Interplay of Insurance and Financial Risks in a Stochastic Environment

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Abstract

Consider an insurer who makes risky investments and hence faces both insurance and financial risks. The insurance business is described by a discrete-time risk model modulated by a stochastic environment that poses systemic and systematic impacts on both the insurance and financial markets. This paper endeavors to quantitatively understand the interplay of the two risks in causing ruin of the insurer. Under the bivariate regular variation framework, we obtain an asymptotic formula to describe the impacts on the insurer's solvency of the two risks and of the stochastic environment.

Keywords: asymptotic estimates; bivariate regular variation; ruin probability; stochastic environment; stochastic ordering; uniformity

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1 Introduction

Since insurers make investments, when conducting an assessment of solvency for insurance business, we need to address two fundamental risks, which are insurance risk that is caused by insurance claims and financial risk that is due to risky investments. The reader is referred to Norberg (1999) and Kalashnikov and Norberg (2002) for related discussions. In this paper, we study the interplay of the insurance and financial risks in causing ruin of an insurer. Ruin theory as a classical research topic has been playing an important role in risk theory and other related fields such as risk management and mathematical finance. In particular, under modern insurance regulatory frameworks, ruin theory has immediate implications on important issues such as solvency capital requirement and insurance risk management; for related discussions, see Nyrhinen (2010) and Trufin et al. (2011), among others.

Throughout the paper, all random variables are assumed to be defined on a common probability space $(\Omega, \mathcal{F}, P)$. We employ a discrete-time risk model to accommodate the two

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risks. Precisely, for each period $i \in \mathbb{N}$, denote by X_i the insurance risk, quantified as the insurer's net loss equal to the total amount of claims less premiums within the period, and denote by Y_i the financial risk quantified as the stochastic present value factor over the same period. In this way, the aggregate stochastic present values of insurance losses become

$$S_0 = 0, \qquad S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j, \qquad n \in \mathbb{N}. \tag{1.1}$$

Denote their running maximum by

$$M_n = \max_{0 \le k < n+1} S_k, \qquad n \in \mathbb{N} \cup \{0\} \cup \{\infty\}. \tag{1.2}$$

In the tradition of ruin theory, the insurer holding an initial capital $x > 0$ is regarded as ruined when its wealth becomes negative, or equivalently, when the running maximum M_n upcrosses level x. Thus, the probability of ruin in either a finite-time or infinite-time horizon is defined by

$$\psi(x; n) = P\left(M_n > x\right), \qquad n \in \mathbb{N} \cup \{\infty\}.$$

Except for few cases under ideal distributional assumptions, a closed-form expression for the ruin probability $\psi(x; n)$ is not available. Thus, the mainstream of the study focuses on characterizing its asymptotic behavior. For example, under this setup, Nyrhinen (1999, 2001) and Tang and Tsitsiashvili (2003, 2004) respectively obtained some crude and precise asymptotic estimates for the ruin probability $\psi(x; n)$ as x becomes large. The reader is referred to Paulsen (2008) and Asmussen and Albrecher (2010) for reviews of early works on this study in both discrete-time and continuous-time models. Recent works under more practical settings include Chen (2011), Fougeres and Mercadier (2012), Yang and Konstantinides (2015), Li and Tang (2015), Lehtomaa (2015), Tang and Yuan (2016), Nyrhinen (2016), Yang et al. (2016), Chen and Yuan (2017), and Chen (2017).

Loosely speaking, most of these works are restricted to a standard framework in which (X_i, Y_i), $i \in \mathbb{N}$, form a sequence of independent and identically distributed (i.i.d.) copies of a generic pair (X, Y) with heavy tails, and a main discovery is that for asymptotically independent X and Y the decay rate of the ruin probability $\psi(x; n)$ as $x \to \infty$ is dominated by whichever tail of X and Y is heavier than the other, while for asymptotically dependent X and Y the decay rate can be much slower, indicating a much more dangerous scenario of the insurance business.

Remarkably, under the i.i.d. assumption above, if $E\left[\ln(|X| \vee 1)\right] < \infty$ and $-\infty \le E\left[\ln Y\right] < 0$, then as $n \to \infty$, the sum S_n in (1.1) converges almost surely to

$$S = \sum_{i=1}^{\infty} X_i \prod_{j=1}^{i} Y_j,$$

the maximum M_n in (1.2) converges almost surely to

$$M = \max_{0 \le k < \infty} S_k,$$

and the limits S and M satisfy the stochastic fixed point equations, respectively,

$$\begin{cases} S =_d (X+S)Y, & \text{with } S \text{ independent of } (X,Y), \\ M =_d (X+M)^+Y, & \text{with } M \text{ independent of } (X,Y), \end{cases} \tag{1.3}$$

where $=_d$ denotes equal in distribution and $x^+ = x \vee 0$ denotes the positive part of a real number x. See Vervaat (1979) for these statements. Note that these are slightly different from ones often appearing in the literature:

$$\begin{cases} S =_d X+SY, & \text{with } S \text{ independent of } (X,Y), \\ M =_d (X+MY)^+, & \text{with } M \text{ independent of } (X,Y). \end{cases} \tag{1.4}$$

See Grey (1994), Buraczewski et al. (2016a, b), and Dyszewski (2016), among many others, for the study of the stochastic fixed point equations (1.4). Under our setting (1.3), a certain restriction on the dependence structure between X and Y usually has to be imposed in order to deal with the products XY, X_1Y_1, …, while under the setting (1.4) such a restriction is often not needed.

Clearly, the i.i.d. assumption on the sequence $\{(X_i, Y_i), i \in \mathbb{N}\}$ is highly impractical, especially for analyzing a business in a relatively long time horizon. In reality, both insurance and financial risks coexist in a stochastic environment that is composed of both changes in the nature such as seasonal effects and global warming, and evolutions of certain macroeconomic factors such as inflation, bank base rate, a country's gross domestic product, and unemployment rate. Thus, it is important to take into account this external stochastic environment when modeling insurance business, and to capture its impact on the insurer's solvency. A natural idea is to assume that the sequence $\{(X_i, Y_i), i \in \mathbb{N}\}$ is modulated by an underlying stochastic process, say, $\Theta = \{\theta_i, i \in \mathbb{N}\}$, that summarizes the stochastic environment. Indeed, this idea has been extensively employed in the mainstream study of insurance and finance since the pioneering works by Asmussen (1989) and Asmussen et al. (1994, 1995). Some recent works in ruin theory along this trend are Lu and Li (2005), Foss et al. (2007), Foss and Richards (2010), and Elliott et al. (2011), among many others. It is discovered that the insurer's solvency is significantly affected by the presence of the external stochastic environment. In the literature, models that incorporate such a stochastic environment Θ are often called regime-switching models and Θ is often assumed to be a Markov process for mathematical tractability.

We continue the study of the asymptotic behavior of the ruin probability by taking into consideration such a stochastic environment quantified by a general stochastic process $\Theta = \{\theta_i, i \in \mathbb{N}\}$. Note that this stochastic environment poses systemic and systematic impacts on both the insurance and financial markets and hence introduces dependence among the insurance and financial risks across consecutive years, which makes the risk model more practical on the one hand but leads to some technical issues to the study on the other. In this paper, we assume that conditional on Θ the random pairs (X_i, Y_i), $i \in \mathbb{N}$, are independent but not identically distributed, each possessing a bivariate regular variation structure governed by two regularly varying tails $\overline{F}$ and $\overline{G}$ with negative indices $-\alpha$ and $-\beta$, respectively. As the main result, we obtain an asymptotic estimate

$$\psi(x;\infty) = (c+o(1))\, x^{-(\alpha\beta)/(\alpha+\beta)} l(x), \qquad x \to \infty,$$

where the constant c is strictly positive for most cases and $l(\cdot)$ is a slowly varying function; see relations (3.2) and (3.4) below. Not surprisingly, the representation for the constant c given on the right-hand side of (3.4) is rather intricate, but it captures the impacts of the two risks and of the stochastic environment, and is still explicit and readily computable.

The rest of this paper consists of four sections. Section 2 collects necessary preliminaries on the stochastic environment and the concept of bivariate regular variation, Section 3 presents our main result after introducing two assumptions, Section 4 constructs an example to illustrate the feasibility of the assumptions and the computability of the obtained formula, and finally Section 5 proves the main result after preparing several lemmas.

2 Preliminaries

2.1 Notational conventions

Throughout this paper, for $x, y \in \mathbb{R} = (-\infty, \infty)$, we write $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, and $x^+ = x \vee 0$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, we write $[\mathbf{x}, \mathbf{y}] = [x_1, y_1] \times [x_2, y_2]$, $[\mathbf{x}, \boldsymbol{\infty}) = [x_1, \infty) \times [x_2, \infty)$, and so on. For a set B, denote by B^c its complement, and by 1_B its indicator function, which is equal to 1 if B occurs and 0 otherwise. We sometimes abbreviate $\nu(B)$, the ν-measure of a set B, to νB as long as no confusion arises.

All limit relationships hold for $x \to \infty$ unless otherwise stated. For two positive functions $h_1(\cdot)$ and $h_2(\cdot)$, we write $h_1(\cdot) \sim h_2(\cdot)$ if $\lim h_1(\cdot)/h_2(\cdot) = 1$. For a non-decreasing function $h : \mathbb{R} \to \mathbb{R}$, denote by $h^{\leftarrow}$ and $h^{\rightarrow}$ its càglàd and càdlàg inverses, defined by, respectively,

$$h^{\leftarrow}(y) = \inf\{x \in \mathbb{R} : h(x) \geq y\} \qquad \text{and} \qquad h^{\rightarrow}(y) = \sup\{x \in \mathbb{R} : h(x) \leq y\}, \qquad y \in \mathbb{R},$$

where $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. Note that $h^{\leftarrow}$ and $h^{\rightarrow}$ are equal almost everywhere with respect to Lebesgue measure. It is easy to see that

$$h^{\leftarrow} \circ h(x) \leq x \qquad \text{and} \qquad h^{\rightarrow} \circ h(x) \geq x, \qquad x \in \mathbb{R}. \tag{2.1}$$

Moreover, if h is right continuous, then

$$h^{\leftarrow}(y) > x \Longleftrightarrow y > h(x). \tag{2.2}$$

See Section A.1.2 of McNeil et al. (2015) for related discussions.

For two random pairs $\boldsymbol{\xi} = (\xi_1, \xi_2)$ and $\boldsymbol{\eta} = (\eta_1, \eta_2)$, we say that $\boldsymbol{\xi}$ is stochastically dominated by $\boldsymbol{\eta}$, written as $\boldsymbol{\xi} \leq_{st} \boldsymbol{\eta}$, if

$$E\left[\phi(\boldsymbol{\xi})\right] \leq E\left[\phi(\boldsymbol{\eta})\right]$$

for every component-wise increasing function $\phi : \mathbb{R}^2 \to \mathbb{R}$ for which the expectations exist. See Section 17.A of Marshall et al. (2011) and references therein for this concept and related discussions. In particular, $\boldsymbol{\xi} \leq_{st} \boldsymbol{\eta}$ if and only if

$$P(\boldsymbol{\xi} \in \Delta) \leq P(\boldsymbol{\eta} \in \Delta)$$

holds for every increasing set $\Delta \subset \mathbb{R}^2$ (namely, a set whose indicator function is component-wise increasing). Restated in the univariate case, $\xi \leq_{st} \eta$ if and only if $P(\xi > x) \leq P(\eta > x)$ for every $x \in \mathbb{R}$. Clearly, if $\boldsymbol{\xi} \leq_{st} \boldsymbol{\eta}$, then $\xi_1 \leq_{st} \eta_1$ and $\xi_2 \leq_{st} \eta_2$.

2.2 The stochastic environment

As mentioned in Section 1, we assume the presence of an external stochastic environment that poses systemic and systematic impacts on both the insurance and financial markets. To quantify this stochastic environment, introduce a general discrete-time stochastic process $\Theta = \{\theta_i, i \in \mathbb{N}\}$ on $(\Omega, \mathcal{F}, P)$, which is not necessarily a Markov process, with each θ_i taking values in $\mathbb{R}$. For each $i \in \mathbb{N}$, let $\mathcal{F}_{\Theta_i}$ be the σ-field generated by $\Theta_i = \{\theta_1, \ldots, \theta_i\}$, representing the history of the stochastic environment up to time i. In this way, $\{\mathcal{F}_{\Theta_i}, i \in \mathbb{N}\}$ forms the natural filtration of the stochastic process Θ.

We say that the sequence of random pairs $\{(X_i, Y_i), i \in \mathbb{N}\}$ is modulated by the stochastic process Θ if

(i) conditional on Θ the random pairs (X_i, Y_i), $i \in \mathbb{N}$, are mutually independent, and

(ii) for each $i \in \mathbb{N}$, the conditional distributions of $(X_1, Y_1), \ldots, (X_i, Y_i)$ on Θ and on Θ_i are equal almost surely.

Verbally, item (ii) means that the insurance and financial risks incurred up to now are not affected by future developments of the stochastic environment, which sounds very natural.

2.3 Bivariate regular variation

We start with the concept of regular variation. A positive function h on $\mathbb{R}_+ = [0, \infty)$ is said to be regularly varying at ∞ with index $\alpha \in \mathbb{R}$, written as $h \in \mathrm{RV}_\alpha$, if

$$\lim_{x \to \infty} \frac{h(xy)}{h(x)} = y^\alpha, \qquad y > 0.$$

When $\alpha = 0$, this defines a slowly varying function at ∞. See Bingham et al. (1987) and Resnick (1987) for textbook treatments of regular variation. Distributions with tail $\overline{F} = 1 - F \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 0$ form a useful class for modeling heavy-tailed risks in insurance and finance.

The concept of multivariate regular variation, which was first introduced by de Haan and Resnick (1981) as a natural extension of univariate regular variation, provides an integrated framework for modeling extreme risks with both heavy tails and asymptotic (in)dependence. One often finds its use in insurance, finance, and risk management when extreme risks are concerned. Recent works in this direction include Embrechts et al. (2009), Mainik and Rüschendorf (2010), Fougeres and Mercadier (2012), Part IV of Rüschendorf (2013), and Tang and Yuan (2013), among many others. We briefly introduce here the two-dimensional version of this concept, and refer the reader to Chapter 6 of Resnick (2007) for more details.

A random pair (X, Y) taking values in $\mathbb{R}_+^2$ is said to follow a distribution with a bivariate regularly varying (BRV) tail if there exist two distribution functions F, G and a non-degenerate (i.e., not identically 0) limit measure ν such that

$$xP\left(\left(\frac{X}{(1/\overline{F})^{\leftarrow}(x)}, \frac{Y}{(1/\overline{G})^{\leftarrow}(x)}\right) \in \cdot\right) \xrightarrow{v} \nu(\cdot) \qquad \text{on } [\mathbf{0}, \boldsymbol{\infty}] \backslash \{\mathbf{0}\}. \tag{2.3}$$

For simplicity, for a distribution F, we abbreviate the càglàd inverse of $1/\overline{F}$ to χ_F, i.e.,

$$\chi_F(x) = \left(\frac{1}{\overline{F}}\right)^{\leftarrow}(x), \qquad x \in \mathbb{R}_+.$$

In (2.3) the notation $\xrightarrow{v}$ denotes vague convergence, meaning that the relation

$$\lim_{x\to\infty} xP\left(\left(\frac{X}{\chi_F(x)}, \frac{Y}{\chi_G(x)}\right) \in B\right) = \nu(B)$$

holds for every Borel set $B \subset [\mathbf{0}, \boldsymbol{\infty}]$ that is away from $\mathbf{0}$ and ν-continuous (namely, its boundary ∂B has ν-measure 0). Discussions on vague convergence can be found in, e.g., Section 3.3.5 of Resnick (2007). Necessarily, both $\overline{F}$ and $\overline{G}$ are regularly varying. Assume that $\overline{F} \in \mathrm{RV}_{-\alpha}$ and $\overline{G} \in \mathrm{RV}_{-\beta}$ for some $\alpha, \beta > 0$, for which case we write $(X, Y) \in \mathrm{BRV}_{-\alpha,-\beta}(\nu, \overline{F}, \overline{G})$. This is actually a non-standard version of bivariate regular variation, but it reduces to a standard one if $\overline{G}(x) \sim \lambda\overline{F}(x)$ for some $\lambda > 0$. By definition, for a random pair $(X, Y) \in \mathrm{BRV}_{-\alpha,-\beta}(\nu, \overline{F}, \overline{G})$, its marginal tails satisfy

$$\lim_{x\to\infty} \frac{\overline{F_X}(x)}{\overline{F}(x)} = \nu((1, 0), \boldsymbol{\infty}] \qquad \text{and} \qquad \lim_{x\to\infty} \frac{\overline{G_Y}(x)}{\overline{G}(x)} = \nu((0, 1), \boldsymbol{\infty}].$$

Thus, the tails $\overline{F}$ and $\overline{G}$ appearing in the definition (2.3) represent, though not necessarily identical to, the marginal tails of (X, Y).

3 The main result

Our main result is established under the following two assumptions. The first one requires that each pair of insurance and financial risks conditional on the stochastic environment Θ possesses a BRV structure.

Assumption 3.1 *For each $i \in \mathbb{N}$, almost surely the modulated pair $(X_i^+, Y_i)|\Theta_i$ possesses $\mathrm{BRV}_{-\alpha,-\beta}(\nu_{\Theta_i}, \overline{F}, \overline{G})$ for some non-degenerate measure ν_{Θ_i} and common representing tails $\overline{F} \in \mathrm{RV}_{-\alpha}$ and $\overline{G} \in \mathrm{RV}_{-\beta}$ for some $\alpha, \beta > 0$.*

The following second assumption requires that each pair of insurance and financial risks conditional on Θ is stochastically dominated by a random pair that possesses a BRV structure.

Assumption 3.2 *There exists a nonnegative random pair $(X^*, Y^*) \in \mathrm{BRV}_{-\alpha,-\beta}(\nu^*, \overline{F}, \overline{G})$ for some non-degenerate measure ν^* and some $\alpha, \beta > 0$, such that, for each $i \in \mathbb{N}$, almost surely,*

$$(X_i^+, Y_i)|\Theta_i \leq_{st} (X^*, Y^*). \tag{3.1}$$

Without loss of generality we can assume that $\nu^*(\mathbf{1}, \boldsymbol{\infty}] > 0$ because otherwise we can modify ν^* by adding mass to the first quadrant.

Recalling (2.3), for any $x \in \mathbb{R}$, define

$$y(x) = (\chi_F \chi_G)^{\leftarrow}(x) \qquad \text{and} \qquad z(x) = \frac{1}{y(x)}.$$

If $\overline{F} \in \mathrm{RV}_{-\alpha}$ and $\overline{G} \in \mathrm{RV}_{-\beta}$ for some $\alpha, \beta > 0$, then by Proposition 0.8(v) of Resnick (1987), it is easy to check that $\chi_F \in \mathrm{RV}_{1/\alpha}$ and $\chi_G \in \mathrm{RV}_{1/\beta}$ and hence that $z(\cdot) \in \mathrm{RV}_{-(\alpha\beta)/(\alpha+\beta)}$. This means that there exists some slowly varying function $l(\cdot)$ such that

$$z(x) \sim x^{-(\alpha\beta)/(\alpha+\beta)} l(x). \tag{3.2}$$

In particular, if $\overline{F} \in \mathrm{RV}_{-\alpha}$ and $\overline{G}(x) \sim \lambda \overline{F}(x)$ for some $\lambda > 0$, then it is not difficult to check that

$$z(x) \sim \lambda^{1/2} \overline{F}(x^{1/2}).$$

Now we are ready to state our main result:

Theorem 3.1 *Consider the discrete-time risk model introduced through (1.1) and (1.2) in which the sequence of random pairs* $\{(X_i, Y_i), i \in \mathbb{N}\}$ *is modulated by the stochastic process* Θ *as described in Subsection 2.2.*

(i) Under Assumptions 3.1 and 3.2, it holds for every $n \in \mathbb{N}$ *that*

$$\lim_{x\to\infty} \frac{\psi(x;n)}{z(x)} = \sum_{i=1}^{n} E\left[\nu_{\Theta_i}(A) \prod_{j=1}^{i-1} Y_j^{(\alpha\beta)/(\alpha+\beta)}\right], \tag{3.3}$$

where $A = \{(s,t) \in [\mathbf{0}, \boldsymbol{\infty}] : st > 1\}$.

(ii) If further $E\left[(Y^*)^{(\alpha\beta)/(\alpha+\beta)}\right] < 1$, *then relation (3.3) holds uniformly for all* $n \in \mathbb{N}$ *and, in particular,*

$$\lim_{x\to\infty} \frac{\psi(x;\infty)}{z(x)} = \sum_{i=1}^{\infty} E\left[\nu_{\Theta_i}(A) \prod_{j=1}^{i-1} Y_j^{(\alpha\beta)/(\alpha+\beta)}\right]. \tag{3.4}$$

The sum on the right-hand side of (3.4) is finite. This can be justified by relation (4.7) below and the almost sure inequalities $\nu_{\Theta_i}(A) \le \nu^*(A)$ and $E\left[Y_i^{(\alpha\beta)/(\alpha+\beta)} | \Theta_i\right] \le E\left[(Y^*)^{(\alpha\beta)/(\alpha+\beta)}\right]$ for each $i \in \mathbb{N}$, which are implied by Assumption 3.2. Moreover, this sum is strictly positive unless all limit measures ν_{Θ_i}, $i \in \mathbb{N}$, assign no mass to the first quadrant, namely, they yield asymptotic independence of the corresponding random pairs (X_i, Y_i), $i \in \mathbb{N}$.

The following is an immediate corollary of Theorem 3.1, showing that under the standard i.i.d. framework relations (3.3) and (3.4) can be greatly simplified:

Corollary 3.1 *Consider the discrete-time risk model in which* (X_i, Y_i), $i \in \mathbb{N}$, *form a sequence of i.i.d. copies of a generic pair* $(X, Y) \in \mathrm{BRV}_{-\alpha,-\beta}(\nu, \overline{F}, \overline{G})$ *for some non-degenerate measure* ν *and marginal tails* $\overline{F} \in \mathrm{RV}_{-\alpha}$ *and* $\overline{G} \in \mathrm{RV}_{-\beta}$ *for some* $\alpha, \beta > 0$.

(i) It holds for every $n \in \mathbb{N}$ that

$$\lim_{x\to\infty} \frac{\psi(x;n)}{z(x)} = \nu(A) \frac{1 - \left(E\left[Y^{(\alpha\beta)/(\alpha+\beta)}\right]\right)^n}{1 - E\left[Y^{(\alpha\beta)/(\alpha+\beta)}\right]},$$

where $A = \{(s,t) \in [\mathbf{0}, \boldsymbol{\infty}] : st > 1\}$.

(ii) If further $E\left[Y^{(\alpha\beta)/(\alpha+\beta)}\right] < 1$, then this relation holds uniformly for all $n \in \mathbb{N}$ and, in particular,

$$\lim_{x\to\infty} \frac{\psi(x;\infty)}{z(x)} = \frac{\nu(A)}{1 - E\left[Y^{(\alpha\beta)/(\alpha+\beta)}\right]}.$$

We would like to point out that, following the proof of Theorem 3.1, the same tail asymptotics as in Theorem 3.1 and Corollary 3.1 can be established for the randomly weighted sums S_n in (1.1) subject to a minor condition on the left tails of the insurance risks X_n.

Under the setting (1.4) and in a profound style, Buraczewski et al. (2016a, b) obtained sharp asymptotic estimates for $\psi(x; \lfloor \tau \ln x \rfloor)$ for some constant $\tau > 0$, where $\lfloor \cdot \rfloor$ denotes the floor function. As our study is carried out under the different setting (1.3), leaving alone our introduction of a stochastic environment, connection between their works and ours is not clear. Nevertheless, the uniformity achieved by us allows the horizon n to vary with x; for example, under Corollary 3.1(ii), it holds for any positive function $n(x) \to \infty$ that

$$\lim_{x\to\infty} \frac{\psi(x; \lfloor n(x) \rfloor)}{z(x)} = \frac{\nu(A)}{1 - E\left[Y^{(\alpha\beta)/(\alpha+\beta)}\right]}.$$

4 Discussions on Theorem 3.1

In this section, we construct an example to illustrate the feasibility of the two assumptions of Theorem 3.1 and the computability of the obtained formula (3.3).

We first show that a random pair with an Archimedean copula and regularly varying marginal tails follows the BRV structure (2.3). Recall an Archimedean copula of the form

$$C(u,v) = \varphi^{-1}\left(\varphi(u) + \varphi(v)\right), \qquad (u,v) \in [\mathbf{0}, \mathbf{1}], \tag{4.1}$$

where $\varphi : (0,1) \mapsto (0,\infty)$, called the generator, is a strictly decreasing and convex function with $\varphi(0+) = \infty$ and $\varphi(1-) = 0$, and the function φ^{-1} is the usual inverse of φ. The following lemma is a slight extension of Lemma 5.2 of Tang and Yuan (2013) to non-standard BRV and we omit its proof here:

Lemma 4.1 *Consider a nonnegative random pair (X,Y) with marginal distribution functions F_X and G_Y and possessing an Archimedean copula C of form (4.1). Assume that*

$$\lim_{s\downarrow 0} \frac{\varphi(1-st)}{\varphi(1-s)} = t^r, \qquad t > 0,$$

for some $1 \le r \le \infty$ (where t^∞ equals 0 when $0 < t < 1$, 1 when $t = 1$, and ∞ when $t > 1$), and that

$$\overline{F_X}(x) \sim c\overline{F}(x) \qquad \text{and} \qquad \overline{G_Y}(x) \sim d\overline{G}(x)$$

for some constants $c, d > 0$, and some representing tails $\overline{F} \in \mathrm{RV}_{-\alpha}$ and $\overline{G} \in \mathrm{RV}_{-\beta}$ for $\alpha, \beta > 0$. Then $(X, Y) \in \mathrm{BRV}_{-\alpha,-\beta}(\nu, \overline{F}, \overline{G})$ with ν defined by

$$\nu[\mathbf{0}, (s,t)]^c = \left(\left(cs^{-\alpha}\right)^r + \left(dt^{-\beta}\right)^r\right)^{1/r}, \qquad (s,t) > \mathbf{0}.$$

We now construct our example. As before, we use a discrete-time stochastic process $\Theta = \{\theta_i, i \in \mathbb{N}\}$ to quantify the external stochastic environment to which the insurance business is exposed. Suppose that Θ affects both the marginal distributions of and the dependence structure between the insurance and financial risks. For each $i \in \mathbb{N}$, denote by F_{Θ_i} and G_{Θ_i} the conditional distribution functions of X_i^+ and Y_i given Θ_i, respectively. Assume that there exist two representing tails $\overline{F} \in \mathrm{RV}_{-\alpha}$ and $\overline{G} \in \mathrm{RV}_{-\beta}$ for some $\alpha, \beta > 0$ and a sequence of nonnegative random pairs $\{(c_{\Theta_i}, d_{\Theta_i}), i \in \mathbb{N}\}$, adapted to $\{\mathcal{F}_{\Theta_i}, i \in \mathbb{N}\}$ and uniformly bounded from above, such that both limit relations

$$\lim_{x \to \infty} \frac{\overline{F_{\Theta_i}}(x)}{\overline{F}(x)} = c_{\Theta_i} \qquad \text{and} \qquad \lim_{x \to \infty} \frac{\overline{G_{\Theta_i}}(x)}{\overline{G}(x)} = d_{\Theta_i} \tag{4.2}$$

hold almost surely and uniformly for $i \in \mathbb{N}$. Taking the first one as an example, its precise meaning is that

$$\lim_{x \to \infty} \sup_{i \in \mathbb{N}} \left| \frac{\overline{F_{\Theta_i\{\omega\}}}(x)}{\overline{F}(x)} - c_{\Theta_i\{\omega\}} \right| = 0$$

holds for all $\omega \in \Omega \backslash O$ for some null set O. Furthermore, for each $i \in \mathbb{N}$, assume that the modulated pair $(X_i^+, Y_i)|\Theta_i$ almost surely possesses a bivariate Gumbel copula of the form

$$C_{\Theta_i}(u, v) = \exp\left\{-\left((-\log u)^{\gamma_{\Theta_i}} + (-\log v)^{\gamma_{\Theta_i}}\right)^{1/\gamma_{\Theta_i}}\right\}, \qquad (u, v) \in [\mathbf{0}, \mathbf{1}], \tag{4.3}$$

with an $\mathcal{F}_{\Theta_i}$-measurable random parameter $\gamma_{\Theta_i} \geq 1$.

For each $i \in \mathbb{N}$, by Lemma 4.1, almost surely the modulated pair $(X_i^+, Y_i)|\Theta_i$ possesses $\mathrm{BRV}_{-\alpha,-\beta}(\nu_{\Theta_i}, \overline{F}, \overline{G})$ with ν_{Θ_i} satisfying

$$\nu_{\Theta_i}[\mathbf{0}, (s,t)]^c = \left((c_{\Theta_i} s^{-\alpha})^{\gamma_{\Theta_i}} + (d_{\Theta_i} t^{-\beta})^{\gamma_{\Theta_i}}\right)^{1/\gamma_{\Theta_i}}, \qquad (s,t) > \mathbf{0}.$$

This verifies Assumption 3.1.

To check Assumption 3.2, we need to construct a dominating random pair $(X^*, Y^*) \in \mathrm{BRV}_{-\alpha,-\beta}(\nu^*, \overline{F}, \overline{G})$ such that the stochastic ordering (3.1) holds for each $i \in \mathbb{N}$. Let c and d be two positive constants greater than the uniform upper bounds of $\{c_{\Theta_i}, i \in \mathbb{N}\}$ and $\{d_{\Theta_i}, i \in \mathbb{N}\}$, respectively. By (4.2), there exists some large x_0 such that, almost surely and uniformly for all $i \in \mathbb{N}$ and $x \geq x_0$,

$$\overline{F_{\Theta_i}}(x) \leq c\overline{F}(x) \leq 1 \qquad \text{and} \qquad \overline{G_{\Theta_i}}(x) \leq d\overline{G}(x) \leq 1.$$

Introduce two distribution functions F_c and G_d on $\mathbb{R}_+$ with respective tails

$$\overline{F_c}(x) = 1_{(x<x_0)} + c\overline{F}(x)1_{(x \geq x_0)} \qquad \text{and} \qquad \overline{G_d}(x) = 1_{(x<x_0)} + d\overline{G}(x)1_{(x \geq x_0)}. \tag{4.4}$$

Clearly, it holds almost surely that

$$\overline{F_{\Theta_i}}(x) \leq \overline{F_c}(x) \qquad \text{and} \qquad \overline{G_{\Theta_i}}(x) \leq \overline{G_d}(x), \qquad i \in \mathbb{N}, x \in \mathbb{R}_+. \tag{4.5}$$

We are satisfied with this general description of the two distribution functions F_c and G_d, but would like to point out that the construction can be made more precise if more information about the conditional distribution functions F_{Θ_i} and G_{Θ_i}, $i \in \mathbb{N}$, is available.

We separate the following two cases of the dependence structure:

(i) Consider a special but still important case that the dependence structure of each pair (X_i^+, Y_i) is not affected by Θ_i. In this case, the copula $C_{\Theta_i}(u, v)$ in (4.3) is reduced to

$$C(u, v) = \exp\left\{ -\left((-\log u)^\gamma + (-\log v)^\gamma\right)^{1/\gamma} \right\}, \qquad (u, v) \in [\mathbf{0}, \mathbf{1}], \tag{4.6}$$

for some deterministic $\gamma \geq 1$. We construct the dominating random pair (X^*, Y^*) to be

$$X^* = F_c^{\leftarrow}(U) \qquad \text{and} \qquad Y^* = G_d^{\leftarrow}(V),$$

where (U, V) is a uniform pair distributed by the copula $C(u, v)$ in (4.6).

(ii) Consider the general case as described by (4.3) with a random parameter $\gamma_{\Theta_i} \geq 1$. In this case, choose $r \in [1, 2]$ to be the uniform upper bound of $\{2^{1/\gamma_{\Theta_i}}, i \in \mathbb{N}\}$ and then construct the two dominating random variables X^* and Y^* to be

$$X^* = F_c^{\leftarrow}(U_r) \qquad \text{and} \qquad Y^* = G_d^{\leftarrow}(U_r),$$

where U_r is a random variable distributed by $P(U_r \leq u) = u^r$ on $[0, 1]$.

The verification of the BRV structure of (X^*, Y^*) for case (i) is given by applying Lemma 4.1. The verification for case (ii) is also easy. Actually, for any $(s, t) > \mathbf{0}$, it follows that

$$\begin{aligned} xP\left(\left(\frac{X^*}{\chi_F(x)}, \frac{Y^*}{\chi_G(x)}\right) \in [\mathbf{0}, (s, t)]^c\right) &= xP\left(U_r > F_c\left(s\chi_F(x)\right) \wedge G_d\left(t\chi_G(x)\right)\right) \\ &\sim rx\left(\overline{F_c}(s\chi_F(x)) \vee \overline{G_d}(t\chi_G(x))\right) \\ &\to r\left((cs^{-\alpha}) \vee (dt^{-\beta})\right), \end{aligned}$$

where the first step can be explained by (2.2) and the last step by (4.4) and the assumed regular variation of $\overline{F}$ and $\overline{G}$. Thus, (X^*, Y^*) follows $\mathrm{BRV}_{-\alpha,-\beta}(\nu^*, \overline{F}, \overline{G})$ with ν^* defined by $\nu^*[\mathbf{0}, (s, t)]^c = r\left((cs^{-\alpha}) \vee (dt^{-\beta})\right)$ for $(s, t) > \mathbf{0}$.

Next we verify that (X^*, Y^*) can indeed serve as the dominating pair described by (3.1). For any increasing set Δ, define

$$\begin{aligned} \tilde{\Delta}_i &= \bigcup_{(x,y)\in\Delta} [F_{\Theta_i}(x), 1] \times [G_{\Theta_i}(y), 1], \qquad i \in \mathbb{N}, \\ \hat{\Delta} &= \bigcup_{(x,y)\in\Delta} [F_c(x), 1] \times [G_d(y), 1], \end{aligned}$$

all being increasing sets restricted to $[\mathbf{0}, \mathbf{1}]$. Moreover, by (4.5), $\tilde{\Delta}_i \subset \hat{\Delta}$ for each $i \in \mathbb{N}$.

For case (i), we have

$$\begin{aligned}
P\left((X_i^+, Y_i) \in \Delta | \Theta_i\right) &\leq P\left((F_{\Theta_i}(X_i^+), G_{\Theta_i}(Y_i)) \in \tilde{\Delta}_i \middle| \Theta_i\right) \\
&\leq P\left((U, V) \in \hat{\Delta}\right) \\
&\leq P\left((F_c^{\rightarrow}(U), G_d^{\rightarrow}(V)) \in \Delta\right) \\
&= P\left((X^*, Y^*) \in \Delta\right),
\end{aligned}$$

where the first step is due to the increase of Δ in $\mathbb{R}^2$, the second step due to $\tilde{\Delta}_i \subset \hat{\Delta}$ and the fact that both $(F_{\Theta_i}(X_i^+), G_{\Theta_i}(Y_i))$ and (U, V) correspond to the same copula (4.6), the third step due to the second inequality in (2.1), and the last step due to the almost sure equalities $F_c^{\leftarrow}(U) = F_c^{\rightarrow}(U)$ and $G_d^{\leftarrow}(V) = G_d^{\rightarrow}(V)$.

For case (ii), similarly,

$$\begin{aligned}
P\left((X_i^+, Y_i) \in \Delta \middle| \Theta_i\right) &\leq P\left((F_{\Theta_i}(X_i^+), G_{\Theta_i}(Y_i)) \in \tilde{\Delta}_i \middle| \Theta_i\right) \\
&\leq P\left((F_{\Theta_i}(X_i^+) \vee G_{\Theta_i}(Y_i), F_{\Theta_i}(X_i^+) \vee G_{\Theta_i}(Y_i)) \in \hat{\Delta} \middle| \Theta_i\right) \\
&\leq P\left(F_{\Theta_i}(X_i^+) \vee G_{\Theta_i}(Y_i) \geq u_0 \middle| \Theta_i\right),
\end{aligned}$$

where the second step is due to $\tilde{\Delta}_i \subset \hat{\Delta}$ and the increase of the set $\hat{\Delta}$ and in the last step $u_0 = \inf\{u \in [0,1] : (u,u) \in \hat{\Delta}\}$. Thus,

$$\begin{aligned}
P\left((X_i^+, Y_i) \in \Delta \middle| \Theta_i\right) &\leq 1 - C_{\Theta_i}(u_0, u_0) \\
&= 1 - u_0^{2^{1/\gamma_{\Theta_i}}} \\
&\leq 1 - u_0^r \\
&= P((U_r, U_r) \in \hat{\Delta}) \\
&\leq P\left((F_c^{\rightarrow}(U_r), G_d^{\rightarrow}(U_r)) \in \Delta\right) \\
&= P\left((X^*, Y^*) \in \Delta\right).
\end{aligned}$$

This verifies Assumption 3.2 for both cases.

To make Theorem 3.1(ii) applicable, we also need to impose the condition that

$$E\left[(Y^*)^{(\alpha\beta)/(\alpha+\beta)}\right] = \int_0^\infty (1 - G_d^r(y)) dy^{(\alpha\beta)/(\alpha+\beta)} < 1,$$

where $r = 1$ for case (i). This may represent a restriction on how much the distributions of the financial risks can be affected by the stochastic environment Θ. As an illustration, assume that each Y_i is independent of Θ (but the copula of each pair (X_i, Y_i) can still be affected by Θ), and that $Y_i, i \in \mathbb{N}$, are stochastically dominated by some nonnegative random variable Y distributed by G. Then the distribution G_d in (4.4) can be simply specified to G. In this way, the condition above reduces to $E\left[(Y^*)^{(\alpha\beta)/(\alpha+\beta)}\right] = rE\left[Y^{(\alpha\beta)/(\alpha+\beta)} G^{r-1}(Y)\right] < 1$. We remark that the assumption of independence between $\{Y_i, i \in \mathbb{N}\}$ and Θ would be reasonable if the stochastic environment does not count information of the financial market and if natural events do not pose systemic risk to the financial market.

Finally, we illustrate the computability of the representation obtained in (3.3). Rewrite each summand on the right-hand side of (3.3) as

$$\begin{aligned} E\left[\nu_{\Theta_i}(A)\prod_{j=1}^{i-1}Y_j^{(\alpha\beta)/(\alpha+\beta)}\right] &= E\left[\nu_{\Theta_i}(A)E\left[\prod_{j=1}^{i-1}Y_j^{(\alpha\beta)/(\alpha+\beta)}\middle|\Theta_i\right]\right] \\ &= E\left[\nu_{\Theta_i}(A)\prod_{j=1}^{i-1}E\left[Y_j^{(\alpha\beta)/(\alpha+\beta)}\middle|\Theta_j\right]\right]. \end{aligned} \tag{4.7}$$

If, as assumed above, each Y_i is independent of Θ (but the copula of each pair (X_i, Y_i) can still be affected by Θ), then the expectation above is decomposed into the product of individual expectations. If we restrict the study to a Markov-modulated risk model, then the formula in (4.7) can be further simplified in an obvious way. Moreover, $\nu_{\Theta_i}(A)$ can be calculated as

$$\begin{aligned} \nu_{\Theta_i}(A) &= -\int_A d\nu_{\Theta_i}[\mathbf{0},(s,t)]^c \\ &= \alpha\beta(\gamma_{\Theta_i}-1)(c_{\Theta_i}d_{\Theta_i})^{\gamma_{\Theta_i}} \\ &\quad\times\iint_{st>1}\left((c_{\Theta_i}s^{-\alpha})^{\gamma_{\Theta_i}}+(d_{\Theta_i}t^{-\beta})^{\gamma_{\Theta_i}}\right)^{1/\gamma_{\Theta_i}-2}s^{-\alpha\gamma_{\Theta_i}-1}t^{-\beta\gamma_{\Theta_i}-1}dsdt, \end{aligned}$$

which is explicit and computable, though a bit tedious.

5 Proof of Theorem 3.1

5.1 Lemmas

The first lemma below shows a certain homogeneity property of the limit measure ν in the BRV structure (2.3).

Lemma 5.1 *Let a nonnegative random pair (X, Y) possess $\mathrm{BRV}_{-\alpha,-\beta}(\nu, \overline{F}, \overline{G})$ for some non-degenerate measure ν and some representing tails $\overline{F} \in \mathrm{RV}_{-\alpha}$ and $\overline{G} \in \mathrm{RV}_{-\beta}$ for $\alpha, \beta > 0$. Then for any $\lambda > 0$ and any Borel set $B \subset [\mathbf{0}, \boldsymbol{\infty}]\backslash\{\mathbf{0}\}$, we have*

$$\nu(B_\lambda) = \lambda^{-1}\nu(B), \tag{5.1}$$

where $B_\lambda = \{(\lambda^{1/\alpha}s, \lambda^{1/\beta}t) : (s,t) \in B\}$.

Proof. For any $\lambda > 0$ and any pair $(s,t) \in [\mathbf{0}, \boldsymbol{\infty}]\backslash\{\mathbf{0}\}$, write

$$\begin{aligned} \nu[\mathbf{0},(\lambda^{1/\alpha}s,\lambda^{1/\beta}t)]^c &= \lim_{x\to\infty} xP\left(\left(\frac{X}{\chi_F(x)},\frac{Y}{\chi_G(x)}\right)\in[\mathbf{0},(\lambda^{1/\alpha}s,\lambda^{1/\beta}t)]^c\right) \\ &= \lim_{x\to\infty} xP\left(\left(\frac{X}{\lambda^{1/\alpha}\chi_F(x)},\frac{Y}{\lambda^{1/\beta}\chi_G(x)}\right)\in[\mathbf{0},(s,t)]^c\right). \end{aligned}$$

For any $0 < \varepsilon < 1$, by $\chi_F \in \mathrm{RV}_{1/\alpha}$ and $\chi_G \in \mathrm{RV}_{1/\beta}$, the right-hand side above is not greater than

$$\lim_{x\to\infty} xP\left(\left(\frac{X}{\chi_F((1-\varepsilon)\lambda x)}, \frac{Y}{\chi_G((1-\varepsilon)\lambda x)}\right) \in [\mathbf{0}, (s,t)]^c\right) = ((1-\varepsilon)\lambda)^{-1}\nu[\mathbf{0}, (s,t)]^c.$$

The other inequality can be established in the same way, and we obtain

$$((1+\varepsilon)\lambda)^{-1}\nu[\mathbf{0}, (s,t)]^c \leq \nu[\mathbf{0}, \left(\lambda^{1/\alpha}s, \lambda^{1/\beta}t\right)]^c \leq ((1-\varepsilon)\lambda)^{-1}\nu[\mathbf{0}, (s,t)]^c.$$

This, upon $\varepsilon \downarrow 0$, shows that relation (5.1) holds for any set B of the form $[\mathbf{0}, (s,t)]^c$. Then following an argument using Dynkin's $\pi - \lambda$ theorem, relation (5.1) holds for any Borel set $B \subset [\mathbf{0}, \boldsymbol{\infty}]\backslash\{\mathbf{0}\}$; see Page 178 of Resnick (2007). ■

The second lemma below rewrites Lemma A.1 of Shi et al. (2017) in terms of non-standard BRV, which is useful for verifying that ν assigns no mass to the boundary of a set when applying vague convergence. This lemma can be proven by following the proof of Lemma A.1 of Shi et al. (2017) and using Lemma 5.1.

Lemma 5.2 *Consider the BRV structure (2.3) with a limit measure ν and indices $\alpha, \beta > 0$. For a Borel set $B \subset [\mathbf{0}, \boldsymbol{\infty}]\backslash\{\mathbf{0}\}$, if $B_\lambda \cap B = \emptyset$ for every $\lambda > 1$ then $\nu(B) = 0$.*

The well-known Breiman's theorem states that for two independent nonnegative random variables ξ and η, if ξ has a tail $\overline{F_\xi} \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 0$, and $E\left[\eta^\beta\right] < \infty$ for some $\beta > \alpha$, then

$$\lim_{x\to\infty} \frac{P\left(\xi\eta > x\right)}{\overline{F_\xi}(x)} = E\left[\eta^\alpha\right].$$

See Cline and Samorodnitsky (1994), who attributed the result to Breiman (1965); see also Denisov and Zwart (2007) and Fougeres and Mercadier (2012) for some enhanced versions or extensions of Breiman's theorem. Now we prepare another enhanced version of Breiman's theorem, which will be used in the proof of Theorem 3.1 but may be interesting in its own right.

Let ξ be a nonnegative random variable with distribution function F_ξ. Given a nonnegative random variable η^* independent of ξ, denote by $\mathcal{D}(\eta^*) = \{\eta\}$ a family of all nonnegative random variables η that are independent of ξ and stochastically dominated by η^*, namely, $\eta \leq_{st} \eta^*$ for all $\eta \in \mathcal{D}(\eta^*)$.

Lemma 5.3 *If there is some distribution function F with tail $\overline{F} \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 0$ such that*

$$\lim_{x\to\infty} \frac{\overline{F_\xi}(x)}{\overline{F}(x)} = r$$

for some $r \geq 0$, and $E\left[(\eta^*)^\beta\right] < \infty$ for some $\beta > \alpha$, then it holds uniformly for all $\eta \in \mathcal{D}(\eta^*)$ that

$$\lim_{x\to\infty} \frac{P\left(\xi\eta > x\right)}{\overline{F}(x)} = rE\left[\eta^\alpha\right]. \tag{5.2}$$

Proof. For $r = 0$, choose some small $\varepsilon > 0$ and large $x_1 > 0$ such that $\overline{F_\xi}(x) \leq \varepsilon \overline{F}(x) \leq 1$ for all $x \geq x_1$. Let ξ^* be a random variable, independent of $\mathcal{D}(\eta^*)$ and with tail

$$\overline{F^*}(x) = 1_{(x<x_1)} + \varepsilon \overline{F}(x) 1_{(x \geq x_1)}.$$

Then $\overline{F^*} \in \mathrm{RV}_{-\alpha}$ and $\xi \leq_{st} \xi^*$. Hence, by Breiman's theorem,

$$\frac{P(\xi\eta > x)}{\overline{F}(x)} \leq \frac{\overline{F^*}(x)}{\overline{F}(x)} \cdot \frac{P(\xi^*\eta^* > x)}{\overline{F^*}(x)} \to \varepsilon E\left[(\eta^*)^\alpha\right],$$

which, by the arbitrariness of $\varepsilon > 0$, results in (5.2) with $r = 0$.

For $r > 0$, we have $\overline{F_\xi} \in \mathrm{RV}_{-\alpha}$, and in this case it suffices to prove that, uniformly for all $\eta \in \mathcal{D}(\eta^*)$,

$$\lim_{x\to\infty} \frac{P\left(\xi\eta > x\right)}{\overline{F_\xi}(x)} = E\left[\eta^\alpha\right]. \tag{5.3}$$

For arbitrarily chosen $c > 1$ and $0 < \varepsilon < (\beta - \alpha) \wedge \alpha$, by Potter's bounds there exists some $x_2 > 0$ such that, uniformly for all $x, y \geq x_2$,

$$\frac{1}{c}\left(\left(\frac{y}{x}\right)^{-\alpha+\varepsilon} \wedge \left(\frac{y}{x}\right)^{-\alpha-\varepsilon}\right) \leq \frac{\overline{F_\xi}(y)}{\overline{F_\xi}(x)} \leq c\left(\left(\frac{y}{x}\right)^{-\alpha+\varepsilon} \vee \left(\frac{y}{x}\right)^{-\alpha-\varepsilon}\right).$$

For the upper bound, we derive

$$P\left(\xi\eta > x\right) \leq P\left(\xi\eta > x, \eta \leq \frac{x}{x_2}\right) + P\left(\eta > \frac{x}{x_2}\right) = I_1 + I_2.$$

For I_1, with $0 < \delta < 1$ and $M > 1$ arbitrarily fixed, we have

$$\begin{aligned}
\frac{I_1}{\overline{F_\xi}(x)} &= \frac{1}{\overline{F_\xi}(x)} \int_0^{x/x_2} \overline{F_\xi}(x/y) P(\eta \in dy) \\
&\leq c \int_0^{x/x_2} \left(y^{\alpha-\varepsilon} \vee y^{\alpha+\varepsilon}\right) P(\eta \in dy) \\
&\leq c\left(\int_0^\delta + \int_\delta^M + \int_M^\infty\right) \left(y^{\alpha-\varepsilon} \vee y^{\alpha+\varepsilon}\right) P(\eta \in dy) \\
&\leq c\left(\delta^{\alpha-\varepsilon} + \left(\delta^{-\varepsilon} \vee M^\varepsilon\right) \int_\delta^M y^\alpha P(\eta \in dy) + \int_M^\infty y^{\alpha+\varepsilon} P(\eta \in dy)\right) \\
&\leq c\left(\delta^{\alpha-\varepsilon} + \left(\delta^{-\varepsilon} \vee M^\varepsilon\right) E\left[\eta^\alpha\right] + E\left[(\eta^*)^{\alpha+\varepsilon} 1_{(\eta^* > M)}\right]\right),
\end{aligned}$$

where in the second step we applied Potter's upper bound and in the last step the replacement in the last term is because $\eta \leq_{st} \eta^*$. By letting $\varepsilon \downarrow 0$, $\delta \downarrow 0$, $M \uparrow \infty$, and $c \downarrow 1$, in turn, the right-hand side above converges to $E\left[\eta^\alpha\right]$ uniformly for $\eta \in \mathcal{D}(\eta^*)$. For I_2, we have

$$\frac{I_2}{\overline{F_\xi}(x)} \leq \frac{P\left(\eta^* > x/x_2\right)}{\overline{F_\xi}(x/x_2)} \cdot \frac{\overline{F_\xi}(x/x_2)}{\overline{F_\xi}(x)} \to 0$$

because $\overline{F_\xi} \in \mathrm{RV}_{-\alpha}$ and $E\left[(\eta^*)^\beta\right] < \infty$ for some $\beta > \alpha$. Simply combining these two estimates together yields that, uniformly for all $\eta \in \mathcal{D}(\eta^*)$,

$$\limsup_{x\to\infty} \frac{P(\xi\eta > x)}{\overline{F_\xi}(x)} \le E[\eta^\alpha],$$

giving the upper-bound version of (5.3). Noting that $P(\xi\eta > x) \ge I_1$, the corresponding lower-bound version of (5.3) can be derived similarly and we omit it here. ■

5.2 Proof of Theorem 3.1(i)

We start with a claim that, for each $i \in \mathbb{N}$ and almost surely,

$$\lim_{x\to\infty} \frac{P(X_i^+ Y_i > x|\Theta_i)}{z(x)} = \nu_{\Theta_i}(A). \tag{5.4}$$

For this purpose, first observe that

$$x \ge (\chi_F \cdot \chi_G)(y(x)) \sim x, \tag{5.5}$$

where the first step is due to the definition of $y(x)$ and the second step due to the regular variation of χ_F and χ_G. By (5.5) and Assumption 3.1 we have, almost surely,

$$\begin{aligned} P\left(X_i^+ Y_i > x|\Theta_i\right) &\le P\left(X_i^+ Y_i > (\chi_F \cdot \chi_G)(y(x)) \middle| \Theta_i\right) \\ &= P\left(\frac{X_i^+}{\chi_F(y(x))} \cdot \frac{Y_i}{\chi_G(y(x))} > 1 \middle| \Theta_i\right) \\ &\sim \nu_{\Theta_i}(A) z(x), \end{aligned}$$

where $\nu_{\Theta_i}(\partial A) = 0$ is verified by Lemma 5.2. On the other hand, by (5.5) again, it holds for any $\varepsilon > 0$ and all large x that $x \le (\chi_F \cdot \chi_G)(y((1+\varepsilon)x))$. Thus, almost surely,

$$\begin{aligned} P\left(X_i^+ Y_i > x \middle| \Theta_i\right) &\ge P\left(X_i^+ Y_i > (\chi_F \cdot \chi_G)(y((1+\varepsilon)x)) \middle| \Theta_i\right) \\ &= P\left(\frac{X_i^+}{\chi_F(y((1+\varepsilon)x))} \cdot \frac{Y_i}{\chi_G(y((1+\varepsilon)x))} > 1 \middle| \Theta_i\right) \\ &\sim \nu_{\Theta_i}(A)\, z((1+\varepsilon)x) \\ &\sim (1+\varepsilon)^{-(\alpha\beta)/(\alpha+\beta)} \nu_{\Theta_i}(A) z(x), \end{aligned}$$

where the last step is due to the regular variation of $z(\cdot)$. The two estimates above lead to (5.4), as claimed.

Since $(X^*, Y^*) \in \mathrm{BRV}_{-\alpha,-\beta}(\nu^*, \overline{F}, \overline{G})$ with $\nu^*(\mathbf{1}, \boldsymbol{\infty}] > 0$, similarly to (5.4),

$$P(X^* Y^* > x) \sim \nu^*(A) z(x). \tag{5.6}$$

Thus, both $X^* Y^*$ and Y^* have regularly varying tails with indices $-(\alpha\beta)/(\alpha+\beta)$ and $-\beta$, respectively, and therefore Y^* has a finite moment of order higher than $(\alpha\beta)/(\alpha+\beta)$.

Let (X_i^*, Y_i^*), $i \in \mathbb{N}$, be i.i.d. copies of the random pair (X^*, Y^*) and independent of all other sources of randomness. Then by Lemma 5.3 and relation (5.4), for each $1 \le i \le n$, it holds that, almost surely,

$$\begin{aligned} \lim_{x\to\infty} \frac{1}{z(x)} P\left(X_i^+ \prod_{j=1}^{i} Y_j > x \middle| \Theta_n \right) &= \lim_{x\to\infty} \frac{1}{z(x)} P\left((X_i^+ Y_i) \prod_{j=1}^{i-1} Y_j > x \middle| \Theta_i \right) \\ &= \lim_{x\to\infty} \frac{P\left(X_i^+ Y_i > x \middle| \Theta_i \right)}{z(x)} E\left[\prod_{j=1}^{i-1} Y_j^{(\alpha\beta)/(\alpha+\beta)} \middle| \Theta_{i-1} \right] \\ &= \nu_{\Theta_i}(A) \prod_{j=1}^{i-1} E\left[Y_j^{(\alpha\beta)/(\alpha+\beta)} \middle| \Theta_j \right]. \end{aligned} \tag{5.7}$$

In the derivation above, the applicability of Lemma 5.3 is justified by the fact that, given Θ_i, the product $\prod_{j=1}^{i-1} Y_j$ is conditionally independent of $X_i^+ Y_i$ and stochastically dominated by $\prod_{j=1}^{i-1} Y_j^*$ that has a finite moment of order higher than $(\alpha\beta)/(\alpha+\beta)$. For each $1 \le i < k \le n$, it is easy to see that, almost surely,

$$\begin{aligned} P\left(X_k^+ \prod_{j=1}^{k} Y_j > x, X_i^+ \prod_{j=1}^{i} Y_j > x \middle| \Theta_n \right) &\le P\left(X_k^* \prod_{j=1}^{k} Y_j^* > x, X_i^* \prod_{j=1}^{i} Y_j^* > x \right) \\ &= o(1) P(X_k^* Y_k^* > x) \\ &= o(z(x)). \end{aligned} \tag{5.8}$$

Actually, in the first step of (5.8) we applied Assumption 3.2, whose applicability can be justified by the facts that the intersection of $x_k \prod_{j=1}^{k} y_j > x$ and $x_i \prod_{j=1}^{i} y_j > x$ increases in each of (x_1, y_1), ..., (x_k, y_k), and that the random pairs (X_1, Y_1), ..., (X_k, Y_k) are conditionally independent given Θ_n. Moreover, in the second step of (5.8), we applied Lemma 7 of Tang and Yuan (2014).

After these preparations, we start to derive upper and lower bounds for the finite-time ruin probability $\psi(x; n)$. First consider the upper bound. For any $0 < \varepsilon < 1$, according to whether or not there is a term $X_i^+ \prod_{j=1}^{i} Y_j$ larger than $(1-\varepsilon)x$, we do the split

$$P\left(\sum_{i=1}^{n} X_i^+ \prod_{j=1}^{i} Y_j > x \middle| \Theta_n \right) = I_1 + I_2.$$

For I_1, by (5.7) and $z(x) \in \mathrm{RV}_{-(\alpha\beta)/(\alpha+\beta)}$, we have, almost surely,

$$\begin{aligned} \frac{I_1}{z(x)} &= \frac{1}{z(x)} P\left(\sum_{i=1}^{n} X_i^+ \prod_{j=1}^{i} Y_j > x, \bigvee_{i=1}^{n} X_i^+ \prod_{j=1}^{i} Y_j > (1-\varepsilon)x \middle| \Theta_n \right) \\ &\le \frac{z((1-\varepsilon)x)}{z(x)} \frac{1}{z((1-\varepsilon)x)} \sum_{i=1}^{n} P\left(X_i^+ \prod_{j=1}^{i} Y_j > (1-\varepsilon)x \middle| \Theta_n \right) \\ &\to (1-\varepsilon)^{-(\alpha\beta)/(\alpha+\beta)} \sum_{i=1}^{n} \nu_{\Theta_i}(A) \prod_{j=1}^{i-1} E\left[Y_j^{(\alpha\beta)/(\alpha+\beta)} \middle| \Theta_j \right]. \end{aligned}$$

For I_2, we derive

$$\begin{aligned}
I_2 &= P\left(\sum_{i=1}^n X_i^+ \prod_{j=1}^i Y_j > x, \bigvee_{i=1}^n X_i^+ \prod_{j=1}^i Y_j \le (1-\varepsilon)x \,\middle|\, \Theta_n\right) \\
&= P\left(\sum_{i=1}^n X_i^+ \prod_{j=1}^i Y_j > x, \bigvee_{i=1}^n X_i^+ \prod_{j=1}^i Y_j \le (1-\varepsilon)x, \bigvee_{k=1}^n X_k^+ \prod_{j=1}^k Y_j > \frac{x}{n} \,\middle|\, \Theta_n\right) \\
&\le \sum_{k=1}^n P\left(\sum_{i=1, i\neq k}^n X_i^+ \prod_{j=1}^i Y_j > \varepsilon x, X_k^+ \prod_{j=1}^k Y_j > \frac{x}{n} \,\middle|\, \Theta_n\right) \\
&\le \sum_{k=1}^n \sum_{i=1, i\neq k}^n P\left(X_i^+ \prod_{j=1}^i Y_j > \frac{\varepsilon x}{n-1}, X_k^+ \prod_{j=1}^k Y_j > \frac{x}{n} \,\middle|\, \Theta_n\right).
\end{aligned}$$

Hence, by (5.8) we have, almost surely,

$$\lim_{x\to\infty} \frac{I_2}{z(x)} = 0.$$

Putting these together, upon $\varepsilon \downarrow 0$, gives that, almost surely,

$$\limsup_{x\to\infty} \frac{1}{z(x)} P\left(\sum_{i=1}^n X_i^+ \prod_{j=1}^i Y_j > x \,\middle|\, \Theta_n\right) \le \sum_{i=1}^n \nu_{\Theta_i}(A) \prod_{j=1}^{i-1} E\left[Y_j^{(\alpha\beta)/(\alpha+\beta)} \,\middle|\, \Theta_j\right].$$

Therefore, applying the dominated convergence theorem we obtain

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\psi(x;n)}{z(x)} &\le \limsup_{x\to\infty} E\left[\frac{1}{z(x)} P\left(\sum_{i=1}^n X_i^+ \prod_{j=1}^i Y_j > x \,\middle|\, \Theta_n\right)\right] \\
&\le E\left[\sum_{i=1}^n \nu_{\Theta_i}(A) \prod_{j=1}^{i-1} E\left[Y_j^{(\alpha\beta)/(\alpha+\beta)} \,\middle|\, \Theta_j\right]\right] \\
&= \sum_{i=1}^n E\left[\nu_{\Theta_i}(A) \prod_{j=1}^{i-1} Y_j^{(\alpha\beta)/(\alpha+\beta)}\right],
\end{aligned}$$

which establishes the upper bound for $\psi(x;n)$. In the derivation above, the applicability of the dominated convergence theorem can be justified as follows. As done in (5.8), by Assumption 3.2 we have, almost surely,

$$\begin{aligned}
\frac{1}{z(x)} P\left(\sum_{i=1}^n X_i^+ \prod_{j=1}^i Y_j > x \,\middle|\, \Theta_n\right) &\le \frac{1}{z(x)} P\left(\sum_{i=1}^n X_i^* \prod_{j=1}^i Y_j^* > x\right) \\
&\sim \frac{1}{z(x)} \sum_{i=1}^n P\left(X_i^* \prod_{j=1}^i Y_j^* > x\right) \\
&\to \nu^*(A) \sum_{i=1}^n \left(E\left[(Y^*)^{(\alpha\beta)/(\alpha+\beta)}\right]\right)^{i-1},
\end{aligned}$$

where in the second step we used Theorem 3.1 of Chen and Yuen (2009) by noticing the fact that $X_i^* \prod_{j=1}^i Y_j^*,\ i \in \mathbb{N}$, are pairwise quasi-asymptotically independent as shown in (5.8), and in the last step we used Breiman's theorem and (5.6).

For the lower bound, introduce the time of ruin

$$T(x) = \inf\left\{k \in \mathbb{N} : \sum_{i=1}^{k} X_i \prod_{j=1}^{i} Y_j > x\right\},$$

so that

$$\psi(x;n) = \sum_{l=1}^{n} P(T(x) = l). \tag{5.9}$$

For any $\varepsilon > 0$ and $1 \le i \le n$,

$$\begin{aligned}
& P\left(T(x) = l\right) \\
= \ & P\left(\bigvee_{k=1}^{l-1} \sum_{i=1}^{k} X_i \prod_{j=1}^{i} Y_j \le x, \sum_{i=1}^{l} X_i \prod_{j=1}^{i} Y_j > x\right) \\
\ge \ & P\left(\bigvee_{k=1}^{l-1} \sum_{i=1}^{k} X_i \prod_{j=1}^{i} Y_j \le x, \sum_{i=1}^{l-1} X_i \prod_{j=1}^{i} Y_j > -\varepsilon x, X_l \prod_{j=1}^{l} Y_j > (1+\varepsilon)x\right) \\
\ge \ & P\left(X_l^+ \prod_{j=1}^{l} Y_j > (1+\varepsilon)x\right) \\
& -P\left(X_l^+ \prod_{j=1}^{l} Y_j > (1+\varepsilon)x, \sum_{i=1}^{l-1} X_i \prod_{j=1}^{i} Y_j \le -\varepsilon x\right) \\
& -P\left(X_l^+ \prod_{j=1}^{l} Y_j > (1+\varepsilon)x, \bigvee_{k=1}^{l-1} \sum_{i=1}^{k} X_i \prod_{j=1}^{i} Y_j > x\right) \\
= \ & J_1 - J_2 - J_3.
\end{aligned}$$

Applying the dominated convergence theorem and relation (5.7) we obtain

$$\begin{aligned}
\lim_{x\to\infty} \frac{J_1}{z(x)} &= \lim_{x\to\infty} \frac{z((1+\varepsilon)x)}{z(x)} \frac{1}{z((1+\varepsilon)x)} E\left[P\left(X_l^+ \prod_{j=1}^{l} Y_j > (1+\varepsilon)x \,\middle|\, \Theta_l\right)\right] \\
&= (1+\varepsilon)^{-(\alpha\beta)/(\alpha+\beta)} E\left[\nu_{\Theta_l}(A) \prod_{j=1}^{l-1} Y_j^{(\alpha\beta)/(\alpha+\beta)}\right].
\end{aligned}$$

For J_2, as done in (5.8), by Assumption 3.2 we have

$$\begin{aligned}
J_2 &= E\left[P\left(X_l^+ \prod_{j=1}^{l} Y_j > (1+\varepsilon)x, \sum_{i=1}^{l-1} X_i \prod_{j=1}^{i} Y_j \le -\varepsilon x \,\middle|\, \Theta_l\right)\right] \\
&\le E\left[P\left((X_l^* Y_l^*) \prod_{j=1}^{l-1} Y_j > (1+\varepsilon)x, \sum_{i=1}^{l-1} X_i \prod_{j=1}^{i} Y_j \le -\varepsilon x \,\middle|\, \Theta_{l-1}\right)\right]
\end{aligned}$$

$$= \quad P\left((X_l^* Y_l^*) \prod_{j=1}^{l-1} Y_j > (1+\varepsilon)x, \sum_{i=1}^{l-1} X_i \prod_{j=1}^{i} Y_j \le -\varepsilon x \right),$$

where the last step is due to the fact that (X_l^*, Y_l^*) is independent of Θ_{l-1} and all other random variables involved. To deal with the last probability above, we observe that $X_l^* Y_l^*$ has a tail in $\mathrm{RV}_{-(\alpha\beta)/(\alpha+\beta)}$, that the product $\prod_{j=1}^{l-1} Y_j$ is independent of $X_l^* Y_l^*$ and stochastically dominated by $\prod_{j=1}^{l-1} Y_j^*$ that has a finite moment of order higher than $(\alpha\beta)/(\alpha+\beta)$, and that the second event is independent of $X_l^* Y_l^*$ and vanishes as $x \to \infty$. Thus, applying Lemma 7 of Tang and Yuan (2014) again,

$$\lim_{x\to\infty} \frac{J_2}{z(x)} = 0.$$

Similarly,

$$\lim_{x\to\infty} \frac{J_3}{z(x)} = 0.$$

Putting these together yields that

$$\liminf_{x\to\infty} \frac{P(T(x)=l)}{z(x)} \ge (1+\varepsilon)^{-(\alpha\beta)/(\alpha+\beta)} E\left[\nu_{\Theta_l}(A) \prod_{j=1}^{l-1} Y_j^{(\alpha\beta)/(\alpha+\beta)} \right].$$

Plugging these estimates into (5.9) and noticing the arbitrariness of $\varepsilon > 0$, we obtain the desired lower bound

$$\lim_{x\to\infty} \frac{\psi(x;n)}{z(x)} \ge \sum_{l=1}^{n} E\left[\nu_{\Theta_l}(A) \prod_{j=1}^{l-1} Y_j^{(\alpha\beta)/(\alpha+\beta)} \right].$$

5.3 Proof of Theorem 3.1(ii)

By Assumption 3.2, it holds for each $i \in \mathbb{N}$ that, almost surely, $\nu_{\Theta_i}(A) \le \nu^*(A)$ and $E\left[Y_i^{(\alpha\beta)/(\alpha+\beta)} | \Theta_i \right] \le E\left[(Y^*)^{(\alpha\beta)/(\alpha+\beta)} \right]$. Thus,

$$\begin{aligned}
\sum_{i=1}^{\infty} E\left[\nu_{\Theta_i}(A) \prod_{j=1}^{i-1} Y_j^{(\alpha\beta)/(\alpha+\beta)} \right] &\le \nu^*(A) \sum_{i=1}^{\infty} E\left[E\left[\prod_{j=1}^{i-1} Y_j^{(\alpha\beta)/(\alpha+\beta)} \middle| \Theta_i \right] \right] \\
&= \nu^*(A) \sum_{i=1}^{\infty} E\left[\prod_{j=1}^{i-1} E\left[Y_j^{(\alpha\beta)/(\alpha+\beta)} \middle| \Theta_j \right] \right] \\
&\le \nu^*(A) \sum_{i=1}^{\infty} \left(E\left[(Y^*)^{(\alpha\beta)/(\alpha+\beta)} \right] \right)^{i-1} \\
&= \frac{\nu^*(A)}{1 - E\left[(Y^*)^{(\alpha\beta)/(\alpha+\beta)} \right]} < \infty.
\end{aligned}$$

To establish the uniformity of relation (3.3) for $n \in \mathbb{N}$, basing on Theorem 3.1(i) and following a standard procedure in the literature (see, e.g., Tang and Yuan (2016)), it suffices to prove that

$$\lim_{n\to\infty}\limsup_{x\to\infty}\frac{1}{z(x)}P\left(\sum_{i=n+1}^{\infty}X_i^+\prod_{j=1}^{i}Y_j>x\right)=0. \tag{5.10}$$

Actually, as done in the first step of (5.8), for each $n\in\mathbb{N}$ we have

$$P\left(\sum_{i=n+1}^{\infty}X_i^+\prod_{j=1}^{i}Y_j>x\right)\leq P\left(\sum_{i=n+1}^{\infty}X_i^*\prod_{j=1}^{i}Y_j^*>x\right).$$

Note that (X_i^*,Y_i^*), $i\in\mathbb{N}$, are i.i.d. random pairs with common BRV structure. The proof of Theorem 20.3.1 of Tang and Yuan (2016) readily shows that

$$\lim_{n\to\infty}\limsup_{x\to\infty}\frac{1}{z(x)}P\left(\sum_{i=n+1}^{\infty}X_i^*\prod_{j=1}^{i}Y_j^*>x\right)=0.$$

Thus, relation (5.10) follows.

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