem formulated as:

$$\begin{cases} \inf_{f \in \mathcal{F}} & \mu_r\left(f(X)\right) \\ \text{s.t.} & L_f(X) \leq R, \end{cases} \tag{4.1}$$

where R is a fixed quantity that can be regarded as the tolerance level of the risk-adjusted liability imposed by the shareholders or regulators of the insurer. The problem is to select the cheapest reinsurance strategy while maintaining the risk-adjusted liability at a level that is acceptable to the shareholders and regulators.

Inspecting Problem (4.1), one can make the following observations:

1. The objective function (resp. constraint function) of Problem (4.1) becomes the constraint function (resp. objective function) of Problem (2.9).

2. The unconstrained version of Problem (4.1) is not meaningful, because the insurer can always purchase no reinsurance so as not to pay any reinsurance premium.

3. By Lemma 2.1, the constraint function can be written equivalently as

$$\int_0^\infty G_X(t) \, df(t) \leq R' := R - [\delta \rho_g(X) + (1 - \delta)\mathbb{E}[X]].$$

4. Unlike Problem (2.9), in which the insurer is eager to purchase reinsurance to reduce its liability, in Problem (4.1) the insurer is disinclined to do so because that will result in a rise in the reinsurance premium, which he/she wishes to minimize. He/she employs reinsurance only to the extent to satisfy the liability constraint.

5. While Problem (2.9) always has a solution, Problem (4.1) may not be well-posed if the value of R (or R') is too low. Specifically, if

$$\int_{\{G_X < 0\}} G_X(t) \, dt = \int_{\{G_X \leq 0\}} G_X(t) \, dt > R',$$

then the feasible set of Problem (4.1) is empty, in which case Problem (4.1) admits no solution. On the other hand, when $R' \geq 0$, then $f \equiv 0$ solves Problem (4.1) trivially.

To seek the optimal reinsurance strategies for Problem (4.1), the insurer is interested in minimizing the cost-to-benefit ratio

$$\frac{r\left(S_X(t)\right)}{\delta g\left(S_X(t)\right) + (1 - \delta)S_X(t)},$$

which is equivalent to maximizing the benefit-to-cost ratio H_X defined in Equation (3.5). In other words, the same technique used to tackle Problem (2.9) can be applied to solve Problem (4.1).

Theorem 4.1. *(Solution of Problem (4.1)) Assume that $\int_{\{G_X < 0\}} G_X(t) \, dt \leq R' < 0$. Define*

$$d^* = \sup \left\{ d \geq 1 \left| \int_{\{H_X > d\}} G_X(t) \, dt \leq R' \right. \right\}.$$

(a) *If $\int_{\{H_X > d^*\}} G_X(t) \, dt = R'$, then the unique optimal solution of Problem (4.1) is given by*

$$f^{*5}(x) = \int_0^x 1_{\{H_X > d^*\}}(t) \, dt.$$

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(b) If $\int_{\{H_X \geq d^*\}} G_X(t) \, dt \leq R' < \int_{\{H_X > d^*\}} G_X(t) \, dt$, *then the optimal solution of Problem (4.1) is given by*

$$f^{*6}(x) = \int_0^x 1_{\{H_X > d^*\}}(t) \, dt + \int_0^x 1_{\{H_X = d^*\}}(t) \, dh^{*6}(t),$$

where h^{*6} is any function in $\mathcal{F}$ such that

$$\int_{\{H_X > d^*\}} G_X(t) \, dt + \int_{\{H_X = d^*\}} G_X(t) \, dh^{*6}(t) = R'.$$

Proof. The proof mirrors that of Theorem 3.3 and is omitted to avoid repetition. □

5 Concluding remarks

In this article, we examine the entire set of ceded loss functions that minimize an insurer's risk-adjusted liability in which the risk margin is quantified by distortion risk measures, possibly under the imposition of a reinsurance premium budget. A transparent argument which lends itself to cost-benefit interpretations and to the exhaustion of all optimal solutions is illustrated. Unlike previous studies, it is shown that the optimal reinsurance treaties can take an arbitrary shape on a certain set in some cases. As a further illustration of the cost-benefit argument, we also completely determine every ceded loss function that minimizes the reinsurance premium while keeping the insurer's risk-adjusted liability below a fixed tolerance level.

Throughout this article, the constraint function is formulated in terms of r, the function that is used to calibrate the reinsurance premium. It is essential to emphasize that our analysis can indeed be extended to more interesting and realistic scenarios with more complicated constraint functions accommodating the interests of both the insurer and reinsurer. Consider, for instance, the constrained minimization problem

$$\begin{cases} \displaystyle\inf_{f \in \mathcal{F}} & L_f(X) \\ \text{s.t.} & \text{VaR}_p \left[f(X) - \mu_r \left(f(X) \right) \right] \leq \pi, \end{cases} \tag{5.1}$$

where $f(X) - \mu_r(f(X))$ represents the total risk of the *reinsurer*, p is a given confidence level in $(0, 1)$, and π is the risk tolerance level of the reinsurer. The constraint function of Problem (5.1) takes the reinsurer's risk preference into account and makes practical sense in that reinsurance is a two-party problem. By virtue of the translation invariance of VaR,

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we can write the constraint function of Problem (5.1) as

$$
\begin{aligned}
\mathrm{VaR}_p\left[f(X) - \mu_r\left(f(X)\right)\right] &= \mathrm{VaR}_p\left[f(X)\right] - \mu_r\left(f(X)\right) \\
&= \int_0^\infty \left[1_{\{S_X(t)>1-p\}} - r\left(S_X(t)\right)\right]\mathrm{d}f(t) \\
&= \int_0^\infty \hat{r}\left(S_X(t)\right)\mathrm{d}f(t),
\end{aligned}
$$

where $\hat{r}(x) := 1_{\{x>1-p\}} - r(x)$. This way of writing expresses both of the objective function and constraint function as appropriate integrals with respect to the ceded loss function. Building upon the idea underlying the proof of Theorem 3.3, one can easily obtain the set of optimal ceded loss functions of Problem (5.1).

Acknowledgements

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A   Appendix

Proof of Lemma 2.1. We first prove Equation (2.10) by writing

$$f(X) = \int_0^X \mathrm{d}f(t) = \int_0^\infty 1_{\{X>t\}}\,\mathrm{d}f(t),$$

which is a well-defined Lebesgue-Stieltjes integral in which the induced measure is absolutely continuous with respect to the Lebesgue measure, as f is a non-decreasing absolutely continuous function with $f(0) = 0$. By Fubini's theorem,

$$\mathbb{E}[f(X)] = \int_0^\infty \mathbb{E}[1_{\{X>t\}}]\,\mathrm{d}f(t) = \int_0^\infty S_X(t)\,\mathrm{d}f(t).$$

Next, as a key step towards Equation (2.11), we prove

$$\rho_g\left(f(X)\right) = \int_0^\infty g\left(S_X(t)\right)\mathrm{d}f(t) \tag{A.1}$$

by showing

$$F_{f(X)}^{-1}(1-p) = \int_0^\infty F_{1_{\{X>t\}}}^{-1}(1-p)\,\mathrm{d}f(t), \qquad 0 \le p \le 1 \tag{A.2}$$

and
$$F_{f(X)}^{-1+}(1-p) = \int_0^\infty F_{1_{\{X>t\}}}^{-1+}(1-p)\,\mathrm{d}f(t), \qquad 0 \le p \le 1. \tag{A.3}$$

The left-hand side of Equation (A.2), by virtue of Equation (2.2), equals $f\left(F_X^{-1}(1-p)\right)$. To calculate the right-hand side of Equation (A.2), note that for each $t \in [0, \operatorname{ess\,sup}(X))$,

$$
\begin{aligned}
F_{1_{\{X>t\}}}^{-1}(1-p) &= \begin{cases} 0, & \text{if } 0 < 1-p \le F_X(t) \\ 1, & \text{if } F_X(t) < 1-p \le 1 \end{cases} \\
&= \begin{cases} 0, & \text{if } t \ge F_X^{-1}(1-p) \\ 1, & \text{if } t < F_X^{-1}(1-p) \end{cases}
\end{aligned}
$$

due to (2.1). Hence

$$\int_0^\infty F_{1_{\{X>t\}}}^{-1}(1-p)\,\mathrm{d}f(t) = \int_0^{F_X^{-1}(1-p)} \mathrm{d}f(t) = f\left(F_X^{-1}(1-p)\right),$$

verifying Equation (A.2). Equation (A.3) can be shown in a completely analogous manner.

We can now establish (A.1) by distinguishing the continuity of g.

Case 1. If g is left-continuous, then it follows from Equations (2.4), (A.2) and Fubini's theorem that

$$
\begin{aligned}
\rho_g\left(f(X)\right) &= \int_0^1 F_{f(X)}^{-1}(1-p)\,\mathrm{d}g(p) \\
&= \int_0^1 \left(\int_0^\infty F_{1_{\{X>t\}}}^{-1}(1-p)\,\mathrm{d}f(t)\right)\mathrm{d}g(p) \\
&= \int_0^\infty \left(\int_0^1 F_{1_{\{X>t\}}}^{-1}(1-p)\,\mathrm{d}g(p)\right)\mathrm{d}f(t) \\
&= \int_0^\infty \left(\int_{[0,S_X(t))} \mathrm{d}g(p)\right)\mathrm{d}f(t) \\
&= \int_0^\infty g\left(S_X(t)\right)\mathrm{d}f(t),
\end{aligned}
$$

in which the last step uses the left-continuity of g.

Case 2. For right-continuous g, we use Equations (2.5) and (A.3) to obtain

$$
\begin{aligned}
\rho_g\left(f(X)\right) &= \int_0^1 F_{f(X)}^{-1+}(1-p)\,\mathrm{d}g(p) \\
&= \int_0^1 \left(\int_0^\infty F_{1_{\{X>t\}}}^{-1+}(1-p)\,\mathrm{d}f(t) \right) \mathrm{d}g(p) \\
&= \int_0^\infty \left(\int_0^1 F_{1_{\{X>t\}}}^{-1+}(1-p)\,\mathrm{d}g(p) \right) \mathrm{d}f(t) \\
&= \int_0^\infty \left(\int_{[0,S_X(t)]} \mathrm{d}g(p) \right) \mathrm{d}f(t) \\
&= \int_0^\infty g\left(S_X(t)\right)\mathrm{d}f(t),
\end{aligned}
$$

where the last step requires the right-continuity of g.

Case 3. For general g, we write, as in Theorem 7 of Dhaene et al. (2012), $g = c_r g_r + c_l g_l$ for some right-continuous distortion function g_r and left-continuous distortion function g_l, and some non-negative constants c_r and c_l such that $c_r + c_l = 1$. Using the results in Cases 1 and 2,

$$
\begin{aligned}
\rho_g\left(f(X)\right) &= c_r \rho_{g_r}\left(f(X)\right) + c_l \rho_{g_l}\left(f(X)\right) \\
&= c_r \int_0^\infty g_r\left(S_X(t)\right)\mathrm{d}f(t) + c_l \int_0^\infty g_l\left(S_X(t)\right)\mathrm{d}f(t) \\
&= \int_0^\infty g\left(S_X(t)\right)\mathrm{d}f(t).
\end{aligned}
$$

Repeating the above derivations with g replaced by r, noting that the condition $g(1) = 1$ is not used, leads to

$$
\mu_r\left(f(X)\right) = \int_0^\infty r\left(S_X(t)\right)\mathrm{d}f(t),
$$

which is the first equality in Equation (2.14).

We now turn to Equation (2.11). Because distortion risk measures are translation invariant, we have

$$
\begin{aligned}
\rho_g(T_f(X)) &= \mu_r\left(f(X)\right) + \rho_g\left(X - f(X)\right) \\
&= \mu_r\left(f(X)\right) + \int_0^\infty g\left(S_X(t)\right)\mathrm{d}(\mathrm{Id}-f)(t) \\
&= \int_0^\infty r\left(S_X(t)\right)\mathrm{d}f(t) + \int_0^\infty g\left(S_X(t)\right)\mathrm{d}t - \int_0^\infty g\left(S_X(t)\right)\mathrm{d}f(t) \\
&= \rho_g(X) + \int_0^\infty \left[r\left(S_X(t)\right) - g\left(S_X(t)\right)\right]\mathrm{d}f(t)
\end{aligned}
$$

where the second equality follows from an application of Equation (A.1) with a replacement of f by $\mathrm{Id} - f$, which also lies in $\mathcal{F}$. Together with Equations (2.6) and (2.10), we obtain

$$
\begin{aligned}
L_f(X) &= (1-\delta)\mathbb{E}[T_f(X)] + \delta\rho_g\left(T_f(X)\right) \\
&= (1-\delta)\left[\mathbb{E}[X] - \mathbb{E}[f(X)] + \mu_r\left(f(X)\right)\right] + \delta\rho_g\left(T_f(X)\right) \\
&= (1-\delta)\left(\mathbb{E}[X] + \int_0^\infty \left[r\left(S_X(t)\right) - S_X(t)\right]\mathrm{d}f(t)\right) \\
&\quad + \delta\left[\rho_g(X) + \int_0^\infty \left[r\left(S_X(t)\right) - g\left(S_X(t)\right)\right]\mathrm{d}f(t)\right] \\
&= \delta\rho_g(X) + (1-\delta)\mathbb{E}[X] + \int_0^\infty G_X(t)\,\mathrm{d}f(t),
\end{aligned}
$$

where $G_X(t) = r\left(S_X(t)\right) - \left[\delta g\left(S_X(t)\right) + (1-\delta)S_X(t)\right]$.

Finally, the two Lebesgue-Stieltjes integrals in Equations (2.11) and (2.14) can be transformed into ordinary Lebesgue integrals with $\mathrm{d}f(t)$ replaced by $f'(t)\,\mathrm{d}t$ because of the absolute continuity of f, whose derivative exists almost everywhere (see Remark (v)(a) on page 285 of Stein and Shakarchi (2005)). $\qquad\square$

Remark A.1. Equation (A.1), when written as

$$
\rho_g\left(\int_0^\infty 1_{\{X>t\}}\,\mathrm{d}f(t)\right) = \int_0^\infty \rho_g(1_{\{X>t\}})\,\mathrm{d}f(t),
$$

can be considered as an integral version of the comonotonic additivity property of distortion risk measures (see Equation (52) of Dhaene et al. (2006)), noting that $(1_{\{X>t\}})_{t\in[0,\infty)}$ is a collection of comonotonic random variables (see Dhaene et al. (2002) for the definition of comonotonicity).