ON PRICING BARRIER OPTIONS AND EXOTIC VARIATIONS

by

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PH.D. THESIS

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To my family

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ABSTRACT

Barrier options have become increasingly popular financial instruments due to the lower costs and the ability to more closely match speculating or hedging needs. In addition, barrier options play a significant role in modeling and managing risks in insurance and finance as well as in refining insurance products such as variable annuities and equity-indexed annuities. Motivated by these immediate applications arising from actuarial and financial contexts, the thesis studies the pricing of barrier options and some exotic variations, assuming that the underlying asset price follows the Black-Scholes model or jump-diffusion processes.

Barrier options have already been well treated in the classical Black-Scholes framework. The first part of the thesis aims to develop a new valuation approach based on the technique of exponential stopping and/or path counting of Brownian motions. We allow the option's boundaries to vary exponentially in time with different rates, and manage to express our pricing formulas properly as combinations of the prices of certain binary options. These expressions are shown to be extremely convenient in further pricing some exotic variations including sequential barrier options, immediate rebate options, multi-asset barrier options and window barrier options. Many known results will be reproduced and new explicit formulas will also be derived, from which we can better understand the impact on option values of various sophisticated barrier structures.

We also consider jump-diffusion models, where it becomes difficult, if not impossible, to obtain the barrier option value in analytical form for exponentially curved boundaries. Our model assumes that the logarithm of the underlying asset price is a Brownian motion plus an independent compound Poisson process. It is quite common to assign a particular distribution (such as normal or double exponential distribution) for the jump size if one wants to pursue closed-form solutions,

whereas our method permits any distributions for the jump size as long as they belong to the exponential family. The formulas derived in the thesis are explicit in the sense that they can be efficiently implemented through Monte Carlo simulations, from which we achieve a good balance between solution tractability and model complexity.

PUBLIC ABSTRACT

The payoff of a barrier option depends on whether the price of the underlying asset ever reaches a pre-specified boundary (or one of two pre-specified boundaries if it is a double-barrier option) during the contract's lifetime. Therefore, the valuation of barrier options can often be a key step in solving many problems in insurance and finance that are related to the so-called "first passage times". We study how to price barrier options and their exotic variations under two fundamental asset models: the Black-Scholes model and jump-diffusion models. The most sophisticated case we consider is that the options have two boundaries that are exponential functions in time.

In the Black-Scholes framework, we propose a new approach to compute the prices of single-barrier and double-barrier options as well as some of their exotic variations. Our method leads to closed-form expressions written in terms of the prices of certain binary options, which are much easier to compute explicitly.

In the last part of the thesis, we consider jump diffusions as the underlying asset price process. The great flexibility of our model as opposed to some old ones comes from the fact that in our model, the jump magnitude of the asset price can follow a wider range of distributions. Fortunately, this generalization does not mean sacrificing the feasibility of our problem because we present a new algorithm to yield the explicit solutions for both single-barrier and double-barrier options. Based on Monte Carlo simulations, we can efficiently implement our pricing formulas.

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CHAPTER 1

INTRODUCTION AND MOTIVATION

1.1 Motivation

Barrier options are a basic type of exotic options and have become frequently traded financial instruments in the US over-the-counter markets. The reasons for the increasing popularity of barrier options are manifold. First, they have lower costs than their plain vanilla counterparts. For example, a knock-in call only pays off when the barrier is breached prior to maturity, and thus is cheaper than a standard call. In addition, barrier options may match an investor's hedging needs in a more suitable manner. For example, buying a down-and-in put with the barrier set below the strike, as opposed to a long standard put, offers an appropriate (and inexpensive) way to protect against large downward movements in the underlying asset price. Moreover, the introduction of barrier features allows the option to more closely match the views about the future market behavior. For example, a long position in a down-and-out call is more consistent with the view that the underlying asset price will rise, as opposed to a long position in a standard call. As a result, when buying a barrier option, you can avoid paying for those states you believe are unlikely to occur; you may also enhance the profit by selling a barrier option that only pays off when you believe are impossible.

The most important motivation for us to spend the whole thesis considering the valuation of barrier options is that barrier options find suitable applications in insurance and finance from a number of aspects. Let us mention some of them. The pricing of barrier options essentially comes down to study the first passage times of certain stochastic processes, which is the building block of the solutions to many problems arising in risk management. Typical examples include the computation of ruin probability and the default risk modeling of an insurance company. Since time-until-ruin and time-until-default random variables can be viewed as first passage times, these problems can be formulated in certain ways based on barrier options. For example, see Wang (2016) which managed to decompose the insurance guaranty scheme introduced in Hwang, Chang and Wu (2015) into down-and-out options with immediate rebate payments.

Barrier options can also be very useful in refining certain insurance products such as modeling dynamic lapsations in variable annuity products. Policyholders are allowed to surrender the contract at their discretion, which usually happens when the embedded guarantee options associated with the variable annuities are deep out-of-money. Let us consider the guaranteed minimum maturity benefits (GMMB) where the account value at time t is denoted by S_t. At the maturity time T, if the contract has not lapsed, the policyholder is entitled the amount

$$S_T \vee G = S_T + (G - S_T)^+,$$

where G is a fixed guaranteed amount to protect against the depreciation of the underlying funds. Hence, the liability to the insurer would be a European put option with the strike G and the maturity T. Now, suppose the account value reaches to a very high point prior to maturity, then it may be worthwhile for the policyholder to surrender the contract to avoid high management fees (subsequent fees are usually set to be proportional to the account value) and invest directly in the underlying funds; there is little reason to continue to pay for the embedded option (put option) with negligible value in the future. This incentive to surrender can obviously be captured by introducing an upper barrier. Let us assume that the contract lapses when the account value rises above a fixed level B. Then the payoff of the embedded guarantee option becomes

$$(G - S_T)^+ \mathbb{1}(\tau_B > T), \tag{1.1}$$

where τ_B denotes the first time the account value rises to the level B. Hence, when valuing GMMB, we will deal with an up-and-out put option instead of the put option we just mentioned. One may also incorporate surrender penalty. Define by ι_t the surrender charge rate as a decreasing function in time. Upon the surrender prior to maturity, a policyholder will receive the amount equal to the account value less the penalty charge. Therefore, we refine our solution of valuing GMMB by adding to (1.1) an immediate rebate option with the payoff at time τ_B equal to

$$(1 - \iota_{\tau_B}) S_{\tau_B} \mathbb{1}(\tau_B < T).$$

Similar discussions about modeling dynamic lapsations in variable annuity products using barriers options can also be found, for example, in Gerber, Shiu and Yang (2013) and Augustyniak and Boudreault (2015).

In summary, the valuation of barrier options can be regarded as a key step in defining and solving many problems in insurance and finance, and hence our discussion in the thesis is highly relevant.

1.2 Thesis subject and our contribution

This thesis studies the pricing of barrier options and their exotic variations in the Black-Scholes (BS) model and jump-diffusion models.

In the first part of the thesis, we mainly focus on the valuation of barrier options restricted by exponentially time-varying boundaries within the classical BS framework. In particular, our contribution is to develop a new pricing method based on the exponential stopping of linear Brownian motions. It is worth pointing out that our approach does not require the use of traditional techniques such as the reflection principle and change of probability measure. Because we are dealing with an arbitrage payoff function, we express the prices of barrier options as combinations of the prices of certain path-independent options (binary options), which are fairly straightforward to determine explicitly. These expressions can be immediately used to recover many well-known results in the literature, and more importantly, based on these expressions, it becomes very convenient in pricing certain- exotic variations of barrier options.

The next contribution is to value some popular variations of barrier options by applying the results obatined in the first part. These options include sequential barrier options, immediate rebate options, multi-asset barrier options and window barrier options, where the boundaries are still exponential functions in time, and the BS economy is considered. First, we show that our formulation of path counting derived in the first part can be structured to deal with the problem where the boundaries may be breached in certain sequential orders, which directly leads to the pricing of sequential barrier options. Second, the pricing formulas in the first part can also be used to identify the distributions of one-sided and two-sided exit times, and thus the values of immediate rebate options can be easily obtained. Furthermore, to value multi-asset barrier options, we propose to adopt the Cholesky decomposition technique to factor out the barrier variable from other source of uncertainty, and the problem reduces to the one under the single-asset model. Finally, we calculate the price of window barrier options by taking repetitive conditional expectations and applying the pricing formula in the first part. In summary, we reproduce some known results for sequential barrier options and immediate rebate options, and for multi-asset barrier options and window barrier options, some new results will be derived.

The third and last major part of this thesis studies the pricing of barrier options under a general jump-diffusion framework, where the logarithm of the underlying asset price is modeled by a linear Brownian motion plus an independent compound Poisson process. The generality of our model comes from the fact that the jump magnitude can follow any distribution as long as it belongs to the exponential family, whereas in some other approaches, certain distribution should be assumed for the jump magnitude for analytical convenience. Following the idea in Shao and Wang (2012), we derive closed-form solutions for the prices of knock-out options with one-sided and two-sided exponential time-varying boundaries. We also show that our pricing formulas, although written in terms of infinite sums and multiple integrals, can be efficiently implemented by the Monte Carlo method.

1.3 Thesis structure

Chapter 2 briefly reviews some well-known results about stochastic processes that we will frequently use or mention throughout the thesis.

Chapter 3 presents a literature review of the existing valuation approaches for barrier options in the BS model. We also revisit three particular methods, density integrations, static hedging and method of images.

In Chapter 4, we develop a new valuation approach to pricing barrier options based on the exponential stopping of linear Brownian motions. We start with single-barrier options with a flat boundary, and then generalize it to the case of an exponentially time-varying boundary. Furthermore, a double knock-out option with exponential boundaries is considered. We evaluate the double knock-out event using path counting technique (See Sidenius (1998) for example), and express the option price as doubly infinite sums of the prices of certain binary options. Finally, we explain that when the payoff function satisfies some mild conditions, the pricing formula as doubly infinite sums is convergent rapidly.

In Chapter 5, we study the pricing of four exotic variations of barrier options. First, we treat sequentail barrier options using the path counting results obtained in Chapter 4. Then we treat immediate rebate options. We apply the martingale approach introduced in Gerber and Shiu (1994b, 1996) to study the case of flat boundaries; furthermore, immediate rebate options with exponential boundaries are valued based on the density functions of the two-sided exit times, which can be recovered from one of the major formulas in Chapter 4 with the payoff function being an identity function. In addition, we study multi-asset barrier options. The Cholesky decomposition is adopted to simplify our derivation, and we also make an extension to the case where the boundaries are stochastic processes. At the end, we derive a closed-form pricing formula for a window double knock-out option with exponential boundaries.

In Chapter 6, we first provide a literature review of the applications of jump-diffusion models in insurance and finance, and some classical pricing methods for barrier options under certain jump-diffusion models will be mentioned. Then we discuss the risk-neutral set-up for our jump-diffusion model. Explicit solutions are derived for the prices of an up-and-out option with an exponential boundary and a double knock-out option with two exponential boundaries. We also show how to numerically implement our formulas using Monte Carlo simulations. At the end, we discuss an application to pricing a double knock-out option restricted by two piecewise exponential boundaries in the BS framework.

1.4 Notation

Table 1.1: Notation and abbreviations

BS	Black-Scholes
$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{Pr})$	filtered probability space
$\{W_t\}$	standard Brownian motion
$\{X_t\}$	$X_t = \mu t + \sigma W_t$ (or Lévy process in Section 2.5)
$\{S_t\}$	price process of a single underlying asset
T	fixed time horizon
m_T	$m_T = \min_{0\le t\le T} X_t$
M_T	$M_T = \max_{0\le t\le T} X_t$
$f_X(x)$	density function of a random variable X
ε_λ	exponential random variable with mean $1/\lambda$
θ_λ^+	positive root of $\frac{\sigma^2}{2}\theta^2 + \mu\theta = \lambda$
θ_λ^-	negative root of $\frac{\sigma^2}{2}\theta^2 + \mu\theta = \lambda$
$\mathbb{1}_\mathcal{A}$	$\mathbb{1}_\mathcal{A} = 1$ when $\mathcal{A}$ occurs and $\mathbb{1}_\mathcal{A} = 0$ otherwise
E	expectation
Pr	probability
E_s	expectation taken under measure Pr given $S_0 = s$
Pr_s	probability given $S_0 = s$
$\mathrm{E}_s[X;\mathcal{A}]$	$\mathrm{E}_s[X;\mathcal{A}] = \mathrm{E}_s[X\mathbb{1}_\mathcal{A}]$ for a random variable X and an event $\mathcal{A}$
$\phi(x)$	$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$
$\phi_t(x)$	$\phi_t(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}$

Table 1.1 continued: Notation and abbreviations

$\Phi(x)$	distribution function of standard normal
$\Phi_2(x, y; \rho)$	distribution function of bivariate standard normal with correlation ρ
$\Phi_3(x, y, z; \rho_{12}, \rho_{13}, \rho_{23})$	distribution function of trivariate standard normal with correlations ρ_{12}, ρ_{13} and ρ_{23}
τ_U	$\tau_U = \inf\{t > 0 \mid S_t = U\}$ With a little abuse of notation, we also use τ_x to denote the hitting time of $\{X_t\}$ for some x.
τ_L	$\tau_L = \inf\{t > 0 \mid S_t = L\}$
$\tilde{\tau}_U$	$\tilde{\tau}_U = \inf\{t > 0 \mid S_t = Ue^{\delta_1 t}\}$
$\tilde{\tau}_L$	$\tilde{\tau}_L = \inf\{t > 0 \mid S_t = Le^{\delta_2 t}\}$
$\tilde{\tau}_{U\mid L}$	$\tilde{\tau}_{U\mid L} = \inf\{t > \tilde{\tau}_U \mid S_t = Le^{\delta_2 t}\}$
$\tilde{\tau}_{L\mid U}$	$\tilde{\tau}_{L\mid U} = \inf\{t > \tilde{\tau}_L \mid S_t = Ue^{\delta_1 t}\}$
$x \wedge y$	$x \wedge y = \min(x, y)$
$x \vee y$	$x \vee y = \max(x, y)$
x^+	$x^+ = x \vee 0$
i.i.d.	independent and identically distributed
$\{\boldsymbol{X}_t\}$	m-dimensional diffusion process with drift vector $\boldsymbol{\mu}$ and diffusion matrix $\boldsymbol{\Sigma}$
$\{\boldsymbol{S}_t\}$	price vector process of m underlying assets
$\boldsymbol{e}$	$\boldsymbol{e} = (1, 0, \ldots, 0)'_{m\times 1}$
$\boldsymbol{I}_m$	$m \times m$ identity matrix
$\mathrm{E}_{\boldsymbol{s}}$	expectation taken under measure Pr given $\boldsymbol{S}_0 = \boldsymbol{s}$

Table 1.1 continued: Notation and abbreviations

$\mathbb{R}$	$(-\infty, \infty)$
$\mathbb{R}^n$	$(-\infty, \infty) \times \cdots \times (-\infty, \infty)$ n times

CHAPTER 2

PRELIMINARIES

In this chapter, we shall recall and prove some classical results about stochastic processes. The following fundamental assumption will be made throughout the thesis. We let $(\Omega, \mathcal{F}, \mathcal{F}_t, \Pr)$ denote a filtered probability space equipped with a filtration $\{\mathcal{F}\}_{t\geq 0}$ satisfying the so-called usual conditions: (1) $\mathcal{F}_0$ contains all null sets under the measure Pr; (2) the filtration is right-continuous, that is, $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$.

2.1 Strong Markov property

Proposition 2.1.1 (Strong Markov Property)**.** *Let $\{W_t\}$ denote a standard Brownian motion. If τ is a stopping time, then the process $\{\bar{W}_t\}$ defined by $\bar{W}_t = W_{\tau+t} - W_\tau$ is also a standard Brownian motion and is independent of $\mathcal{F}_\tau$.*

2.2 Reflection principle

Proposition 2.2.1 (Reflection Principle)**.** *Let $\{W_t\}$ denote a standard Brownian motion. If τ is a stopping time, then the reflected process $\{\hat{W}_t\}$ defined by $\hat{W}_t = W_t \mathbb{1}(t < \tau) + (2W_\tau - W_t)\mathbb{1}(t \geq \tau)$ is also a standard Brownian motion.*

Figure 2.1 illustrates the reflected sample path of $\{W_t\}$ starting at the first time it reaches the level b $(b > 0)$ from below. Proposition 2.2.1 implies that the solid path and the dashed path in Figure 2.1 starting at the hitting time occur with the same probability.

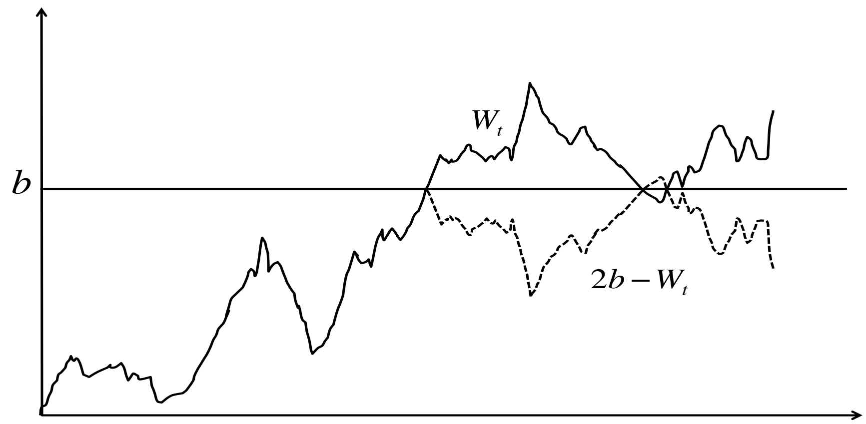

Figure 2.1: The reflection of a standard Brownian motion at the hitting time

The examples below are immediate applications of the reflection principle.

Example 2.2.1 (Joint distributions of Brownian motion and its extremum)**.** Let $X_t = \mu t + \sigma W_t$ where $\{W_t\}$ is a standard Brownian motion, and define $m_T = \min\limits_{0\le t\le T} X_t$ and $M_T = \max\limits_{0\le t\le T} X_t$ for some $T > 0$. The following formulas are the classical results based on the reflection principle.

$$\Pr\left(X_T > x, m_T \le y\right) = e^{\kappa y}\Pr(X_T > x - 2y), \quad y < x \wedge 0, \tag{2.1}$$

$$\Pr\left(X_T \le x, M_T \ge y\right) = e^{\kappa y}\Pr(X_T \le x - 2y), \quad y > x \vee 0, \tag{2.2}$$

where $\kappa = \frac{2\mu}{\sigma^2}$. The case where $\mu = 0$ is easy and can be obtained directly from Proposition 2.2.1 if one considers τ_y, the first time the process $\{X_t\}$ reaches the level y. The derivation becomes less straightforward when $\mu \neq 0$. A simple proof is given in Section 2.6.1 based on the Esscher transform factorization.

Example 2.2.2 (Brownian bridge)**.** The two formulas in Example 2.2.1 can be used to obtain some fundamental results for Brownian bridge.

$$\Pr(m_T \le y | X_T = x) = \frac{\Pr(X_T \in \mathrm{d}x, m_T \le y)}{\Pr(X_T \in \mathrm{d}x)} = \frac{e^{\kappa b}\phi_T(x-2b)}{\phi_T(x)}, \quad y < x \wedge 0,$$
$$\Pr(M_T \ge y | X_T = x) = \frac{\Pr(X_T \in \mathrm{d}x, M_T \ge y)}{\Pr(X_T \in \mathrm{d}x)} = \frac{e^{\kappa b}\phi_T(x-2b)}{\phi_T(x)}, \quad y > x \vee 0,$$

where $\phi_T(x)$ denotes the density function of X_T. By a straightforward calculation,

$$\Pr(m_T \le y | X_T = x) = e^{-\frac{2(-y)(x-y)}{\sigma^2 T}}, \quad y < x \wedge 0, \tag{2.3}$$
$$\Pr(M_T \ge y | X_T = x) = e^{-\frac{2y(y-x)}{\sigma^2 T}}, \quad y > x \vee 0. \tag{2.4}$$

Note that the expressions on the right-hand sides of (2.3) and (2.4) do not contain the drift parameter μ. As expected, when we let $y \to x$, these two conditional probabilities tend to 1. There is a simple way to see why these expressions do not contain μ. In fact, the Brownian bridge tied at x at time T can be constructed by $\{X_t\}$ in the following manner.

$$Y_t = X_t - \frac{t}{T}(X_T - x), \quad 0 \le t \le T.$$

Substituting $X_t = \mu t + \sigma W_t$ into the definition above, we have

$$Y_t = \sigma W_t - \frac{t}{T}(\sigma W_T - x), \quad 0 \le t \le T.$$

Hence, the drift parameter is irrelevant, and we can derive formulas (2.3) and (2.4) assuming $\mu = 0$. We only need (2.1) and (2.2) with $\kappa = 0$, the easy case.

2.3 Optional stopping theorem

Proposition 2.3.1 (Optional Stopping Theorem)**.** *Let* $\{\mathcal{M}_t\}$ *denote a martingale and* τ *denote a stopping time. Then* $\mathrm{E}[\mathcal{M}_\tau] = \mathrm{E}[\mathcal{M}_0]$ *if either of the two following* *conditions is satisfied: (1) τ is bounded almost surely; (2) $\{\mathcal{M}_t\}$ is uniformly integrable, that is, $\sup_{t\geq 0} \mathrm{E}\left[|\mathcal{M}_t|\mathbb{1}(\mathcal{M}_t > x)\right] \to 0$ as $x \to \infty$.*

2.4 Laplace transform of one-sided exit time

Proposition 2.4.1. *Let $X_t = \mu t + \sigma W_t$ where $\{W_t\}$ is a standard Brownian motion. Define by τ_b the first time the process $\{X_t\}$ reaches level b. For $\lambda > 0$, the Laplace transform of τ_b is given by*

$$\mathrm{E}\left[e^{-\lambda\tau_b}\right] = \begin{cases} e^{-\theta_\lambda^+ b} & b > 0 \\ e^{-\theta_\lambda^- b} & b < 0 \end{cases} \tag{2.5}$$

where

$$\theta_\lambda^+ = \frac{-\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2} \quad \textit{and} \quad \theta_\lambda^- = \frac{-\mu - \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}$$

are the two solutions of the quadratic equation $\frac{\sigma^2}{2}\theta^2 + \mu\theta - \lambda = 0$.

The proof of Proposition 2.4.1 is given in Section 2.6.2.

2.5 Esscher transforms

The concept of Esscher transforms was first introduced by Esscher (1932) for a single random variable. Gerber and Shiu (1994a) then extended this definition to the class of Lévy processes. Let $\{X_t\}$ denote a Lévy process. An Esscher transform induces a new probability measure on $\{X_t\}$. Let a be a real number such that $\mathrm{E}\left[e^{aX_1}\right]$ exists. The expectation of $h(X_t, 0 \leq t \leq T)$ for some function $h(\cdot)$ under the Esscher-transformed measure with index a is defined by

$$\mathrm{E}\left[h(X_t, 0 \leq t \leq T); a\right] = \frac{\mathrm{E}\left[h(X_t, 0 \leq t \leq T)e^{aX_T}\right]}{\mathrm{E}\left[e^{aX_T}\right]}. \tag{2.6}$$

The Esscher transform can also be defined in multivariate cases. Let m-dimensional vectors $\boldsymbol{X}_t = (X_{1t}, X_{2t}, \cdots, X_{mt})'$ and $\boldsymbol{a} = (a_1, a_2, \cdots, a_m)'$. In an analogous manner, we can have

$$\mathrm{E}\left[h(\boldsymbol{X}_t, 0 \le t \le T); \boldsymbol{a}\right] = \frac{\mathrm{E}\left[h(\boldsymbol{X}_t, 0 \le t \le T)e^{\boldsymbol{a}'\boldsymbol{X}_T}\right]}{\mathrm{E}\left[e^{\boldsymbol{a}'\boldsymbol{X}_T}\right]},$$

where abusing notation, we still use the same function $h(\cdot)$ for vectors. The two factorization formulas below can be very useful in simplifying certain expectations.

$$\mathrm{E}\left[e^{aX_T}h(X_t, 0 \le t \le T); b\right] = \mathrm{E}\left[e^{aX_T}; b\right] \cdot \mathrm{E}\left[h(X_t, 0 \le t \le T); a + b\right], \tag{2.7}$$

$$\mathrm{E}\left[e^{\boldsymbol{a}'\boldsymbol{X}_T}h(\boldsymbol{X}_t, 0 \le t \le T); \boldsymbol{b}\right] = \mathrm{E}\left[e^{\boldsymbol{a}'\boldsymbol{X}_T}; \boldsymbol{b}\right] \cdot \mathrm{E}\left[h(\boldsymbol{X}_t, 0 \le t \le T); \boldsymbol{a} + \boldsymbol{b}\right], \tag{2.8}$$

where $\boldsymbol{b} = (b_1, b_2, \cdots, b_m)'$.

It is not difficult to show that a Lévy process under the Esscher transform is still a Lévy process. Consider one-dimensional case for example. For $0 \le s < t$ and each Borel set $\mathcal{C}$, the independent and stationary increments property of $\{X_t\}$ under the original measure leads to

$$\begin{aligned}
\Pr\left(X_t - X_s \in \mathcal{C} | \mathcal{F}_s; a\right) &= \frac{\mathrm{E}\left[\mathbb{1}(X_t - X_s \in \mathcal{C})e^{aX_t} | \mathcal{F}_s\right]}{\mathrm{E}\left[e^{aX_t} | \mathcal{F}_s\right]} \\
&= \frac{\mathrm{E}\left[\mathbb{1}(X_t - X_s \in \mathcal{C})e^{a(X_t - X_s)} | \mathcal{F}_s\right]}{\mathrm{E}\left[e^{a(X_t - X_s)} | \mathcal{F}_s\right]} \\
&= \frac{\mathrm{E}\left[\mathbb{1}(X_{t-s} \in \mathcal{C})e^{aX_{t-s}}\right]}{\mathrm{E}\left[e^{aX_{t-s}}\right]} \\
&= \Pr\left(X_{t-s} \in \mathcal{C}; a\right),
\end{aligned}$$

which shows that the independent and stationary increments property is satisfied under the transformed measure with index a. The case of a multivariate Lévy process can be treated in a similar fashion.

The following example considers Brownian motions, a special case of Lévy processes, under Esscher transforms, and shows how to determine the drift and volatility parameters under the new measure.

Example 2.5.1 (Brownian motions under Esscher transforms)**.** Let $X_t = \mu t + \sigma W_t$ where $\{W_t\}$ is a standard Brownian motion. For $0 \leq s < t$, it can be shown

$$\mathrm{E}\left[e^{z(X_t - X_s)} | \mathcal{F}_s; a\right] = \mathrm{E}\left[e^{zX_{t-s}}; a\right] = \exp\left((\mu + a\sigma^2)z(t-s) + \tfrac{1}{2}\sigma^2 z^2 (t-s)\right),$$

from which we conclude that, under the transformed measure with index a, the process $\{X_t\}$ is still a Brownian motion with the modified drift $\mu + a\sigma^2$ and the same volatility σ. Likewise, one can also show that for an m-dimensional Brownian motion with drift vector $\boldsymbol{\mu}$ and diffusion matrix $\boldsymbol{\Sigma}$, its distribution under the transformed measure with index $\boldsymbol{a}$ becomes an m-dimensional Brownian motion with drift vector $\boldsymbol{\mu} + \boldsymbol{\Sigma a}$ and the same diffusion matrix $\boldsymbol{\Sigma}$.

Example 2.5.2 (Option pricing by Esscher transforms)**.** Let $X_t = \mu t + \sigma W_t$ where $\{W_t\}$ is a standard Brownian motion. In Gerber and Shiu (1994a), the method of Esscher transforms was introduced as a powerful tool in option pricing. Assume the risk-free interest rate is constant, denoted by r. Let S_t be the time-t price of a non-dividend-paying asset and

$$S_t = S_0 e^{X_t},$$

where $\{X_t\}$ is a Lévy process. According to Gerber and Shiu (1994a), to find a risk-neutral measure, an index a^* is determined such that the discounted asset price process $\{e^{-rt}S_t\}$ is a martingale under the transformed measure with the index a^*. Equivalently, this means that we need to solve the equation

$$\mathrm{E}\left[e^{X_t}; a^*\right] = e^{rt}, \quad t \geq 0 \tag{2.9}$$

to get a^*, which is unique (See Gerber and Shiu (1994b)). Then an option price is calculated as the expectation, with respect to this particular risk-neutral measure, of the discounted payoffs. When multiple risk-neutral measures are present, the Esscher transforms method can yield a simple and unambiguous solution. However, in most part of this thesis except for Chapter 6, we will only consider a geometric Brownian motion for the asset price process, in which case the risk-neutral measure is unique.

2.6 Appendix

2.6.1 Proof of Example 2.2.1

Proof. We only derive (2.1). When $\mu = 0$, an immediate consequence of the reflection principle (Proposition 2.2.1) is that the two events $\{X_T > x, m_T \le y\}$ and $\{2y - X_T > x\}$ occur with the same probability. When $\mu \ne 0$, we utilize the Esscher tranform factorization formula (2.7). According to the discussion in Example 2.5.1, under the transformed measure with index $-\frac{\kappa}{2}$, the drift term of $\{X_t\}$ becomes $\mu - \frac{\kappa}{2}\sigma^2 = 0$. Therefore,

$$\begin{aligned}
\Pr\left(X_T > x, m_T \le y\right) &= \mathrm{E}\left[e^{-\frac{\kappa}{2}X_T} e^{\frac{\kappa}{2}X_T} \mathbb{1}(X_T > x, m_T \le y)\right] \\
&= \mathrm{E}\left[e^{-\frac{\kappa}{2}X_T}\right] \mathrm{E}\left[e^{\frac{\kappa}{2}X_T} \mathbb{1}(X_T > x, m_T \le y); -\tfrac{\kappa}{2}\right] \\
&= \mathrm{E}\left[e^{-\frac{\kappa}{2}X_T}\right] \mathrm{E}\left[e^{\frac{\kappa}{2}(2y - X_T)} \mathbb{1}(2y - X_T > x); -\tfrac{\kappa}{2}\right] \\
&= e^{\kappa y}\mathrm{E}\left[e^{-\frac{\kappa}{2}X_T}\right] \mathrm{E}\left[e^{-\frac{\kappa}{2}X_T} \mathbb{1}(2y - X_T > x); -\tfrac{\kappa}{2}\right] \\
&= e^{\kappa y}\mathrm{E}\left[e^{-\frac{\kappa}{2}X_T}\right] \mathrm{E}\left[e^{\frac{\kappa}{2}X_T} \mathbb{1}(2y + X_T > x); -\tfrac{\kappa}{2}\right] \\
&= e^{\kappa y}\Pr\left(X_T > x - 2y\right).
\end{aligned}$$

The second to last step is due to the fact that X_T and $-X_T$ have identical distribution when $\mu = 0$. $\square$

2.6.2 Proof of Proposition 2.4.1

Proof. First note that the two stochastic processes $\{e^{-\lambda t + \theta_\lambda^+ X_t}\}$ and $\{e^{-\lambda t + \theta_\lambda^- X_t}\}$ are martingales. When $b > 0$, we apply the optional stopping theorem to the first martingale $\{e^{-\lambda t+\theta_\lambda^+ X_t}\}$ for the bounded stopping time $\tau_b \wedge t$ with some positive t. In particular, we have

$$\mathrm{E}\left[e^{-\lambda(\tau_b\wedge t)+\theta_\lambda^+ X_{\tau_b\wedge t}}\right] = 1.$$

The equation above can be expanded as

$$\mathrm{E}\left[e^{-\lambda t+\theta_\lambda^+ X_t}\mathbb{1}(t<\tau_b)\right] + \mathrm{E}\left[e^{-\lambda\tau_b+\theta_\lambda^+ X_{\tau_b}}\mathbb{1}(t\geq\tau_b)\right] = 1. \tag{2.10}$$

Observe that $e^{\theta_\lambda^+ X_t} < e^{\theta_\lambda^+ b}$ when $t < \tau_b$. Then we let $t \to \infty$ in the (2.10), and apply the dominated convergence theorem. It follows that the first expectation tends to zero and the second expectation tends to $e^{\theta_\lambda^+ b}\mathrm{E}\left[e^{-\lambda\tau_b}\right]$ because $X_{\tau_b} = b$. Hence, when $t \to \infty$, (2.10) becomes

$$e^{\theta_\lambda^+ b}\mathrm{E}\left[e^{-\lambda\tau_b}\right] = 1,$$

from which the result follows immediately. When $b < 0$, we choose the second martingale and use a similar argument. $\square$

CHAPTER 3

REVIEW OF EXISTING VALUATION APPROACHES

3.1 A brief overview

The purpose of this chapter is to provide the readers a brief literature review of valuation approaches to pricing barrier options. Three particular methods will be discussed. Our attention will be only given to those common types of barrier options within the classical BS framework (A review of option pricing in jump-diffusion models will be given separately in Chapter 6). Due to our interest, some popular barrier options, such as discretely monitored barrier options, will not be mentioned. Before the survey, let us briefly review the basic set-up for option pricing.

We start with the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{Pr})$ to describe the uncertainty of the financial world, where $\mathcal{F}_t$ can be regarded as the information available up to time t and Pr is the physical probability. In the BS model, the asset price process $\{S_t\}$ can be expressed as a geometric Brownian motion

$$S_t = S_0 \exp(\mu t + \sigma W_t), \quad 0 \leq t \leq T, \tag{3.1}$$

where μ and σ are constants, $\{W_t\}$ is a standard Brownian motion and T is a fixed time horizon. We assume the asset pays no dividends, but nevertheless a proportional constant dividend yield can be easily incorporated into the general drift parameter. We also assume that the risk-free interest rate is constant, denoted by r. Then according to Harrison and Kreps (1979) and Harrison and Pliska (1981), no arbitrage argument leads to the existence of a risk-neutral measure Pr^* equivalent to Pr such that the option value is calculated as the expectation of the discounted payoff under $\Pr^*$. The dynamic of $\{S_t\}$ under $\Pr^*$ is given by

$$S_t = S_0 \exp((r - \sigma^2/2)t + \sigma W_t^*), \tag{3.2}$$

where $\{W_t^*\}$ is a standard Brownian motion under $\Pr^*$. We remark that the risk-neutral measure can also be found by the method of Esscher transforms (See Example 2.5.2). Because the asset price process has the same form under $\Pr^*$ and Pr, one can, without loss of generality, use (3.1) when deriving pricing formulas. Now let us begin the literature review.

The valuation of single-barrier options can be traced back to the seminal paper Merton (1973), which derived a closed-form formula for the price of a down-and-out call by solving the BS equation subject to certain boundary conditions. The application of binomial tree methodology to knock-out options with rebates can be found in Cox and Rubinstein (1985). Static hedging was pioneered by Peter Carr for pricing path-dependent options. It was shown in, for example, Carr and Chou (1997, 2002) that barrier options can be replicated staticly by a portfolio of path-independent options, the prices of which are fairly easy to calculate. The problem of a two-sided barrier is far less straightforward to tackle. Based on the classical results in Anderson (1960), Kunitomo and Ikeda (1992) expressed the prices of knock-out call and put options with two exponential boundaries in terms of doubly infinite sums of normal probabilities. Kolkiewicz (2002) studied the exit times distributions using a new method and provided a general valuation for a large class of double-barrier options. Using path counting and the reflection principle, Sidenius (1998) and Li (1998) obtained closed-form solutions of double knock-out options with flat boundaries and exponential boundaries, respectively. The Laplace transform approach was considered in Geman and Yor (1996) and Pelsser (2000). Buchen and Konstandatos (2009) developed the method of images to price double knock-out options with exponential boundaries and arbitrary payoffs. The partial differential equation approach can be found, for example, in Zvan, Vetzal and Forsyth (2000) to price barrier options in a unified framework.

Two exotic variations of barrier options worth mentioning are outside barrier options and partial barrier options, which were proposed by Heynen and Kat (1994a) and Heynen and Kat (1994b), respectively. In an outside barrier option, the asset (called an outside asset or an external barrier variable) associated with the barrier provision is not the same asset underlying the payoff. Heynen and Kat (1994a) and Carr (1995) evaluated this type of options based on a bivariate assumption. Kwok, Wu and Yu (1998) extended to a multi-asset model with a single external barrier variable, and the problem for double barriers was studied in Wong and Kwok (2003). Skipper (2007) applied the method of images to a very general barrier variable which depends on multiple asset prices through a power function. In a partial barrier option, the monitoring period either starts after the initiation time of the contract or ends before the maturity time. Carr and Chou (2002) applied the static hedging technique to price this type of options. Guillaume (2003) derived a closed-form formula for a window double-barrier option where the monitoring period is a strict subset of the option's lifetime.

In particular, we revisit three time-honored pricing methods in the literature and present them in the remaining of this chapter. As pointed out earlier, in the BS framework, no-arbitrage pricing only requires us to modify the drift parameter of the asset price process and the problem reduces to the calculation of a general discounted expectation. Therefore, most of the time in this thesis when we are in the BS world, we treat a general drift parameter μ and calculate related expectations. Since we assume a constant risk-free interest rate r, one can always obtain the time-0 no-arbitrage price by multiplying back the discount factor e^{-rT} and replacing μ by $r - \frac{1}{2}\sigma^2$.

3.2 Density integrations

The most direct way to determine the price of barrier options is to identify the joint distribution of the underlying asset price at maturity and its extremum restricted by the boundary up to the maturity time. Let m_T denote the running minimum of $X_t = \ln \frac{S_t}{S_0}$ from time 0 up to time T: $m_T = \min\limits_{0 \le t \le T} X_t$. Given an initial asset price s_0 and an arbitrary payoff function $\pi(s)$, the forward price of a down-and-out option with a barrier B can be expressed as

$$\mathrm{E}\left[\pi(s_0 e^{X_T}) \mathbb{1}\left(s_0 e^{m_T} > B\right)\right], \quad s_0 > B. \tag{3.3}$$

It is sufficient to find the joint distribution of X_T and m_T. Define the density function $g^+(x;y)$ by

$$g^+(x;y)\mathrm{d}x = \Pr\left(X_T \in \mathrm{d}x, m_T > y\right), \quad y < x \wedge 0.$$

Differentiating both sides of equation (2.1) with respect to x, we are able to express $g^+(x;y)$ solely in terms of the density function of X_T, denoted by $\phi_T(x)$. In particular, we have

$$g^+(x;y) = \phi_T(x) - e^{\kappa y}\phi_T(x - 2y),$$

where $\kappa = \frac{2\mu}{\sigma^2}$. Then expectation (3.3) becomes

$$\begin{aligned}
&\mathrm{E}_{s_0}\left[\pi(s_0 e^{X_T}) \mathbb{1}\left(s_0 e^{m_T} > B\right)\right] \\
&= \int_{\ln \frac{B}{s_0}}^{\infty} \pi(s_0 e^x) g^+\left(x; \ln \frac{B}{s_0}\right) \mathrm{d}x \\
&= \int_{\ln \frac{B}{s_0}}^{\infty} \pi(s_0 e^x)\phi_T(x)\mathrm{d}x - e^{\kappa \ln \frac{B}{s_0}} \int_{\ln \frac{B}{s_0}}^{\infty} \pi(s_0 e^x)\phi_T\left(x - 2\ln\frac{B}{s_0}\right)\mathrm{d}x.
\end{aligned}$$

The first integral is

$$\begin{aligned}\int_{\ln\frac{B}{s_0}}^{\infty}\pi(s_0e^x)\phi_T(x)\mathrm{d}x &= \int_{-\infty}^{\infty}\pi(s_0e^x)\mathbb{1}(s_0e^x > B)\phi_T(x)\mathrm{d}x\\ &= \mathrm{E}\left[\pi(s_0e^{X_T})\mathbb{1}(s_0e^{X_T} > B)\right]\\ &= \mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}(S_T > B)\right],\end{aligned}$$

where $\mathrm{E}_{s_0}[\cdot]$ means the expectation is computed given that the initial asset price $S_0 = s_0$. By the change of variable $z = x - 2\ln\frac{B}{s_0}$, the second integral becomes

$$\begin{aligned}\int_{\ln\frac{B}{s_0}}^{\infty}\pi(s_0e^x)\phi_T\left(x - 2\ln\frac{B}{s_0}\right)\mathrm{d}x &= \int_{-\ln\frac{B}{s_0}}^{\infty}\pi\left(\frac{B^2}{s_0}e^z\right)\phi_T(z)\,\mathrm{d}z\\ &= \int_{-\infty}^{\infty}\pi\left(\frac{B^2}{s_0}e^z\right)\mathbb{1}\left(\frac{B^2}{s_0}e^z > B\right)\phi_T(z)\,\mathrm{d}z\\ &= \mathrm{E}_{\frac{B^2}{s_0}}\left[\pi(S_T)\mathbb{1}(S_T > B)\right],\end{aligned}$$

where $\mathrm{E}_{\frac{B^2}{s_0}}[\cdot]$ means the expectation is computed given that the initial asset price $S_0 = \frac{B^2}{s_0}$. Note that $e^{\kappa\ln\frac{B}{s_0}} = \left(\frac{B}{s_0}\right)^{\kappa}$. Hence, we arrive at obtaining a representation formula for the time-0 forward price of the down-and-out option:

$$\begin{aligned}&\mathrm{E}\left[\pi(s_0e^{X_T})\mathbb{1}\left(s_0e^{m_T} > B\right)\right]\\ &= \mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}(S_T > B)\right] - \left(\frac{B}{s_0}\right)^{\kappa}\mathrm{E}_{\frac{B^2}{s_0}}\left[\pi(S_T)\mathbb{1}(S_T > B)\right], \quad s_0 > B. \qquad (3.4)\end{aligned}$$

Remark 3.2.1. To evaluate the right-hand side of (3.4), one only needs to determine the first expectation. In fact, if we consider the first expectation as a function of the initial price s_0, the second expectation is the function just obtained but evaluated at $\frac{B^2}{s_0}$.

Remark 3.2.2. The calculations can be quite tedious if we choose to use the "real" joint density function of X_T and m_T, that is, $\Pr(X_T \in \mathrm{d}x, m_T \in \mathrm{d}y)$. See, for example, Section 7.3.3 in Shreve (2010) which used several pages to integrate the joint density of X_T and its running maximum to derive the formula of the price of an up-and-out call option.

Remark 3.2.3. To handle up-and-out or up-and-in options, one will need to find the joint distribution of X_T and M_T, which can be derived from (2.2). Some may be more interested in the joint distribution of the triplet (X_T, m_T, M_T), which is the key to pricing double-barrier options. However, it turns out to be far less straightforward to identify this joint distribution, especially when the drift parameter μ is not zero. The closed-form expression of this trivariate density has been available for quite some time in the literature of probability theory (See, for example, formula (4.1.48) in Kwok (2008)). Define the density function $g(x; y, z)$ such that

$$g(x; y, z)\mathrm{d}x = \Pr(X_T \in \mathrm{d}x, m_T > y, M_T < z).$$

Then $g(x; y, z)$ can be expressed as the doubly infinite sum

$$g(x; y, z) = \sum_{n=-\infty}^{\infty} e^{(y-z)n\kappa} \big[\phi_T(x - 2(y-z)n) - e^{y\kappa}\phi_T(x - 2y - 2(y-z)n)\big],$$

where $\phi_T(x)$ is the density function of X_T.

3.3 Static hedging

Static hedging was pioneered by Professor Peter Carr for the purpose of valuing barrier options through a portfolio of path-independent options (See, for example, Carr (1995), Carr and Chou (1997, 2002)). This technique is essentially derived from applying the reflection principle to the underlying asset price process at its first passage time of a pre-specified barrier. Let us first describe a different version of the reflection principle for geometric Brownian motion.

Proposition 3.3.1. *Let $\pi(s)$ denote a general payoff function, and define its reflected payoff function with respect to the barrier B as*

$$\pi^*(s) = \left(\frac{B}{s}\right)^{\kappa} \pi\left(\frac{B^2}{s}\right), \tag{3.5}$$

where $\kappa = \frac{2\mu}{\sigma^2}$. Let $t \in [0,T]$. Then given $S_t = B$, the payoffs $\pi(S_T)$ and $\pi^*(S_T)$ have the same conditional expectation.

Remark 3.3.1. Proposition 3.3.1 can be viewed as a more generalized version of the reflection principle (Proposition 2.2.1) because the payoff function is arbitrary. The expression $\left(\frac{B}{s}\right)^{\kappa}$ accounts for the non-zero drift parameter.

Proof of Proposition 3.3.1. By the Esscher transform factorization (2.7), we have

$$\mathrm{E}\left[\left(\frac{B}{S_T}\right)^{\kappa} \pi\left(\frac{B^2}{S_T}\right) \middle| S_t = B\right] = \mathrm{E}\left[\left(\frac{B}{S_T}\right)^{\kappa} \middle| S_t = B\right] \mathrm{E}\left[\pi\left(\frac{B^2}{S_T}\right) \middle| S_t = B; -\kappa\right].$$

Because $\{(S_t)^{-\kappa}\}$ is a martingale,

$$\mathrm{E}\left[\left(\frac{B}{S_T}\right)^{\kappa} \middle| S_t = B\right] = \left(\frac{B}{B}\right)^{\kappa} = 1.$$

Under the transformed measure with index $-\kappa$, $X_t = \ln \frac{S_t}{S_0}$ becomes a linear Brownian motion with modified drift $\mu - \kappa\sigma^2 = -\mu$ and the same volatility σ. It then follows that $\{X_t, t \leq T\}$ under the transformed measure has the same distribution as $\{-X_t, t \leq T\}$ under the original measure. Hence,

$$\begin{aligned}
&\mathrm{E}\left[\pi\left(\frac{B^2}{S_T}\right) \middle| S_t = B; -\kappa\right] \\
&= \mathrm{E}\left[\pi\left(\frac{B^2}{Be^{X_T - X_t}}\right); -\kappa\right] = \mathrm{E}\left[\pi\left(Be^{X_T - X_t}\right)\right] = \mathrm{E}\left[\pi\left(S_T\right) \middle| S_t = B\right].
\end{aligned}$$

Here we also use the fact that $\{X_t\}$ has independent increments. □

Now let us explain the procedure of static hedging of, for example, a down-and-out option. We claim that a down-and-out option can be replicated using a path-independent option with maturity payoff

$$\pi(S_T)\mathbb{1}(S_T > B) - \left(\frac{B}{S_T}\right)^\kappa \pi\left(\frac{B^2}{S_T}\right)\mathbb{1}(S_T < B).$$

Here, the word "static" means that we do not need to dynamically rebalance the replicating portfolio prior to maturity. Define $\pi_B(s) = \pi(s)\mathbb{1}(s > B)$, then the payoff above can be expressed as $\pi_B(S_T) - \pi_B^*(S_T)$ where $\pi_B^*(s)$ is the reflected payoff function of $\pi_B(s)$ as defined in (3.5). To hedge the down-and-out options, one can at time 0 hold a portfolio consisting of a long position on the options with payoff $\pi_B(S_T)$ and a short position on the options with payoff $\pi_B^*(S_T)$. If the barrier is never breached prior to maturity, the portfolio delivers a payoff of $\pi(S_T)$, matching the payoff of the barrier options. Otherwise, the barrier options expire worthless upon breaching the barrier, and one can then close both positions in the portfolio, also resulting in a zero value at the time of breaching, which follows from Proposition 3.3.1.

Therefore, given $S_0 = s_0$, we can write the time-0 forward price of down-and-out options as

$$\begin{aligned}
&\mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}\left(\min_{0<t\le T} S_t > B\right)\right] \\
&= \mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}(S_T > B)\right] - \mathrm{E}_{s_0}\left[\left(\frac{B}{S_T}\right)^\kappa \pi\left(\frac{B^2}{S_T}\right)\mathbb{1}(S_T < B)\right]. \qquad (3.6)
\end{aligned}$$

Following the same argument used in the proof of Proposition 3.3.1, we can easily demonstrate that the second expectation on the right-hand side of (3.6) is equal to

$$\left(\frac{B}{s_0}\right)^\kappa \mathrm{E}_{\frac{B^2}{s_0}}\left[\pi\left(S_T\right)\mathbb{1}(S_T > B)\right].$$

Hence, formula (3.6) indeed agrees with formula (3.4). The other three types of single-barrier options can be hedged and valued in a similar fashion. Table 3.1

summarizes the replicating portfolios for all types of single-barrier options. For a comprehensive discussion of static hedging to price various types of barrier options (and exotic variations), we refer to Carr and Chou (2002).

Table 3.1: Static hedging of single-barrier options

Option type	Maturity payoff	Static portfolio payoff
Up-and-out	$\pi(S_T)\mathbb{1}\left(\max\limits_{0<t\leq T} S_t < B\right)$	$\begin{cases} \pi(S_T) & S_T < B \\ -\left(\dfrac{B}{S_T}\right)^{\kappa} \pi\left(\dfrac{B^2}{S_T}\right) & S_T > B \end{cases}$
Up-and-in	$\pi(S_T)\mathbb{1}\left(\max\limits_{0<t\leq T} S_t > B\right)$	$\begin{cases} 0 & S_T < B \\ \pi(S_T) + \left(\dfrac{B}{S_T}\right)^{\kappa} \pi\left(\dfrac{B^2}{S_T}\right) & S_T > B \end{cases}$
Down-and-out	$\pi(S_T)\mathbb{1}\left(\min\limits_{0<t\leq T} S_t > B\right)$	$\begin{cases} \pi(S_T) & S_T > B \\ -\left(\dfrac{B}{S_T}\right)^{\kappa} \pi\left(\dfrac{B^2}{S_T}\right) & S_T < B \end{cases}$
Down-and-in	$\pi(S_T)\mathbb{1}\left(\min\limits_{0<t\leq T} S_t < B\right)$	$\begin{cases} 0 & S_T > B \\ \pi(S_T) + \left(\dfrac{B}{S_T}\right)^{\kappa} \pi\left(\dfrac{B^2}{S_T}\right) & S_T < B \end{cases}$

3.4 Method of images

The prices of barrier options satisfy the BS partial differential equation subject to some boundary conditions modified to account for the barrier event. Merton (1973) is considered as the first one appeared in the literature to solve the BS equation and derive an explicit formula for the price of down-and-out call options. The procedure is very similar to that for plain vanilla options such as calls and puts. Under certain variable transformations, one would be able to translate the BS equation to a heat equation subject to some semi-infinite boundary conditions, which can be solved through a routine procedure.

Now we treat a down-and-out option as an example. Let $v(s, t)$ denote its time-t price given that $S_t = s$. Also, let $u(s,t)$ denote the price of the corresponding plain vanilla options without the barrier feature. We know that $v(s,t)$ satisfies the BS equation

$$\frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} - rv = 0, \quad s > B, 0 \le t < T, \tag{3.7}$$

with the boundary conditions

$$v(B,t) = 0, \tag{3.8}$$

$$v(s,T) = u(s,T) = \pi(s). \tag{3.9}$$

One can formulate similar equations for the other three types of single-barrier options according to Table 3.2. Let us briefly explain the boundary conditions (3.8) and (3.9): the down-and-out options become void upon breaching the barrier prior to maturity, which yields (3.8); condition (3.9) basically means that the payoff $\pi(s)$ is granted at the time of maturity if the barrier has never been reached before.

Table 3.2: Active domains and boundary conditions for $v(s,t)$

Option type	Active domain of (s,t)	Boundary conditions
Up-and-out	$(0,B) \times [0,T)$	$\begin{cases} v(B,t) = 0 \\ v(s,T) = u(s,T) \end{cases}$
Up-and-in	$(0,B) \times [0,T)$	$\begin{cases} v(B,t) = u(B,t) \\ v(s,T) = 0 \end{cases}$
Down-and-out	$(B,\infty) \times [0,T)$	$\begin{cases} v(B,t) = 0 \\ v(s,T) = u(s,T) \end{cases}$
Down-and-in	$(B,\infty) \times [0,T)$	$\begin{cases} v(B,t) = u(B,t) \\ v(s,T) = 0 \end{cases}$

Now consider the following variable transformations:

$$x = \ln \frac{s}{B}, \quad z = \sigma^2(T-t), \quad H^v(x,z) = e^{\frac{1}{2}\kappa x + \zeta(T-t)} v(s,t),$$

where $\kappa = \frac{2\mu}{\sigma^2}$ with $\mu = r - \frac{1}{2}\sigma^2$ and $\zeta = r + \frac{1}{8}\kappa^2\sigma^2$. It is easy to verify that the BS equation (3.7) to (3.9), after the transformation, reduce to a heat equation for $H^v(x,z)$:

$$\frac{1}{2}\frac{\partial^2 H^v}{\partial x^2} - \frac{\partial H^v}{\partial z} = 0, \quad x > 0, 0 < z \leq \sigma^2 T, \tag{3.10}$$

with the boundary conditions

$$H^v(0,z) = 0, \tag{3.11}$$

$$H^v(x,0) = e^{\frac{1}{2}\kappa x}\pi(Be^x). \tag{3.12}$$

This is a semi-infinite boundary problem which can be solved using variable separation or the Fourier transform inversion. It is called "semi-infinite" because the active domain of x is the positive real line.

We now introduce an approach called method of images, which can significantly simplify the derivation when dealing with the heat equation. Note that if $H^v(x,z)$ is a solution to (3.10), so is $-H^v(-x,z)$. Hence, we can somehow extend the domain of x to the whole real line so that we will be dealing with an "infinite" boundary problem instead, and we also want the condition (3.11) satisfied automatically. Let us see how to achieve this. Recall that $u(s,t)$ denotes the price of the corresponding plain vanilla options. The it also satisfies the BS equation (3.7) but the active domain of (s,t) becomes $(0,\infty)\times[0,T)$ and it is only subject to the second condition (3.9). We can make the exactly same change of variables for $u(s,t)$ and obtain its corresponding heat equation of $H^u(x,z)$:

$$\frac{1}{2}\frac{\partial^2 H^u}{\partial x^2} - \frac{\partial H^u}{\partial z} = 0, \quad -\infty < x < \infty, 0 < z \leq \sigma^2 T,$$

with the boundary condition

$$H^u(x,0) = e^{\frac{1}{2}\kappa x}\pi(Be^x).$$

Notice that the equations above lead to an infinite boundary problem. We further refine the boundary condition so that the function vanishes on the negative real line. To do this, let $\hat{u}(x,z)$ denote a function such that its corresponding transformed function $H^{\hat{u}}(x,z)$ satisfies

$$\frac{1}{2}\frac{\partial^2 H^{\hat{u}}}{\partial x^2} - \frac{\partial H^{\hat{u}}}{\partial z} = 0, \quad -\infty < x < \infty, 0 < z \leq \sigma^2 T,$$

with the boundary condition

$$H^{\hat{u}}(x,0) = e^{\frac{1}{2}\kappa x}\pi(Be^x)\mathbb{1}(x>0).$$

Therefore, $\hat{u}(s,t)$ corresponds to the time-t price of a plain vanilla option with payoff function equal to $\pi(s)\mathbb{1}(s>B)$. We now can observe the following relation between $H^{\hat{u}}(x,z)$ and $H^v(x,z)$:

$$H^v(x,z) = H^{\hat{u}}(x,z) - H^{\hat{u}}(-x,z).$$

In particular, one can verify that the right-hand side of the equation above satisfies equations (3.10) to (3.12). To obtain the relation between $v(s,t)$ and $\hat{u}(s,t)$, we reverse the change of variables. It follows that

$$\begin{aligned}
v(s,t) &= e^{-\frac{1}{2}\kappa x - \zeta(T-t)} H^v(x,z) \\
&= e^{-\frac{1}{2}\kappa x - \zeta(T-t)} \left[H^{\hat{u}}(x,z) - H^{\hat{u}}(-x,z)\right] \\
&= e^{-\frac{1}{2}\kappa x - \zeta(T-t)} \left[e^{\frac{1}{2}\kappa x + \zeta(T-t)}\hat{u}(s,t) - e^{-\frac{1}{2}\kappa x + \zeta(T-t)}\hat{u}(Be^{-x},t)\right] \\
&= \hat{u}(s,t) - e^{-\kappa x}\hat{u}(Be^{-x},t) \\
&= \hat{u}(s,t) - \left(\frac{B}{s}\right)^{\kappa} \hat{u}\left(\frac{B^2}{s}, t\right), \qquad (3.13)
\end{aligned}$$

where we have defined $\hat{u}(s,t)$ as the time-t price of a binary option with payoff function $\pi(s)\mathbb{1}(s > B)$. Hence,

$$\hat{u}(s,t) = \mathrm{E}\left[e^{-r(T-t)}\pi(S_T)\mathbb{1}(S_T > B)|\mathcal{F}_t, S_t = s\right],$$

and formula (3.13) agrees with (3.4) and (3.6) when $t = 0$.

Remark 3.4.1. The method of images approach can also be used to evaluate more sophisticated barrier options. See the book Buchen (2012) for the development of this approach in pricing a variety of exotic options. In Buchen and Konstandatos (2009), the method of images approach was extended to price double knock-out options with exponential boundaries and arbitrary payoffs.

CHAPTER 4

PRICING BARRIER OPTIONS VIA EXPONENTIAL STOPPING: A NEW VALUATION APPROACH

4.1 Introduction

It seems that little can be further achieved in option pricing under the celebrated BS framework. We show an opposite view by presenting a new valuation approach to pricing a large class of barrier options. The pricing of path-dependent options such as barrier options usually requires one to take advantage of the symmetry property of Brownian motions, which is typically expressed by the reflection principle. In addition, when dealing with more complicated barrier options, changing the drift of a Brownian motion from one to another is inevitable, and this is usually done by change of measure using the Girsanov theorem.

In this chapter, we shall price a variety of barrier options using a new method based on exponential stopping, which obviates the explicit needs for the reflection principle and change of measure. The exponential stopping basically replaces the fixed maturity time by an independent exponential random variable, leading to the Laplace transform of the option value with respect to the maturity time. Partially thanks to the memoryless property, this Laplace transform is fairly easy to calculate in an explicit form (See Geman and Yor (1996)). However, we will not invert the Laplace transform, instead we shall manage to show that the exponential stopping directly yields the option value for fixed maturity time. The most exotic case we consider in this chapter is a double knock-out option with exponentially time-varying boundaries and arbitrary payoffs. We refine the path counting technique (See Anderson (1960), Li (1998) and Sidenius (1998) for example) to express the double knock-out event as doubly infinite sums, and our results can be structured to price more exotic variations of barrier options, which will be discussed in detail in Chapter 5. We should point out that all the formulas derived in this chapter using our new method have been available in the literature in some alternative expressions.

Let us describe the organization of the remainder of this chapter. Section 4.2 treats barrier options with a single flat boundary. Section 4.3 treats barrier options with an exponential boundary. Section 4.4 treats a double knock-out option with exponential boundaries, discusses the convergence about one of our major pricing formulas and provides some numerical examples.

4.2 Single-barrier options

We follow the setting for the BS model described at the beginning of Chapter 3. Define the hitting time of $\{S_t\}$ with respect to a barrier B as

$$\tau_B = \inf\left\{t > 0 | S_t = B\right\}.$$

Given an initial asset price $s_0 < B$, the time-0 forward price of an up-and-in option corresponds to the expectation

$$\mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}(\tau_B < T)\right]. \tag{4.1}$$

Depending on whether the terminal asset price is above or below the barrier, the expectation (4.1) can be further written as the sum of the following two terms,

$$\mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}(S_T > B, \tau_B < T)\right] + \mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}(S_T < B, \tau_B < T)\right]. \tag{4.2}$$

Note that every continuous path of the underlying asset price process that starts at $s_0 < B$ and terminates at $S_T > B$ must breach the barrier before maturity, and thus the first expectation in (4.2) simply reduces to

$$\mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}(S_T > B)\right]. \tag{4.3}$$

To determine the second expectation in (4.2), we resort to the exponential stopping of Brownian motion. In particular, let ε_λ denote an exponential random variable with mean $1/\lambda$ and independent of the process $\{S_t\}$. We replace the fixed maturity T by ε_λ and evaluate the expectation

$$\mathrm{E}_{s_0}\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(S_{\varepsilon_\lambda} < B, \tau_B < \varepsilon_\lambda)\right]. \tag{4.4}$$

Conditioning on the stopping time τ_B, we apply the memoryless property of exponential distribution and the strong Markov property, and hence arrive at the equation

$$\mathrm{E}_{s_0}\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(S_{\varepsilon_\lambda} < B, \tau_B < \varepsilon_\lambda)|\mathcal{F}_{\tau_B}, \tau_B\right] = e^{-\lambda\tau_B}\mathrm{E}_B\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(S_{\varepsilon_\lambda} < B)\right].$$

By the law of iterated expectations, (4.4) is rewritten as

$$\begin{aligned}\mathrm{E}_{s_0}\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(S_{\varepsilon_\lambda} < B, \tau_B < \varepsilon_\lambda)\right] &= \mathrm{E}_{s_0}\left[e^{-\lambda\tau_B}\right]\cdot\mathrm{E}_B\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(S_{\varepsilon_\lambda} < B)\right] \\ &= \left(\frac{s_0}{B}\right)^{\theta_\lambda^+}\mathrm{E}_B\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(S_{\varepsilon_\lambda} < B)\right],\end{aligned} \tag{4.5}$$

where θ_λ^+ denotes the positive root of the equation of θ

$$\frac{\sigma^2}{2}\theta^2 + \mu\theta - \lambda = 0. \tag{4.6}$$

The last step is an immediate consequence of Proposition 2.4.1 by setting $b = \ln\frac{B}{s_0}$. Note that the expression $\left(\frac{s_0}{B}\right)^{\theta_\lambda^+}$ in (4.5) depends on the parameter λ. Now we let s_1 be a number such that $s_1 > B$. Then every continuous sample path of the underlying asset price process that starts at s_1 and terminates below the barrier must breach the barrier before maturity. Therefore, we have

$$\begin{aligned}
\mathrm{E}_{s_1}\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(S_{\varepsilon_\lambda} < B)\right] &= \mathrm{E}_{s_1}\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(S_{\varepsilon_\lambda} < B, \tau_B < \varepsilon_\lambda)\right] \\
&= \mathrm{E}_{s_1}\left[e^{-\lambda\tau_B}\right] \cdot \mathrm{E}_B\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(S_{\varepsilon_\lambda} < B)\right] \\
&= \left(\frac{s_1}{B}\right)^{\theta_\lambda^-} \mathrm{E}_B\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(S_{\varepsilon_\lambda} < B)\right], \qquad (4.7)
\end{aligned}$$

where θ_λ^- denotes the negative root of the equation (4.6). By the memoryless property and the Markov property, the derivation of (4.7) is essentially the same as that of (4.5). We specifically choose

$$s_1 = \frac{B^2}{s_0}. \qquad (4.8)$$

Then $s_1 > B$ because we assume $s_0 < B$. It is implied by (4.7) that

$$\mathrm{E}_B\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(S_{\varepsilon_\lambda} < B)\right] = \left(\frac{s_0}{B}\right)^{\theta_\lambda^-} \mathrm{E}_{\frac{B^2}{s_0}}\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(S_{\varepsilon_\lambda} < B)\right].$$

Substituting the equation above back to (4.5) yields

$$\begin{aligned}
\mathrm{E}_{s_0}\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(S_{\varepsilon_\lambda} < B, \tau_B < \varepsilon_\lambda)\right] &= \left(\frac{s_0}{B}\right)^{\theta_\lambda^+ + \theta_\lambda^-} \mathrm{E}_{\frac{B^2}{s_0}}\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(S_{\varepsilon_\lambda} < B)\right] \\
&= \left(\frac{B}{s_0}\right)^{\kappa} \mathrm{E}_{\frac{B^2}{s_0}}\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(S_{\varepsilon_\lambda} < B)\right], \qquad (4.9)
\end{aligned}$$

where

$$\kappa = -\left(\theta_\lambda^+ + \theta_\lambda^-\right) = \frac{2\mu}{\sigma^2}$$

because θ_λ^+ and θ_λ^- are the solutions of the equation (4.6).

Note that the choice of s_1 given by (4.8) makes the expression $\left(\frac{B}{s_0}\right)^{\kappa}$ independent of the parameter λ. Therefore, we claim that the exponential random variable ε_λ can be changed back to a positive fixed time T, and we obtain a formula for the second expectation in (4.2):

$$\mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}(S_T < B, \tau_B < T)\right] = \left(\frac{B}{s_0}\right)^{\kappa} \mathrm{E}_{\frac{B^2}{s_0}}\left[\pi(S_T)\mathbb{1}(S_T < B)\right]. \qquad (4.10)$$

This can be heuristically derived from the fact that the collection of combinations of exponential distributions is weakly dense in the set of all distributions defined on the positive real line (See, for example, Dufresne (2007)). As a consequence, equation (4.9) still holds if ε_λ on both sides is replaced by an arbitrary positive random variable independent of $\{S_t\}$, and of course, by a positive fixed time T as a special case. Combining (4.2), (4.3) and (4.10), we arrive at getting a representation pricing formula for up-and-in options:

$$\mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}(\tau_B < T)\right] = \mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}(S_T > B)\right] + \left(\frac{B}{s_0}\right)^{\kappa} \mathrm{E}_{\frac{B^2}{s_0}}\left[\pi(S_T)\mathbb{1}(S_T < B)\right],$$

where the expectations on the right-hand side are the forward prices of some binary options (all-or-nothing options), which are easy to calculate.

Remark 4.2.1. There is an altenative way to see why equation (4.9) also holds for any positive fixed time. Define an auxiliary function Λ_t such that

$$\Lambda_t = \mathrm{E}_{s_0}\left[\pi(S_t)\mathbb{1}(S_t < B, \tau_B < t)\right] - \left(\frac{B}{s_0}\right)^{\kappa} \mathrm{E}_{\frac{B^2}{s_0}}\left[\pi(S_t)\mathbb{1}(S_t < B)\right].$$

Then equation (4.9) implies that the Laplace transform of Λ_t is zero for every positive λ:

$$\int_0^\infty e^{-\lambda t}\Lambda_t \mathrm{d}t = 0.$$

The Laplace transform is in general not a one-to-one operator, so the function whose Laplace transform is zero may not necessarily be a zero function. Here we want to further assume that the function Λ_t is continuous in t, then one can conclude that $\Lambda_t = 0$ for every positive t, which leads to (4.10). The proof of this claim is a simple exercise in real analysis and will be provided in Section 4.5.1.

For down-and-in options, we do not need to repeat the entire procedure described above; instead, we can simply switch the positions of the process $\{S_t\}$ and the barrier. It then follows that with $s_0 > B$, the time-0 forward price of down-and-in options is given by

$$\mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}(\tau_B < T)\right] = \mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}(S_T < B)\right] + \left(\frac{B}{s_0}\right)^{\kappa} \mathrm{E}_{\frac{B^2}{s_0}}\left[\pi(S_T)\mathbb{1}(S_T > B)\right].$$

The corresponding knock-out options can be valued by the in-out parity. The following theorem provides the prices of all types of single-barrier options.

Theorem 4.2.1. *Given initial asset value s_0, the time-0 forward prices of up-and-in, up-and-out, down-and-in and down-and-out options are respectively given by*

$$V_{\mathrm{ui}} = V_\pi^+(s_0, B) + \left(\frac{B}{s_0}\right)^{\kappa} V_\pi^-\left(\frac{B^2}{s_0}, B\right), \quad s_0 < B, \tag{4.11}$$

$$V_{\mathrm{uo}} = V_\pi^-(s_0, B) - \left(\frac{B}{s_0}\right)^{\kappa} V_\pi^-\left(\frac{B^2}{s_0}, B\right), \quad s_0 < B, \tag{4.12}$$

$$V_{\mathrm{di}} = V_\pi^-(s_0, B) + \left(\frac{B}{s_0}\right)^{\kappa} V_\pi^+\left(\frac{B^2}{s_0}, B\right), \quad s_0 > B, \tag{4.13}$$

$$V_{\mathrm{do}} = V_\pi^+(s_0, B) - \left(\frac{B}{s_0}\right)^{\kappa} V_\pi^+\left(\frac{B^2}{s_0}, B\right), \quad s_0 > B, \tag{4.14}$$

where

$$V_\pi^\pm(s, x) = \mathrm{E}_s\left[\pi(S_T)\mathbb{1}(S_T \gtrless x)\right] \tag{4.15}$$

denote the time-0 forward prices of some binary options and $\kappa = \frac{2\mu}{\sigma^2}$.

Remark 4.2.2. It is worth mentioning that our approach does not use traditional techniques such as the reflection principle and change of probability measure.

Remark 4.2.3. We also remark that Theorem 4.2.1 delivers representation formulas for single-barrier options which are written as combinations of the prices of binary options (all-or-nothing options). These expressions show their advantage in being extended to study the cases of a variety of exotic variations of barrier options, which will be discussed in the next chapter.

Example 4.2.1 (Up-and-in call and put options)**.** We make the formulation a general one by treating an arbitrary payoff function $\pi(s)$. In this example, the explicit solutions of up-and-in call and put will be given. Let $\pi(s) = (s-K)^+$ or $(K-s)^+$ where K is the strike price. We assume the underlying asset does not pay any dividends. To derive the time-0 no-arbitrage prices, we multiply the discount factor e^{-rT} and let the drift term $\mu = r - \frac{1}{2}\sigma^2$ in the formulas in Theorem 4.2.1. The time-0 no-arbitrage price of up-and-in call options is given by

$$V_{\text{uic}} = \begin{cases} d(s_0; K, B) + \left(\dfrac{B}{s_0}\right)^{\kappa} \left[d\left(\dfrac{B^2}{s_0}; K, K\right) - d\left(\dfrac{B^2}{s_0}; K, B\right) \right] & K < B \\ d(s_0; K, K) & K \geq B \end{cases}$$

and the time-0 no-arbitrage price of up-and-in put options is given by

$$V_{\text{uip}} = \begin{cases} \left(\dfrac{B}{s_0}\right)^{\kappa} \left[d\left(\dfrac{B^2}{s_0}; K, K\right) + Ke^{-rT} - \dfrac{B^2}{s_0} \right] & K < B \\ d(s_0; K, K) - d(s_0; K, B) + \left(\dfrac{B}{s_0}\right)^{\kappa} \left[d\left(\dfrac{B^2}{s_0}; K, B\right) + Ke^{-rT} - \dfrac{B^2}{s_0} \right] & K \geq B \end{cases}$$

where

$$d(s; x, y) = s\Phi\left(\frac{\ln\frac{s}{y} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - xe^{-rT}\Phi\left(\frac{\ln\frac{s}{y} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

denotes the value of a gap option with initial asset price s, strike price x and trigger price y, and $\Phi(\cdot)$ is the distribution function of standard normal random variable.

4.3 Single-barrier options with exponential boundary

We can easily extend to the case where the asset price process is restricted by an exponential boundary. Assume the boundary $B_t = Be^{\delta t}$, and thus the time-0 forward price of an up-and-in option is expressed as

$$\mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}\left(\max_{0\leq t\leq T}(S_t - Be^{\delta t}) > 0\right)\right], \quad s_0 < B.$$

Define $\tilde{S}_t = S_t e^{-\delta t}$ and $\tilde{\pi}(s) = \pi(s e^{\delta T})$. One can rewrite the expectation above as

$$\mathrm{E}_{s_0}\left[\tilde{\pi}(\tilde{S}_T)\mathbb{1}\left(\max_{0\le t\le T}(\tilde{S}_t - B) > 0\right)\right], \quad \tilde{S}_0 = s_0 < B.$$

This reduces to the problem of up-and-in options with flat boundary B where the asset price process is $\{\tilde{S}_t\}$ and the payoff function is $\tilde{\pi}(s)$. Applying equation (4.11) in Theorem 4.2.1 yields

$$\begin{aligned}
&\mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}\left(\max_{0\le t\le T}(S_t - Be^{\delta t}) > 0\right)\right] \\
&= V_\pi^+\left(s_0, Be^{\delta T}\right) + \left(\frac{B}{s_0}\right)^{\frac{2(\mu-\delta)}{\sigma^2}} V_\pi^-\left(\frac{B^2}{s_0}, Be^{\delta T}\right).
\end{aligned}$$

The other three types of barrier options can be valued in a similar fashion. We obtain the following corollary.

Corollary 4.3.1. *Given initial asset value s_0, the time-0 forward prices of up-and-in, up-and-out, down-and-in and down-and-out options with the exponential boundary $B_t = Be^{\delta t}$ are respectively given by*

$$V_{\text{eui}} = V_\pi^+\left(s_0, B_T\right) + \left(\frac{B}{s_0}\right)^{\tilde{\kappa}} V_\pi^-\left(\frac{B^2}{s_0}, B_T\right), \quad s_0 < B, \tag{4.16}$$

$$V_{\text{euo}} = V_\pi^-\left(s_0, B_T\right) - \left(\frac{B}{s_0}\right)^{\tilde{\kappa}} V_\pi^-\left(\frac{B^2}{s_0}, B_T\right), \quad s_0 < B, \tag{4.17}$$

$$V_{\text{edi}} = V_\pi^-\left(s_0, B_T\right) + \left(\frac{B}{s_0}\right)^{\tilde{\kappa}} V_\pi^+\left(\frac{B^2}{s_0}, B_T\right), \quad s_0 > B, \tag{4.18}$$

$$V_{\text{edo}} = V_\pi^+\left(s_0, B_T\right) - \left(\frac{B}{s_0}\right)^{\tilde{\kappa}} V_\pi^+\left(\frac{B^2}{s_0}, B_T\right), \quad s_0 > B, \tag{4.19}$$

where we have defined $V_\pi^\pm(s,x) = \mathrm{E}_s\left[\pi(S_T)\mathbb{1}(S_T \gtrless x)\right]$ and $\tilde{\kappa} = \frac{2(\mu-\delta)}{\sigma^2}$.

Remark 4.3.1. As a check, we can let $\delta = 0$ in Corollary 4.3.1 to recover the results in Theorem 4.2.1.

4.4 Double-barrier options with exponential boundaries

We in this section tackle the problem of double-barrier options delimited by two non-parallel curved boundaries that vary exponentially in time. Denote by $U_t = Ue^{\delta_1 t}$ and $L_t = Le^{\delta_2 t}$ the upper boundary and the lower boundary respectively. If $\delta_1 = \delta_2 = 0$, it reduces to the case of two flat boundaries. We assume $Ue^{\delta_1 T} > Le^{\delta_2 T}$ to guarantee that these two boundaries do not intersect before maturity.

A double knock-out option comes into being if the underlying asset price never breaches either barrier prior to maturity. Otherwise, the option expires worthless upon the time of breaching. Therefore, its time-0 forward price is given by

$$\mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}\left(\max_{0\le t\le T}(S_t - Ue^{\delta_1 t}) < 0, \min_{0\le t\le T}(S_t - Le^{\delta_2 t}) > 0\right)\right], \quad L < s_0 < U.$$

Define two hitting times of $\{S_t\}$

$$\tilde{\tau}_U = \inf\{t > 0 | S_t = Ue^{\delta_1 t}\} \quad \text{and} \quad \tilde{\tau}_L = \inf\{t > 0 | S_t = Le^{\delta_2 t}\}.$$

Let $\tilde{\tau}_U \wedge \tilde{\tau}_L$ denote the minimum between $\tilde{\tau}_U$ and $\tilde{\tau}_L$, then the time-0 forward price of double knock-out options can be rewritten as

$$\mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}\left(\tilde{\tau}_U \wedge \tilde{\tau}_L > T\right)\right], \quad L < s_0 < U. \tag{4.20}$$

The evaluation of (4.20) is far less straightforward than that for a single-barrier case because the path-dependent component is characterized by two barriers instead of one. Therefore, the relative order of the process $\{S_t\}$ in breaching the two barriers should be carefully considered. Our approach further modifies the results in Li (1998) and Sidenius (1998) and decomposes the double knock-out event $\tilde{\tau}_U \wedge \tilde{\tau}_L > T$ according to a series of breaching patterns.

4.4.1 Path counting of double knock-out

Definition 4.4.1. Let $\mathcal{B}_0$ denote the sample space of all continuous paths of the underlying asset price process $\{S_t\}$ over the time interval $[0, T]$. We introduce $\{\mathcal{B}_n\}_{n\geq 1}$ and $\{\mathcal{B}_{-n}\}_{n\geq 1}$, two sequences of subsets of $\mathcal{B}_0$. For $S_0 < U$, let $\mathcal{B}_n$ be the event that there exist n time points $0 < t_1 < \cdots < t_k < \cdots < t_n \leq T$, such that $S_{t_k} = U_{t_k}$ when k is odd and $S_{t_k} = L_{t_k}$ when k is even. Analogously, for $S_0 > L$, let $\mathcal{B}_{-n}$ be the event that there exist n time points $0 < t_1 < \cdots < t_k < \cdots < t_n \leq T$, such that $S_{t_k} = L_{t_k}$ when k is odd and $S_{t_k} = U_{t_k}$ when k is even.

Remark 4.4.1. Sidenius (1998) and Li (1998) also introduced analogous concepts as in Defintion 4.4.1; the former treated flat boundaries and the latter treated exponential boundaries.

Figure 4.1 shows possible examples of the events of interest, $\mathcal{B}_{2n}$, $\mathcal{B}_{2n-1}$, $\mathcal{B}_{-2n}$ and $\mathcal{B}_{-(2n-1)}$, for $L < S_0 < U$ and two flat barriers. Obviously, $\mathcal{B}_n$ is the event that the process $\{S_t\}$ alternates breaching the two barriers for n times and starts with an upcrossing; $\mathcal{B}_{-n}$ is the event that the process $\{S_t\}$ alternates breaching the two barriers for n times and starts with a downcrossing. The following nesting relations can be obtained by inspection:

$$\mathcal{B}_0 \supset \mathcal{B}_n \supset \mathcal{B}_{n+1}, \quad \mathcal{B}_0 \supset \mathcal{B}_{-n} \supset \mathcal{B}_{-(n+1)}, \quad n = 1, 2, \cdots, \tag{4.21}$$

and when $L < S_0 < U$,

$$\mathcal{B}_n \supset \mathcal{B}_{-(n+1)}, \quad \mathcal{B}_{-n} \supset \mathcal{B}_{n+1}, \quad n = 1, 2, \cdots. \tag{4.22}$$

Using the concept of path counting, we obtain the following identities, which serve as the basic ingredients in deriving the prices of double-barrier options.

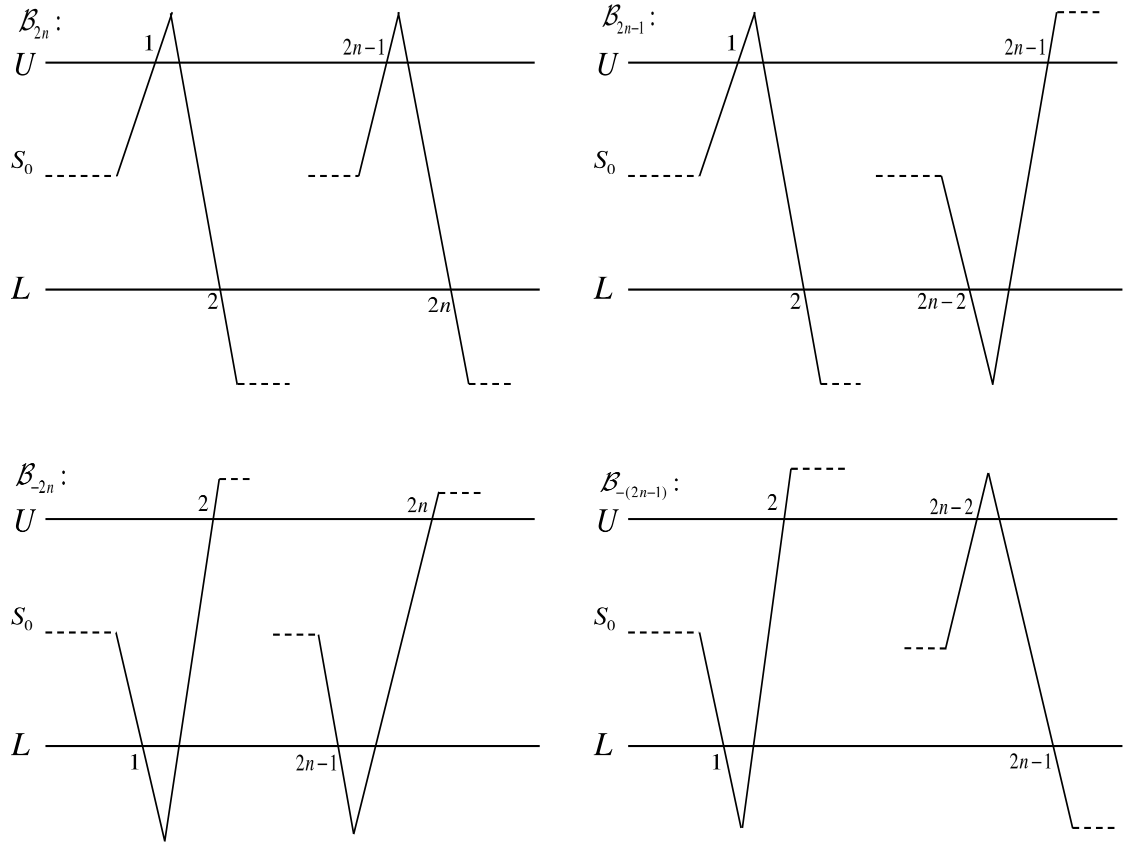

Figure 4.1: Examples of $\mathcal{B}_{\pm n}, n \geq 1$.

Proposition 4.4.1. *When $L < S_0 < U$, almost surely,*

$$\mathbb{1}(\tilde{\tau}_U < \tilde{\tau}_L \wedge T) = \sum_{n=1}^{\infty} \mathbb{1}_{\mathcal{B}_{2n-1}} - \sum_{n=1}^{\infty} \mathbb{1}_{\mathcal{B}_{-2n}}, \tag{4.23}$$

$$\mathbb{1}(\tilde{\tau}_L < \tilde{\tau}_U \wedge T) = \sum_{n=1}^{\infty} \mathbb{1}_{\mathcal{B}_{-(2n-1)}} - \sum_{n=1}^{\infty} \mathbb{1}_{\mathcal{B}_{2n}}. \tag{4.24}$$

As a result,

$$\mathbb{1}(\tilde{\tau}_U \wedge \tilde{\tau}_L > T) = \sum_{n=-\infty}^{\infty} \mathbb{1}_{\mathcal{B}_{2n}} - \sum_{n=-\infty}^{\infty} \mathbb{1}_{\mathcal{B}_{2n-1}}. \tag{4.25}$$

Remark 4.4.2. The intuition behind Proposition 4.4.1 is not difficult to understand if one uses the inclusion-exclusion reasoning and notices the nesting relations (4.22). For example, the left-hand side of identity (4.23) means that the process $\{S_t\}$ hits the upper barrier first and it occurs before the maturity time T. It is natural to include the event $\mathcal{B}_1$. But $\mathcal{B}_1$ may contain the possibility that the lower barrier is hit first, which should be excluded. Thus we subtract $\mathcal{B}_{-2}$. But $\mathcal{B}_{-2}$ also contains the possibility that the upper barrier is hit first and should be included. Thus we add back $\mathcal{B}_3$. Note that $\mathcal{B}_1 \supset \mathcal{B}_{-2} \supset \mathcal{B}_3 \supset \mathcal{B}_{-4} \supset \cdots$. We continue this procedure for indefinitely many times and obtain

$$\mathbb{1}(\tilde{\tau}_U < \tilde{\tau}_L \wedge T) = \mathbb{1}_{\mathcal{B}_1} - \mathbb{1}_{\mathcal{B}_{-2}} + \mathbb{1}_{\mathcal{B}_3} - \mathbb{1}_{\mathcal{B}_{-4}} + \cdots = \sum_{n=1}^{\infty} \mathbb{1}_{\mathcal{B}_{2n-1}} - \sum_{n=1}^{\infty} \mathbb{1}_{\mathcal{B}_{-2n}}.$$

Switching the two barriers in (4.23) yields the second identity (4.24).

Remark 4.4.3. The utilization of path counting or similar techniques can be traced back to, for example, Anderson (1960) which obtained some fundamental results such as the probabilities of a standard Brownian motion hitting one linear boundary before hitting the other one. To derive the corresponding probabilities for a geometric Brownian motion, one can use the Girsanov theorem to perform change of measure and take an exponential transformation of the Brownian motion. For a detailed discussion about the case of geometric Brownian motion, we refer to the proof of Theorem 2.1 in Kunitomo and Ikeda (1992).

It is worthwhile to point out that the right-hand sides of (4.23), (4.24) and (4.25) are all well-defined because the infinite sums on the right-hand sides of the identities are all finite almost surely. This can be guaranteed by the following proposition.

Lemma 4.4.2. *For every outcome $\omega \in \mathcal{B}_0$, there exists an integer N such that the given outcome is not in $\mathcal{B}_{\pm n}$ for all $n \geq N$.*

Lemma 4.4.2 follows from the result given in Sidenius (1998), which considered two flat barriers. For the sake of completeness, we also give an outline of the proof here.

Proof of Lemma 4.4.2. Without loss of generality, we show the existence of N for $\mathcal{B}_n, n \geq 1$. For a given outcome ω in $\mathcal{B}_0$, it induces a continuous function $S_t(\omega)$ over the finite interval $[0, T]$. If the conclusion is not true, then there exist two increasing sequences of time points $\{t_n^+\}$ and $\{t_n^-\}, n = 1, 2, ...,$ in $[0, T]$ such that

$$S_{t_n^+}(\omega) = U_{t_n^+}, \quad S_{t_n^-}(\omega) = L_{t_n^-}, \quad t_n^+ < t_n^- < t_{n+1}^+. \tag{4.26}$$

The sequence $\{t_n^+\}$ on the compact set $[0, T]$ guarantees a subsequence that coverges to a time point, say t^+. By the continuity of the functions $S_t(\omega)$ and U_t, we have $S_{t^+}(\omega) = U_{t^+}$. However, for every neighborhood of t^+, there will be infinite many points of $\{t_n^+\}$ in it according to the definition of limiting point. Thus, by (4.26), the given neighborhood will also contain a point, say t^- such that $S_{t^-}(\omega) = L_{t^-}$, which violates the continuity of $S_t(\omega)$. $\square$

Now, let us rigorously demonstrate Proposition 4.4.1. We shall present two different proofs. The first one is based on Lemma 4.4.2.

First proof of Proposition 4.4.1. We only prove (4.23), and (4.24) is readily obtained by switching the two barriers. Note that

$$\mathbb{1}(\tilde{\tau}_U \wedge \tilde{\tau}_L > T) = 1 - \mathbb{1}(\tilde{\tau}_U < \tilde{\tau}_L \wedge T) - \mathbb{1}(\tilde{\tau}_L < \tilde{\tau}_U \wedge T),$$

and $\mathbb{1}_{\mathcal{B}_0} = 1$ almost surely, then identity (4.25) follows immediately. It suffices to show that both sides of (4.23) are identical for every outcome $\omega \in \mathcal{B}_0$. We basically have two cases.

The first case: When the left-hand side of (4.23) is one, either the event $\{\tilde{\tau}_U < T < \tilde{\tau}_L\}$ or the event $\{\tilde{\tau}_U < \tilde{\tau}_L < T\}$ occurs. If $\tilde{\tau}_U < T < \tilde{\tau}_L$, then the sample path stays above the lower barrier before maturity, and hence, each indicator function on the right-hand side of (4.23) vanishes except that $\mathbb{1}_{\mathcal{B}_1} = 1$. If $\tilde{\tau}_U < \tilde{\tau}_L < T$, we apply Lemma 4.4.2. There exists a minimal integer N such that $\mathbb{1}_{\mathcal{B}_{\pm n}} = 0$ when $n > N$. Note that $N \geq 2$ because the sample path breaches both barriers at least once before maturity. Assume N is an odd number for the given sample path and in particular, $N = 2M + 1$ for some $M \geq 1$. Therefore, the right-hand side of (4.23) reduces to a finite sum

$$\sum_{n=1}^{M+1} \mathbb{1}_{\mathcal{B}_{2n-1}} - \sum_{n=1}^{M} \mathbb{1}_{\mathcal{B}_{-2n}}. \tag{4.27}$$

The minimality of N ensures that at least one of $\mathbb{1}_{\mathcal{B}_N}$ and $\mathbb{1}_{\mathcal{B}_{-N}}$ is equal to one. In fact, one can show that for the given sample path such that $\tilde{\tau}_U < \tilde{\tau}_L < T$, $\mathbb{1}_{\mathcal{B}_N} = 1$ and $\mathbb{1}_{\mathcal{B}_{-N}} = 0$. Suppose $\mathbb{1}_{\mathcal{B}_{-N}} = 1$, for example. Since the sample path breaches the upper barrier first, then $\mathbb{1}_{\mathcal{B}_{N+1}} = 1$, which violates the minimality of N. The nesting relations (4.21) and (4.22) imply that

$$\mathbb{1}_{\mathcal{B}_n} \geq \mathbb{1}_{\mathcal{B}_N} = 1, \quad 1 \leq n \leq N - 1.$$

Therefore, the expression (4.27) becomes $(M + 1) - M = 1$, which gives the right-hand side of (4.23). When N is an even number, the argument follows similarly.

The second case: On the other hand, when the left-hand side of (4.23) is zero, either the event $\tilde{\tau}_U \wedge \tilde{\tau}_L > T$ occurs or the event $\tilde{\tau}_L < \tilde{\tau}_U \wedge T$ occurs. If $\tilde{\tau}_U \wedge \tilde{\tau}_L > T$, then the two barriers are never breached before maturity time T, and hence each indicator function on the right-hand side of (4.23) is zero. If $\tilde{\tau}_L < \tilde{\tau}_U \wedge T$, one can repeat those steps for the first case. □

Second proof of Proposition 4.4.1. This proof is simpler than the first one and is derived from the following two recursion formulas. When $L < S_0 < U$, the following identities hold almost surely for $n \geq 0$:

$$\mathbb{1}_{\mathcal{B}_{n+1}} = \mathbb{1}_{\mathcal{B}_{-n}\cap\mathcal{B}^+} + \mathbb{1}_{\mathcal{B}_{n+1}\cap\mathcal{B}^-}, \tag{4.28}$$

$$\mathbb{1}_{\mathcal{B}_{-(n+1)}} = \mathbb{1}_{\mathcal{B}_{n}\cap\mathcal{B}^-} + \mathbb{1}_{\mathcal{B}_{-(n+1)}\cap\mathcal{B}^+}, \tag{4.29}$$

where we define $\mathcal{B}^+ = \{\tilde{\tau}_U < \tilde{\tau}_L \wedge T\}$ and $\mathcal{B}^- = \{\tilde{\tau}_L < \tilde{\tau}_U \wedge T\}$. We only show (4.28), and identity (4.29) can be derived in a similar fashion. By Definition 4.4.1, the event $\mathcal{B}_{n+1}$ can be decomposed as the union of two disjoint events, $\mathcal{B}_{-n}\cap\mathcal{B}^+$ and $\mathcal{B}_{n+1}\cap\mathcal{B}^-$, depending on which barrier is breached first before time T. Apparently, for every ω in $\mathcal{B}_0$, if the sample path $\{S_t(\omega), 0 \leq t \leq T\}$ breaches the upper barrier first, then w is in $\mathcal{B}_{-n}\cap\mathcal{B}^+$; otherwise, if the sample path $\{S_t(\omega), 0 \leq t \leq T\}$ breaches the lower barrier first, then ω is in $\mathcal{B}_{n+1}\cap\mathcal{B}^-$. Now, to evaluate $\mathbb{1}_{\mathcal{B}^+}$ and $\mathbb{1}_{\mathcal{B}^-}$, one simply recursively apply (4.28) and (4.29) starting at $n = 0$. For example, we have

$$\begin{aligned}
\mathbb{1}_{\mathcal{B}^+} &= \mathbb{1}_{\mathcal{B}_1} - \mathbb{1}_{\mathcal{B}_1\cap\mathcal{B}^-} \\
&= \mathbb{1}_{\mathcal{B}_1} - (\mathbb{1}_{\mathcal{B}_{-2}} - \mathbb{1}_{\mathcal{B}_{-2}\cap\mathcal{B}^+}) \\
&= \mathbb{1}_{\mathcal{B}_1} - \mathbb{1}_{\mathcal{B}_{-2}} + \mathbb{1}_{\mathcal{B}_3} - \mathbb{1}_{\mathcal{B}_3\cap\mathcal{B}^-} \\
&= \cdots \\
&= \sum_{n=1}^{\infty} \mathbb{1}_{\mathcal{B}_{2n-1}} - \sum_{n=1}^{\infty} \mathbb{1}_{\mathcal{B}_{-2n}}.
\end{aligned}$$

The indicator $\mathbb{1}_{\mathcal{B}^-}$ can be evaluated in a similar fashion. □

4.4.2 Pricing formula for double knock-out options based on exponential stopping

Identity (4.25) demonstrates how double knock-out options can be hedged and valued by a series of options that are activated given the two barriers are breached in certain patterns. We shall show that each such option can reduce to a combination of plain vanilla options, and as a consequence, we represent the price of double knock-out options in terms of the prices of a combination of plain vanilla options.

Define $B_0^\pi(s) = \mathrm{E}_s\left[\pi(S_T)\right]$ and for $n \geq 1$,

$$B_n^\pi(s) = \mathrm{E}_s\left[\pi(S_T)\mathbb{1}_{\mathcal{B}_n}\right], \quad s < U, \tag{4.30}$$

$$B_{-n}^\pi(s) = \mathrm{E}_s\left[\pi(S_T)\mathbb{1}_{\mathcal{B}_{-n}}\right], \quad s > L. \tag{4.31}$$

The recursion formulas for $B_n^\pi(s_0)$ and $B_{-n}^\pi(s_0)$ are presented below.

Lemma 4.4.3. *For $n \geq 2$ and $L < s_0 < U$,*

$$B_n^\pi(s_0) = \left(\frac{U}{s_0}\right)^{\kappa_1} B_{-(n-1)}^\pi\left(\frac{U^2}{s_0}\right), \tag{4.32}$$

$$B_{-n}^\pi(s_0) = \left(\frac{L}{s_0}\right)^{\kappa_2} B_{n-1}^\pi\left(\frac{L^2}{s_0}\right), \tag{4.33}$$

where $\kappa_1 = \frac{2(\mu-\delta_1)}{\sigma^2}$ and $\kappa_2 = \frac{2(\mu-\delta_2)}{\sigma^2}$.

We shall later give a proof of Lemma 4.4.3 via the exponential stopping of Brownian motion, which does not require the traditional techniques including the reflection principle and change of probability measure. For an alternative proof that applies these techniques, one can directly read Section 4.5.2.

Now let us combine Lemma 4.4.3 and identity (4.25) in Proposition 4.4.1 to derive the pricing formula for the double knock-out options, which leads to the following theorem.

Theorem 4.4.4. *Given initial asset value s_0 where $L < s_0 < U$, the time-0 forward price of double knock-out options with exponential boundaries $U_t = Ue^{\delta_1 t}$ and $L_t = Le^{\delta_2 t}$ is given by the doubly infinite sum*

$$V_{\text{edko}} = \sum_{n=-\infty}^{\infty} \left(\frac{s_0\beta^n}{L}\right)^{\gamma_n} \beta^{n\kappa_2} \left[V_\pi^*\left(s_0\beta^{2n}\right) - \left(\frac{L}{s_0\beta^{2n}}\right)^{\kappa_2} V_\pi^*\left(\frac{L^2}{s_0\beta^{2n}}\right)\right], \tag{4.34}$$

where $\beta = \frac{L}{U}, \gamma_n = n(\kappa_2 - \kappa_1), \kappa_1 = \frac{2(\mu-\delta_1)}{\sigma^2}, \kappa_2 = \frac{2(\mu-\delta_2)}{\sigma^2}$ and

$$V_\pi^*(s) = \mathrm{E}_s\left[\pi(S_T)\mathbb{1}(L_T < S_T < U_T)\right] \tag{4.35}$$

denotes the time-0 forward price of a binary option.

Proof of Theorem 4.4.4. Applying inductions of (4.32) and (4.33) yields, for $n \geq 1$,

$$B^\pi_{2n}(s_0) = \left(\frac{s_0 L^{n-1}}{U^n}\right)^{n(\kappa_2-\kappa_1)-\kappa_2} B^\pi_{-1}\left(\frac{U^{2n}}{s_0 L^{2n-2}}\right), \tag{4.36}$$

$$B^\pi_{2n-1}(s_0) = \left(\frac{s_0 L^{n-2}}{U^{n-1}}\right)^{(n-1)(\kappa_2-\kappa_1)+\kappa_2} \left(\frac{L}{s_0}\right)^{\kappa_2} B^\pi_1\left(\frac{s_0 L^{2n-2}}{U^{2n-2}}\right). \tag{4.37}$$

The expressions of $B^\pi_{-2n}(s_0)$ and $B^\pi_{-(2n-1)}(s_0)$ can be directly obtained by switching the two barriers in (4.36) and (4.37), respectively. Hence,

$$B^\pi_{-2n}(s_0) = \left(\frac{s_0 U^{n-1}}{L^n}\right)^{n(\kappa_1-\kappa_2)-\kappa_1} B^\pi_1\left(\frac{L^{2n}}{s_0 U^{2n-2}}\right), \tag{4.38}$$

$$B^\pi_{-(2n-1)}(s_0) = \left(\frac{s_0 U^{n-2}}{L^{n-1}}\right)^{(n-1)(\kappa_1-\kappa_2)+\kappa_1} \left(\frac{U}{s_0}\right)^{\kappa_1} B^\pi_{-1}\left(\frac{s_0 U^{2n-2}}{L^{2n-2}}\right). \tag{4.39}$$

One can notice that when $L < s < U$, $B^\pi_1(s)$ represents the forward price of an up-and-in option with the upper barrier $U_t = Ue^{\delta_1 t}$ and $B^\pi_{-1}(s)$ represents the forward price of a down-and-in option with the lower barrier $L_t = Le^{\delta_2 t}$. As an immediate result of Corollary 4.3.1, we have

$$B^\pi_1(s) = V^+_\pi\left(s, U_T\right) + \left(\frac{U}{s}\right)^{\kappa_1} V^-_\pi\left(\frac{U^2}{s}, U_T\right), \quad s < U, \tag{4.40}$$

$$B^\pi_{-1}(s) = V^-_\pi\left(s, L_T\right) + \left(\frac{L}{s}\right)^{\kappa_2} V^+_\pi\left(\frac{L^2}{s}, L_T\right), \quad s > L. \tag{4.41}$$

Let us define $\hat{\pi}(s)$ such that

$$\hat{\pi}(s) = \pi(s)\mathbb{1}(L_T < s < U_T).$$

The double knock-out event $\tilde{\tau}_U \wedge \tilde{\tau}_L > T$ implies that the terminal value of the asset price must be in the range (L_T, U_T), otherwise the option will be knocked out and the payoff becomes zero. Hence, we have

$$\pi(S_T)\mathbb{1}(\tilde{\tau}_U \wedge \tilde{\tau}_L > T) = \hat{\pi}(S_T)\mathbb{1}(\tilde{\tau}_U \wedge \tilde{\tau}_L > T), \tag{4.42}$$

and we can feel free to replace $\pi(s)$ by $\hat{\pi}(s)$ in all equations (4.36) to (4.41). Note that when $\pi(s)$ is replaced by $\hat{\pi}(s)$, the first terms on the right-hand sides of (4.40) and (4.41) vanish. Combining equations (4.36) to (4.41) leads to

$$B^{\hat{\pi}}_{2n}(s_0) = \left(\frac{s_0 L^{n-1}}{U^n}\right)^{n(\kappa_2-\kappa_1)+\kappa_2} \left(\frac{L}{s_0}\right)^{\kappa_2} V^*_\pi\left(\frac{s_0 L^{2n}}{U^{2n}}\right), \tag{4.43}$$

$$B^{\hat{\pi}}_{2n-1}(s_0) = \left(\frac{s_0 L^{n-1}}{U^n}\right)^{n(\kappa_2-\kappa_1)-\kappa_2} V^*_\pi\left(\frac{U^{2n}}{s_0 L^{2n-2}}\right), \tag{4.44}$$

$$B^{\hat{\pi}}_{-2n}(s_0) = \left(\frac{s_0 U^{n-1}}{L^n}\right)^{n(\kappa_1-\kappa_2)+\kappa_1} \left(\frac{U}{s_0}\right)^{\kappa_1} V^*_\pi\left(\frac{s_0 U^{2n}}{L^{2n}}\right), \tag{4.45}$$

$$B^{\hat{\pi}}_{-(2n-1)}(s_0) = \left(\frac{s_0 U^{n-1}}{L^n}\right)^{n(\kappa_1-\kappa_2)-\kappa_1} V^*_\pi\left(\frac{L^{2n}}{s_0 U^{2n-2}}\right). \tag{4.46}$$

By identity (4.42) and the path counting (4.25) in Proposition 4.4.1,

$$\begin{aligned} \mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}(\tilde{\tau}_U \wedge \tilde{\tau}_L > T)\right] &= \mathrm{E}_{s_0}\left[\hat{\pi}(S_T)\mathbb{1}(\tilde{\tau}_U \wedge \tilde{\tau}_L > T)\right] \\ &= \sum_{n=-\infty}^{\infty} B^{\hat{\pi}}_{2n}(s_0) - \sum_{n=-\infty}^{\infty} B^{\hat{\pi}}_{2n-1}(s_0). \end{aligned}$$

Here, the order of expectation and summation can be exchanged due to the fact that the corresponding indicator functions and the payoff function are all non-negative. We then substitute (4.43) to (4.46) into the last equation and formula (4.34) follows after some simple rearrangements. $\square$

In Theorem 4.4.4, the price of double knock-out options is expressed as doubly infinite sums of the prices of path-independent options. This formula was first derived by Kunitomo and Ikeda (1992) for call and put options. Once the path-independent options are valued, the representation formula (4.34) becomes explicit. In fact, closed-form solutions of the binary option price $V_\pi^*(s)$ are available for most payoff types. Due to the rapid decay of normal probabilities, we shall explain that the doubly infinite series converge under some mild conditions such as boundedness of the payoff function over the finite interval (L_T, U_T).

Buchen and Konstandatos (2009) also discussed the pricing of double knock-out options with exponential boundaries and arbitrary payoffs, and they used the method of images approach to derive a pricing formula equivalent to (4.34). When the payoff is of call or put type, our formula (4.34) reproduces the results obtained in Kunitomo and Ikeda (1992). The well-known formula for double knock-out options with two flat boundaries is clearly contained in Theorem 4.4.4 as a special case if we let $\delta_1 = \delta_2 = 0$. Then $\kappa_1 = \kappa_2 = \kappa = \frac{2\mu}{\sigma^2}, \gamma_n = 0$ and the following corollary is obtained.

Corollary 4.4.5. *Given initial asset value s_0 where $L < s_0 < U$, the time-0 forward price of double knock-out options with flat boundaries U and L is given by the doubly infinite sum*

$$V_{\text{dko}} = \sum_{n=-\infty}^{\infty} \beta^{n\kappa} \left[\bar{V}_\pi \left(s_0 \beta^{2n} \right) - \left(\frac{L}{s_0 \beta^{2n}} \right)^{\kappa} \bar{V}_\pi \left(\frac{L^2}{s_0 \beta^{2n}} \right) \right], \tag{4.47}$$

where $\beta = \frac{L}{U}, \kappa = \frac{2\mu}{\sigma^2}$ and $\bar{V}_\pi(s) = \mathrm{E}_s \left[\pi(S_T) \mathbb{1}(L < S_T < U) \right]$.

We shall show in Section 4.5.3 a shortcut to get (4.47), which obviates the need for Theorem 4.4.4 or path counting like (4.25). The proof will stand on its own and basically utilizes the geometric expansion of the barrier option value after we take the exponential stopping of the asset price process $\{S_t\}$. Unfortunately, this approach fails when the two boundaries are non-flat and exponentially curved with different curvatures ($\delta_1 \neq \delta_2$).

Example 4.4.1 (Double knock-out call and put options)**.** We first observe the following relations

$$V_\pi^*(s) = V_\pi^-(s, U_T) - V_\pi^-(s, L_T) = V_\pi^+(s, L_T) - V_\pi^+(s, U_T), \tag{4.48}$$

where $V_\pi^*(s)$ is given by (4.35) and we have defined $V_\pi^\pm(s, x) = \mathrm{E}_s\left[\pi(S_T)\mathbb{1}(S_T \gtrless x)\right]$ when studying single-barrier options. Let us consider call and put options for the pricing formula (4.34), where $\pi(s) = (s - K)^+$ or $\pi(s) = (K - s)^+$ for a strike price K which is assumed to be between L_T and U_T. We use relations in (4.48) to calculate the binary option price $V_\pi^*(s)$. For example, if $\pi(s) = (K-s)^+$, $V_\pi^-(s, U_T)$ reduces to the forward price of a K-strike put and $V_\pi^-(s, L_T)$ reduces to the forward price of a K- strike gap put with trigger price L_T. Similar argument follows if $\pi(s) = (s - K)^+$. We shall not write down the explicit solutions of the double knock-out call and put since they are too lengthy. Interested readers can refer to Kunitomo and Ikeda (1992) for the corresponding explicit formulas. These examples will be numerically implemented in Section 4.4.3.

Now let us go back to prove Lemma 4.4.3 based on the exponential stopping of Brownian motions.

Proof of Lemma 4.4.3. We only demonstrate (4.32), and (4.33) follows by switching the two barriers. Without loss of generality, it is sufficient to deal with one flat boundary and one curved boundary. Let $\delta_1 = 0$ in (4.32), in which case, $\kappa_1 = \frac{2\mu}{\sigma^2}$ and $\mathcal{B}_n$ becomes the event associated with an upper barrier $U_t = U$ and a lower barrier

$L_t = Le^{\delta_2 t}$. In light of the idea used in the derivation for single-barrier options, we replace the fixed maturity time T by an exponential random variable ε_λ, which is independent of the process $\{S_t\}$ and has mean $1/\lambda$. Denote the corresponding event by $\hat{\mathcal{B}}_n$ when the maturity time is ε_λ. For notional convenience, we write

$$\mathrm{E}\left[Y \mathbb{1}_{\mathcal{A}}\right] = \mathrm{E}\left[Y; \mathcal{A}\right]$$

for a random variable Y and an event $\mathcal{A}$. Define

$$\hat{B}_n^\pi(s) = \mathrm{E}_s\left[\pi(S_{\varepsilon_\lambda}); \hat{\mathcal{B}}_n, \varepsilon_\lambda \leq T\right], \quad n \geq 2,$$

The event $\varepsilon_\lambda \leq T$ is included to avoid the situation where the two barriers intersect before the maturity time T. We aim at showing

$$\hat{B}_n^\pi(s_0) = \left(\frac{U}{s_0}\right)^{\kappa_1} \hat{B}_{-(n-1)}^\pi\left(\frac{U^2}{s_0}\right) \tag{4.49}$$

with $\kappa_1 = \frac{2\mu}{\sigma^2}$. By definition, the event $\hat{\mathcal{B}}_n, n \geq 2$, implies that the process $\{S_t\}$ must breach the upper barrier at least once prior to maturity. Hence, we have

$$\hat{B}_n^\pi(s_0) = \mathrm{E}_{s_0}\left[\pi(S_{\varepsilon_\lambda}); \hat{\mathcal{B}}_n, \varepsilon_\lambda \leq T, \tau_U < \varepsilon_\lambda\right], \tag{4.50}$$

where τ_U is the hitting time. Because of the memoryless property of ε_λ and the strong Markov property,

$$\begin{aligned}
&\mathrm{E}_{s_0}\left[\pi(S_{\varepsilon_\lambda}); \hat{\mathcal{B}}_n, \varepsilon_\lambda \leq T, \tau_U < \varepsilon_\lambda \middle| \mathcal{F}_{\tau_U}, \tau_U\right] \\
&= e^{-\lambda\tau_U}\mathrm{E}_{s_0}\left[\pi(S_{\varepsilon_\lambda}); \hat{\mathcal{B}}_n, \varepsilon_\lambda \leq T \middle| \mathcal{F}_{\tau_U}, \tau_U, \tau_U < \varepsilon_\lambda\right] \\
&= e^{-\lambda\tau_U}\mathrm{E}_U\left[\pi(S_{\varepsilon_\lambda}); \hat{\mathcal{B}}_{-(n-1)}^{(\tau_U)}, \varepsilon_\lambda \leq T - \tau_U \middle| \tau_U\right],
\end{aligned} \tag{4.51}$$

where $\hat{\mathcal{B}}_n^{(t)}, t \geq 0$, is defined as the event associated with a lower barrier starting at the level $Le^{\delta_2 t}$. When $t = 0$, the event $\hat{\mathcal{B}}_n^{(t)}$ simply reduces to $\hat{\mathcal{B}}_n$. See Figure 4.2 as a visual aid to understand (4.51).

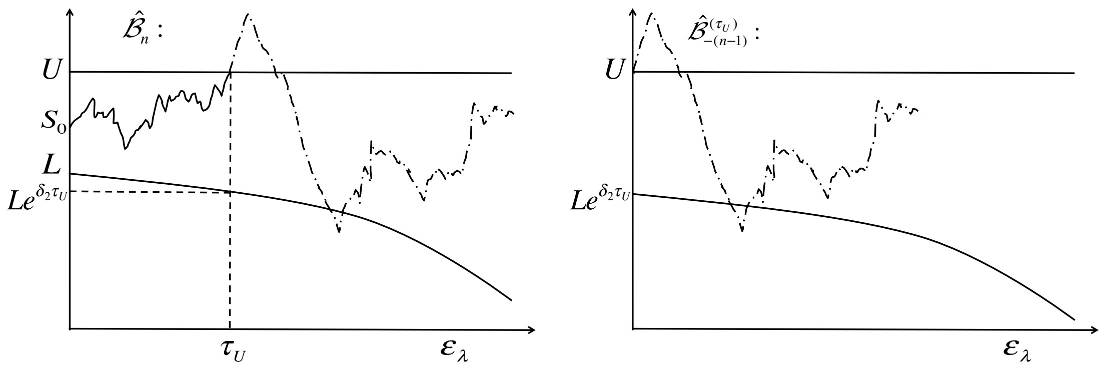

Figure 4.2: An illustration of equation (4.51)

Now we introduce an auxiliary function Δ_t such that

$$\Delta_t = \mathrm{E}_U \left[\pi(S_{\varepsilon_\lambda}); \hat{\mathcal{B}}^{(t)}_{-(n-1)}, \varepsilon_\lambda \le T - t\right], \quad t \ge 0.$$

Then (4.51) can be expressed as $e^{-\lambda\tau_U}\Delta_{\tau_U}$, and by the law of iterated expectations as well as the definition (4.50),

$$\hat{B}^{\pi}_{n}(s_0) = \mathrm{E}_{s_0}\left[e^{-\lambda\tau_U}\Delta_{\tau_U}\right]. \tag{4.52}$$

Now let us evaluate the right-hand side of (4.49). Let $s_1 = \frac{U^2}{s_0}$. Then $s_1 > U$ because $s_0 < U$. Given $S_0 = s_1$, $\hat{\mathcal{B}}_{-(n-1)}, n \ge 2$, implies that the process $\{S_t\}$ must breach the upper barrier from above prior to maturity. It then follows that

$$\begin{aligned}
\hat{B}^{\pi}_{-(n-1)}(s_1) &= \mathrm{E}_{s_1}\left[\pi(S_{\varepsilon_\lambda}); \hat{\mathcal{B}}_{-(n-1)}, \varepsilon_\lambda \le T\right] \\
&= \mathrm{E}_{s_1}\left[\pi(S_{\varepsilon_\lambda}); \hat{\mathcal{B}}_{-(n-1)}, \varepsilon_\lambda \le T, \tau_U < \varepsilon_\lambda\right] \\
&= \mathrm{E}_{s_1}\left[e^{-\lambda\tau_U}\Delta_{\tau_U}\right].
\end{aligned}$$

In the last step we also apply the memoryless property of ε_λ and the strong Markov property. Therefore, in terms of Δ_t, equation (4.49) is equivalent to

$$\mathrm{E}_{s_0}\left[e^{-\lambda\tau_U}\Delta_{\tau_U}\right] = \left(\frac{U}{s_0}\right)^{\kappa_1} \mathrm{E}_{s_1}\left[e^{-\lambda\tau_U}\Delta_{\tau_U}\right]. \tag{4.53}$$

It is not difficult to verify the equation above. Note that the Laplace transform of the hitting time τ_U is given in Proposition 2.4.1. Hence, we have

$$\mathrm{Pr}_{s_0}\left(\tau_U < \varepsilon_\lambda\right) = \mathrm{E}_{s_0}\left[e^{-\lambda\tau_U}\right] = \left(\frac{s_0}{U}\right)^{\theta_\lambda^+},$$
$$\mathrm{Pr}_{s_1}\left(\tau_U < \varepsilon_\lambda\right) = \mathrm{E}_{s_1}\left[e^{-\lambda\tau_U}\right] = \left(\frac{s_1}{U}\right)^{\theta_\lambda^-} = \left(\frac{U}{s_0}\right)^{\theta_\lambda^-}.$$

Because $\theta_\lambda^+ + \theta_\lambda^- = -\frac{2\mu}{\sigma^2} = -\kappa_1$, it is obvious that

$$\mathrm{Pr}_{s_0}\left(\tau_U < \varepsilon_\lambda\right) = \left(\frac{U}{s_0}\right)^{\kappa_1}\mathrm{Pr}_{s_1}\left(\tau_U < \varepsilon_\lambda\right).$$

The coefficient κ_1 is independent of the parameter λ. Because the class of combinations of exponential distributions is weakly dense in the set of all distributions defined on the positive real line, the equation above still holds when ε_λ is replaced by a positive time t. Hence,

$$\mathrm{Pr}_{s_0}\left(\tau_U < t\right) = \left(\frac{U}{s_0}\right)^{\kappa_1}\mathrm{Pr}_{s_1}\left(\tau_U < t\right), \quad t \geq 0. \tag{4.54}$$

Taking derivatives on both sides of (4.54) with respect to t, one can see that the density functions of τ_U given $S_0 = s_0$ and s_1 satisfy the same equation

$$f_{\tau_U}(t; s_0) = \left(\frac{U}{s_0}\right)^{\kappa_1} f_{\tau_U}(t; s_1), \quad t \geq 0.$$

Thus, (4.53), and equivalently (4.49) are readily obtained. Again, we can change ε_λ back to fixed T in (4.49) and this yields equation (4.32). The general case where $\delta_1 \neq 0$ is simply a trivial extension by considering $\tilde{S}_t = S_t e^{-\delta_1 t}$, $\tilde{\pi}(s) = \pi\left(se^{\delta_1 T}\right)$, an upper barrier $U_t = U$ and a lower barrier $L_t = Le^{(\delta_2-\delta_1)t}$. $\square$

Our results can be used to recover the well-known expressions of some important densities and probabilities.

Corollary 4.4.6. *Define density functions* $f(x;t), f^+(x;t)$ *and* $f^-(x;t)$ *by*

$$f(x;t)\mathrm{d}x = \Pr_{s_0}(S_t \in \mathrm{d}x, \tilde{\tau}_U \wedge \tilde{\tau}_L > t),$$

$$f^+(x;t)\mathrm{d}x = \Pr_{s_0}(S_t \in \mathrm{d}x, \tilde{\tau}_U < \tilde{\tau}_L \wedge t),$$

$$f^-(x;t)\mathrm{d}x = \Pr_{s_0}(S_t \in \mathrm{d}x, \tilde{\tau}_L < \tilde{\tau}_U \wedge t),$$

for some $t > 0$ *where* $\Pr_{s_0}(\cdot)$ *denotes some probability given* $S_0 = s_0$*. Then we have*

$$f(x;t) = \sum_{n=-\infty}^{\infty} [a_n(x,t) - b_n(x,t)] \qquad L_t < x < U_t \tag{4.55}$$

$$f^+(x;t) = \begin{cases} \sum_{n=1}^{\infty} \left[a_{n-1}(x,t) - b_{-(n-1)}(x,t)\right] & x > U_t \\ \sum_{n=1}^{\infty} [b_n(x,t) - a_{-n}(x,t)] & 0 < x < U_t \end{cases} \tag{4.56}$$

$$f^-(x;t) = \begin{cases} \sum_{n=1}^{\infty} \left[b_{-(n-1)}(x,t) - a_n(x,t)\right] & x > L_t \\ \sum_{n=1}^{\infty} \left[a_{-(n-1)}(x,t) - b_n(x,t)\right] & 0 < x < L_t \end{cases} \tag{4.57}$$

where

$$a_n(x,t) = \left(\frac{s_0\beta^n}{L}\right)^{\gamma_n} \beta^{n\kappa_2} \frac{1}{x\sigma\sqrt{t}}\phi\left(\frac{\ln x - \ln s_0\beta^{2n} - \mu t}{\sigma\sqrt{t}}\right),$$

$$b_n(x,t) = \left(\frac{s_0\beta^n}{L}\right)^{\gamma_n-\kappa_2} \frac{1}{x\sigma\sqrt{t}}\phi\left(\frac{\ln x - \ln(L^2/s_0\beta^{2n}) - \mu t}{\sigma\sqrt{t}}\right),$$

with $\phi(\cdot)$ *being the standard normal density function.*

Proof of Corollary 4.4.6. Note that we already obtained the formulas for $B^{\pi}_{\pm n}(s_0)$, given by (4.36) to (4.39) in the proof of Theorem 4.4.4. Now simply let the payoff function $\pi(s) = \mathbb{1}(s \in \mathrm{d}x)$ and apply Proposition 4.4.1. Note that the density function of S_t given $S_0 = s_0$ is $\Pr_{s_0}(S_t \in \mathrm{d}x) = \frac{1}{x\sigma\sqrt{t}}\phi\left(\frac{\ln x - \ln s_0 - \mu t}{\sigma\sqrt{t}}\right)\mathrm{d}x$. □

Remark 4.4.4. Using some fundamental results in Anderson (1960) as ingredients, Kunitomo and Ikeda (1992) derived formula (4.55). Following the same procedure, Kolkiewicz (2002) also obtained formulas (4.56) and (4.57).

Remark 4.4.5. One should expect that the sum $f(x;t) + f^{+}(x;t) + f^{-}(x,t)$ is equal to the density function of S_t. Using the formulas (4.55) to (4.57), it is not difficult to show that $f(x;t)+f^{+}(x;t)+f^{-}(x,t) = a_0(x,t) = \frac{1}{x\sigma\sqrt{t}}\phi\left(\frac{\ln x - \ln s_0 - \mu t}{\sigma\sqrt{t}}\right) = \Pr_{s_0}(S_t \in \mathrm{d}x)/\mathrm{d}x$.

Corollary 4.4.7. *For some $t > 0$,*

$$\begin{aligned}\Pr_{s_0}\left(\tilde{\tau}_U < \tilde{\tau}_L \wedge t\right) = \sum_{n=1}^{\infty} \Bigg[& \left(\frac{s_0\beta^{n-1}}{L}\right)^{\gamma_{n-1}} \beta^{(n-1)\kappa_2} G_t\left(\ln\frac{s_0\beta^{2n-1}}{L}, \mu-\delta_1\right) \\ & - \left(\frac{L\beta^{n-1}}{s_0}\right)^{\gamma_{n-1}+\kappa_2} G_t\left(\ln\frac{L\beta^{2n-1}}{s_0}, \mu-\delta_1\right)\Bigg],\end{aligned} \tag{4.58}$$

and

$$\begin{aligned}\Pr_{s_0}\left(\tilde{\tau}_L < \tilde{\tau}_U \wedge t\right) = \sum_{n=1}^{\infty} \Bigg[& \left(\frac{U\beta^{n-1}}{s_0}\right)^{\gamma_{n-1}} \beta^{-(n-1)\kappa_1} G_t\left(\ln\frac{s_0}{L\beta^{2n-2}}, \mu-\delta_2\right) \\ & - \left(\frac{s_0\beta^{n-1}}{U}\right)^{\gamma_{n-1}-\kappa_1} G_t\left(\ln\frac{U}{s_0\beta^{2n-1}}, \mu-\delta_2\right)\Bigg],\end{aligned} \tag{4.59}$$

where

$$G_t(x,y) = \Phi\left(\frac{x+yt}{\sigma\sqrt{t}}\right) + e^{-\frac{2xy}{\sigma^2}}\Phi\left(\frac{x-yt}{\sigma\sqrt{t}}\right) \tag{4.60}$$

with $\Phi(\cdot)$ being the standard normal distribution function.

Proof of Corollary 4.4.7. We can show (4.58) and (4.59) by integrating with respect to x the density functions $f^{+}(x;t)$ and $f^{-}(x;t)$ given by (4.56) and (4.57) respectively. For example,

$$\Pr_{s_0}\left(\tilde{\tau}_U < \tilde{\tau}_L \wedge t\right) = \int_0^{\infty} \Pr_{s_0}\left(S_t \in \mathrm{d}x, \tilde{\tau}_U < \tilde{\tau}_L \wedge t\right) = \int_0^{\infty} f^{+}(x;t)\mathrm{d}x,$$

The order of integration and summation can be interchanged in $\int_0^{\infty} f^{+}(x;t)\mathrm{d}x$ since each $a_n(x,t)$ and $b_n(x,t)$ are non-negative, and thus the integrations can be performed termwise. Then we have to perform some elementary but cumbersome calculations to get (4.58) and (4.59). A more direct method is to use identities (4.23) and (4.24) and simply choose $\pi(s) = 1$ in equations (4.37), (4.38) and (4.40). In particular, it is not difficult to show when $\pi(s) = 1$,

$$B_1^\pi(s_0) = \Pr_{s_0}(\tilde{\tau}_U < T) = G_T\left(\ln\frac{s_0}{U}, \mu - \delta_1\right).$$

Then the remaining work is to rearrange equations (4.37) and (4.38). □

Remark 4.4.6. The two density functions

$$\Pr_{s_0}(\tilde{\tau}_U \in \mathrm{d}t, \tilde{\tau}_U < \tilde{\tau}_L) = \frac{\partial}{\partial t}\Pr_{s_0}(\tilde{\tau}_U < \tilde{\tau}_L \wedge t),$$
$$\Pr_{s_0}(\tilde{\tau}_L \in \mathrm{d}t, \tilde{\tau}_L < \tilde{\tau}_U) = \frac{\partial}{\partial t}\Pr_{s_0}(\tilde{\tau}_L < \tilde{\tau}_U \wedge t),$$

are the key ingredients to value immediate rebate options restricted by two exponential barriers, which will be deferred for discussion until Section 5.3.

The study of exponential boundaries can be extended to deal with more general curved boundaries as we may be able to approximate a smooth nonlinear function using a set of piecewise exponential functions. This problem was briefly mentioned in Kunitomo and Ikeda (1992) for a double-barrier option with n-period piecewise exponential time-varying boundaries, but no explicit solution was given. We shall delay the detailed consideration of this option until Section 6.6, where a closed-form formula will be derived as an application of our pricing method for jump-diffusion models.

4.4.3 Convergence of the pricing formula and numerical examples

We will explain rigorously that the pricing formula (4.34) as doubly infinite sums is convergent under certain mild condition and we will also present some numerical results for different choices of parameter values.

Suppose the payoff function $\pi(s)$ is bounded by a constant C over the finite interval (L_T, U_T). This admits a fairly wide range of payoff types. We only demonstrate the convergence of the first doubly infinite series in (4.34) and leave it to readers to show the convergence of the second series in an analogous manner. Without loss of generality, we assume $s_0 = 1$ and $C = 1$. The first infinite series in (4.34) is

$$\sum_{n=-\infty}^{\infty} \left(\frac{\beta^n}{L}\right)^{\gamma_n} \beta^{n\kappa_2} V_\pi^*\left(\beta^{2n}\right). \tag{4.61}$$

The convergence follows from the rapid decay of normal distribution. Note that

$$\begin{aligned} V_\pi^*(s) &= \mathrm{E}_s\left[\pi(S_T)\mathbb{1}(L_T < S_T < U_T)\right] \\ &\le \Pr_s\left(L_T < S_T < U_T\right) \\ &= \Phi\left(\frac{\ln(s/L_T) + \mu T}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{\ln(s/U_T) + \mu T}{\sigma\sqrt{T}}\right), \end{aligned}$$

where $\Phi(\cdot)$ is the distribution function of standard normal random variable. Hence, one can observe that $V_\pi^*(s) \to 0$ when $s \to 0$ or ∞. In fact, $V_\pi^*(s)$ decreases to zero at a much higher speed. We use the L'Hôpital's rule to obtain the following asymptotic property: For $\xi < \frac{1}{2\sigma^2 T}$ and $\eta \in \mathbb{R}$,

$$s^{(\xi \ln s)+\eta} \cdot V_\pi^*(s) \to 0 \tag{4.62}$$

as $s \to 0$ or ∞. For notional convenience, we define $\beta_n = \beta^{2n} = \frac{L^{2n}}{U^{2n}}$. Then the summand in (4.61) can be rewritten as

$$\left(\frac{L^n}{U^n}\right)^{n(\kappa_2-\kappa_1-4\xi\ln\beta)} \cdot \left(\frac{L^{\kappa_1-2\eta}}{U^{\kappa_2-2\eta}}\right)^n \cdot {\beta_n}^{(\xi \ln \beta_n)+\eta} \cdot V_\pi^*(\beta_n) \tag{4.63}$$

with ξ and η being some undetermined parameters. In particular, we choose ξ such that $\kappa_2 - \kappa_1 - 4\xi \ln \beta = 0$. Then $\xi = \frac{\kappa_2 - \kappa_1}{4 \ln \beta}$ and (4.63) reduces to

$$\left(\frac{L^{\kappa_1 - 2\eta}}{U^{\kappa_2 - 2\eta}} \right)^n \cdot {\beta_n}^{(\xi \ln \beta_n) + \eta} \cdot V_\pi^*(\beta_n).$$

We also choose η such that $\frac{L^{\kappa_1 - 2\eta}}{U^{\kappa_2 - 2\eta}} < 1$. Note that $\beta_n \to 0$ as $n \to \infty$. According to (4.62), for every finite ϵ, there exists a positive integer N depending on ξ and η, such that ${\beta_n}^{(\xi \ln \beta_n) + \eta} \cdot V_\pi^*(\beta_n) < \epsilon$ when $n > N$. For a given ϵ, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(\frac{L^{\kappa_1 - 2\eta}}{U^{\kappa_2 - 2\eta}} \right)^n \cdot {\beta_n}^{(\xi \ln \beta_n) + \eta} \cdot V_\pi^*(\beta_n) \\ &< \sum_{n=0}^{N} \left(\frac{L^{\kappa_1 - 2\eta}}{U^{\kappa_2 - 2\eta}} \right)^n \cdot {\beta_n}^{(\xi \ln \beta_n) + \eta} \cdot V_\pi^*(\beta_n) + \epsilon \cdot \sum_{n>N} \left(\frac{L^{\kappa_1 - 2\eta}}{U^{\kappa_2 - 2\eta}} \right)^n < \infty, \end{aligned}$$

because $\frac{L^{\kappa_1 - 2\eta}}{U^{\kappa_2 - 2\eta}} < 1$. This means that for a pre-specified number ϵ, we can truncate the infinite sum with a finite sum of $N + 1$ terms, and the approximating result has an error of order ϵ. To show the convergence of

$$\sum_{n=-1}^{-\infty} \left(\frac{L^{\kappa_1 - 2\eta}}{U^{\kappa_2 - 2\eta}} \right)^n \cdot {\beta_n}^{(\xi \ln \beta_n) + \eta} \cdot V_\pi^*(\beta_n),$$

we simply choose η such that $\frac{L^{\kappa_1 - 2\eta}}{U^{\kappa_2 - 2\eta}} > 1$ and notice that $\beta_n \to \infty$ as $n \to -\infty$. Finally, we verify that the choice of ξ satisfies the condition $\xi < \frac{1}{2\sigma^2 T}$. In fact, $\xi = \frac{\kappa_2 - \kappa_1}{4 \ln \beta} < \frac{1}{2\sigma^2 T}$ is equivalent to $Ue^{\delta_1 T} > Le^{\delta_2 T}$, which is our primary assumption.

Now let us show some numerical examples. We revisit several parameter values considered in Kunitomo and Ikeda (1992); it shows that our results are the same as theirs except for some negligible errors in a few entries.

We compute the prices of double knock-out (at-the-money) call options and put options for various choices of barrier levels, curvature rates and volatilities. To obtain the arbitrage-free price at time 0, we multiply the discount factor e^{-rT} and let the drift parameter $\mu = r - \frac{1}{2}\sigma^2$ in our calculation. The parameter values used in the calculations are: $s_0 = 1000, K = 1000, r = 0.05$ and $T = 0.5$. In particular, we let (δ_1, δ_2) be $(0.1, -0.1)$, $(0, 0)$ or $(-0.1, 0.1)$. $(\delta_1, \delta_2) = (0.1, -0.1)$ corresponds to the case of two diverging barriers. When $(\delta_1, \delta_2) = (0, 0)$, it reduces to the case where the two barriers are flat (see formula (4.47)), and when $(\delta_1, \delta_2) = (-0.1, 0.1)$, the barriers are converging to each other. The most extreme case is also taken into account where the two barriers can never be reached, and in this case the barrier option simply reduces to vanilla call or put option; hence, the prices are identical regardless of the values of δ_1 and δ_2. The results are provided in Table 4.1 and 4.2.

Table 4.1: Double knock-out call with exponential boundaries

δ_1/δ_2	L/U	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
0.1/-0.1	$0/\infty$	68.89	96.35	123.85
	400/1600	68.64	85.88	81.60
	500/1500	67.78	76.57	64.85
	600/1400	64.63	61.48	45.23
	700/1300	55.20	40.54	25.08
0/0	$0/\infty$	68.89	96.35	123.85
	400/1600	68.14	80.06	71.05
	500/1500	66.13	67.88	53.35
	600/1400	60.06	50.23	34.22
	700/1300	45.65	28.90	16.49
-0.1/0.1	$0/\infty$	68.89	96.35	123.85
	400/1600	66.93	72.22	59.59
	500/1500	62.75	57.30	41.70
	600/1400	52.50	38.10	24.05
	700/1300	33.45	18.22	9.45

Our observations agree with those found in Kunitomo and Ikeda (1992) and Buchen and Konstandatos (2009). The doubly infinite series in (4.34) will converge as long as a small number of terms are involved. In many cases, it is sufficient to

Table 4.2: Double knock-out put with exponential boundaries

δ_1/δ_2	L/U	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
0.1/-0.1	$0/\infty$	44.20	71.66	99.16
	400/1600	44.20	71.66	98.66
	500/1500	44.20	71.42	93.78
	600/1400	44.18	68.10	75.73
	700/1300	43.13	52.82	42.72
0/0	$0/\infty$	44.20	71.66	99.16
	400/1600	44.20	71.65	98.30
	500/1500	44.20	71.15	91.13
	600/1400	44.12	65.58	68.32
	700/1300	41.62	45.12	32.69
-0.1/0.1	$0/\infty$	44.20	71.66	99.16
	400/1600	44.20	71.64	97.71
	500/1500	44.20	70.63	87.49
	600/1400	43.94	61.78	59.61
	700/1300	38.64	35.98	22.87

truncate the infinite sum only up to $n = \pm 2$ terms. The option values tend to decline as the two barriers approach towards each other, since there is a greater chance for the barriers to be breached and thus for the options to expire worthless.

4.5 Appendix

4.5.1 Proof of Remark 4.2.1

Proof. We want to show that when Λ_t is continuous in $[0, \infty)$, the following equation implies that Λ_t is a zero function:

$$\int_0^\infty e^{-\lambda t} \Lambda_t \mathrm{d}t = 0, \quad \lambda > 0. \tag{4.64}$$

Make change of variables $s = e^{-t}$ and $\tilde{\Lambda}_s = \Lambda_t$, then (4.64) leads to

$$\int_0^1 s^{\lambda-1}\tilde{\Lambda}_s \mathrm{d}s = 0, \quad \lambda > 0. \tag{4.65}$$

Note that $\tilde{\Lambda}_s$ is also continuous in $[0, 1]$. By Weierstrass approximation theorem, for every positve ϵ, there exists a polynomial P_s^ϵ such that $|\tilde{\Lambda}_s - P_s^\epsilon| < \epsilon$. From equation (4.65), we also have

$$\int_0^1 P_s^\epsilon \tilde{\Lambda}_s \mathrm{d}s = 0.$$

It then follows that

$$\int_0^1 \tilde{\Lambda}_s^2 \mathrm{d}s = \int_0^1 \tilde{\Lambda}_s^2 \mathrm{d}s - \int_0^1 P_s^\epsilon \tilde{\Lambda}_s \mathrm{d}s = \int_0^1 \tilde{\Lambda}_s(\tilde{\Lambda}_s - P_s^\epsilon)\mathrm{d}s,$$

and as a result,

$$\int_0^1 \tilde{\Lambda}_s^2 \mathrm{d}s \le \int_0^1 |\tilde{\Lambda}_s||\tilde{\Lambda}_s - P_s^\epsilon|\mathrm{d}s < \epsilon \int_0^1 |\tilde{\Lambda}_s|\mathrm{d}s.$$

Because $\tilde{\Lambda}_s$ is continuous in $[0, 1]$, it is bounded, and we can let ϵ tend to zero and obtain $\int_0^1 \tilde{\Lambda}_s^2 \mathrm{d}s = 0$. Hence, $\tilde{\Lambda}_s = 0$ for all $0 \le s \le 1$, and as a consequence, $\Lambda_t = 0$ for all $t \ge 0$. □

4.5.2 An alternative proof of Lemma 4.4.3

Proof. Recall that we need to show for $n \ge 2$ that

$$B_n^\pi(s_0) = \left(\frac{U}{s_0}\right)^{\kappa_1} B_{-(n-1)}^\pi\left(\frac{U^2}{s_0}\right), \quad s_0 < U, \tag{4.66}$$

$$B_{-n}^\pi(s_0) = \left(\frac{L}{s_0}\right)^{\kappa_2} B_{n-1}^\pi\left(\frac{L^2}{s_0}\right), \quad s_0 > L. \tag{4.67}$$

We only present a sketch of the proof of (4.66) based on the reflection principle (Propostion 2.2.1) and the Esscher transform factorization (Section 2.5). It is actually sufficient to consider the case where $\delta_1 = 0$, that is, the upper barrier is flat and $\kappa_1 = \frac{2\mu}{\sigma^2}$. According to the discussion in Example 2.5.1, under the transformed measure with index $-\frac{\kappa_1}{2}$, the drift term of $\{X_t\}$ becomes $\mu - \frac{\kappa_1}{2}\sigma^2 = 0$. For $n \geq 2$, $\mathcal{B}_n$ implies $\{\tau_U < T\}$. Then

$$B_n^\pi(s_0) = \mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}_{\mathcal{B}_n}\right] = \mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}(\tau_U < T)\mathbb{1}_{\mathcal{B}_n}\right].$$

The Esscher transform factorization formula (2.7) yields

$$\begin{aligned} B_n^\pi(s_0) &= \mathrm{E}_{s_0}\left[e^{-\frac{\kappa_1}{2}X_T} e^{\frac{\kappa_1}{2}X_T}\pi(S_T)\mathbb{1}(\tau_U < T)\mathbb{1}_{\mathcal{B}_n}\right] \\ &= \mathrm{E}\left[e^{-\frac{\kappa_1}{2}X_T}\right]\mathrm{E}_{s_0}\left[e^{\frac{\kappa_1}{2}X_T}\pi(S_T)\mathbb{1}(\tau_U < T)\mathbb{1}_{\mathcal{B}_n}; -\frac{\kappa_1}{2}\right], \end{aligned} \tag{4.68}$$

where by Definition 4.4.1, $\mathcal{B}_n$ can be viewed as the event that there exist $n-1$ time points $\tau_U < t_1 < \cdots < t_{n-1} \leq T$ such that $S_{t_k} = L_{t_k}$ when k is odd and $S_{t_k} = U$ when k is even. Note that the process $\{X_t; 0 \leq t \leq T\}$ in the second expectation of (4.68) has no drift, and hence, $\{X_t; 0 \leq t \leq T\}$ and $\{-X_t; 0 \leq t \leq T\}$ have the same distribution. We first reflect the process at τ_U and then change $\{X_t; 0 \leq t \leq T\}$ to $\{-X_t; 0 \leq t \leq T\}$. It follows from the reflection principle that the second expectation on the right-hand side of (4.68) can be rewritten as

$$\mathrm{E}_{\frac{U^2}{s_0}}\left[e^{\frac{\kappa_1}{2}(2b+X_T)}\pi(S_T)\mathbb{1}(\tau_U < T)\mathbb{1}_{\mathcal{B}_{-(n-1)}}; -\frac{\kappa_1}{2}\right],$$

where $b = \ln\frac{U}{s_0}$. Actually the indicator $\mathbb{1}(\tau_U < T)$ can be removed. The reason is that if the initial asset price is $\frac{U^2}{s_0}$ and the event $\mathcal{B}_{-(n-1)}$ occurs for $n \geq 2$, the asset price process must breach the upper barrier before time T. Reversing the Esscher transform factorization, we obtain

$$\begin{aligned} B_n^\pi(s_0) &= e^{b\kappa_1}\mathrm{E}\left[e^{-\frac{\kappa_1}{2}X_T}\right]\mathrm{E}_{\frac{U^2}{s_0}}\left[e^{\frac{\kappa_1}{2}X_T}\pi(S_T)\mathbb{1}_{\mathcal{B}_{-(n-1)}}; -\frac{\kappa_1}{2}\right] \\ &= \left(\frac{U}{s_0}\right)^{\kappa_1}\mathrm{E}_{\frac{U^2}{s_0}}\left[\pi(S_T)\mathbb{1}_{\mathcal{B}_{-(n-1)}}\right] \\ &= \left(\frac{U}{s_0}\right)^{\kappa_1} B_{-(n-1)}^\pi\left(\frac{U^2}{s_0}\right), \end{aligned}$$

which yields (4.66). Then we switch the two barriers, from which (4.67) follows. $\square$

4.5.3 An alternative proof of Corollary 4.4.5 and related remarks

Proof. Recall that Corollary 4.4.5 claims

$$V_{\text{dko}} = \sum_{n=-\infty}^{\infty} \beta^{n\kappa} \left[\bar{V}_\pi \left(s_0 \beta^{2n} \right) - \left(\frac{L}{s_0 \beta^{2n}} \right)^{\kappa} \bar{V}_\pi \left(\frac{L^2}{s_0 \beta^{2n}} \right) \right], \tag{4.69}$$

where $\beta = \frac{L}{U}, \kappa = \frac{2\mu}{\sigma^2}$ and $\bar{V}_\pi(s) = \mathrm{E}_s \left[\pi(S_T) \mathbb{1}(L < S_T < U) \right]$. This is a special case of Theorem 4.4.4 where the barriers are flat. Now let us present a self-contained proof that does not depend on Theorem 4.4.4. We still use the exponential stopping of Brownian motions, but the path counting is no longer required. Define by $\tau_{U,L}$ the first time the asset price process exits the interval $[L, U]$, that is, $\tau_{U,L} = \inf\{t > 0 | S_t \leq L \text{ or } S_t \geq U\}$. Again, we let $L < s_0 < U$. Then V_{dko} can be expressed as

$$V_{\text{dko}} = \mathrm{E}_{s_0} \left[\pi(S_T) \mathbb{1}(\tau_{U,L} > T) \right].$$

Let us first consider the exponential stopping of the asset price process and evaluate the knock-in option value

$$\mathrm{E}_{s_0} \left[\pi(S_{\varepsilon_\lambda}) \mathbb{1}(\tau_{U,L} < \varepsilon_\lambda) \right],$$

where ε_λ is an independent exponential random variable with mean $1/\lambda$. If we define $\hat{V}_\pi(s) = \mathrm{E}_s \left[\pi(S_{\varepsilon_\lambda}) \right]$, then using the memoryless property of ε_λ, the strong Markov property and the law of iterated expectations, we can show

$$\mathrm{E}_{s_0} \left[\pi(S_{\varepsilon_\lambda}) \mathbb{1}(\tau_{U,L} < \varepsilon_\lambda) \right] = \mathrm{E}_{s_0} \left[e^{-\lambda \tau_{U,L}} \hat{V}_\pi(S_{\tau_{U,L}}) \right].$$

Note that $S_{\tau_{U,L}} = U$ when $\tau_U < \tau_L$ and $S_{\tau_{U,L}} = L$ when $\tau_L < \tau_U$. Therefore,

$$\begin{aligned}
&\mathrm{E}_{s_0} \left[e^{-\lambda \tau_{U,L}} \hat{V}_\pi(S_{\tau_{U,L}}) \right] \\
&= \mathrm{E}_{s_0} \left[e^{-\lambda \tau_U} \mathbb{1}(\tau_U < \tau_L) \right] \hat{V}_\pi(U) + \mathrm{E}_{s_0} \left[e^{-\lambda \tau_L} \mathbb{1}(\tau_L < \tau_U) \right] \hat{V}_\pi(L) \\
&= \frac{\left(\frac{s_0}{L}\right)^{\theta_\lambda^+} - \left(\frac{s_0}{L}\right)^{\theta_\lambda^-}}{\left(\frac{U}{L}\right)^{\theta_\lambda^+} - \left(\frac{U}{L}\right)^{\theta_\lambda^-}} \hat{V}_\pi(U) + \frac{\left(\frac{s_0}{U}\right)^{\theta_\lambda^+} - \left(\frac{s_0}{U}\right)^{\theta_\lambda^-}}{\left(\frac{L}{U}\right)^{\theta_\lambda^+} - \left(\frac{L}{U}\right)^{\theta_\lambda^-}} \hat{V}_\pi(L).
\end{aligned} \tag{4.70}$$

The last step follows immediately from equations (5.14) and (5.15) in Theorem 5.3.2 where we let $T \to \infty$ and $r = \lambda$. Because $\{\tau_{U,L} > T\}$ implies $\{L < S_T < U\}$, we can express V_{dko} in an alternative form given by

$$V_{\text{dko}} = \mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}(L < S_T < U, \tau_{U,L} > T)\right].$$

Hence, if we define $\hat{\hat{V}}_\pi(s_0) = \mathrm{E}_{s_0}\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(L < S_{\varepsilon_\lambda} < U)\right]$, the put-call parity and the formula (4.70) lead to the following equation.

$$\begin{aligned}
&\mathrm{E}_{s_0}\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(\tau_{U,L} > \varepsilon_\lambda)\right] \\
&= \mathrm{E}_{s_0}\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(L < S_{\varepsilon_\lambda} < U, \tau_{U,L} > \varepsilon_\lambda)\right] \\
&= \hat{\hat{V}}_\pi(s_0) - \frac{\left(\frac{s_0}{L}\right)^{\theta_\lambda^+} - \left(\frac{s_0}{L}\right)^{\theta_\lambda^-}}{\left(\frac{U}{L}\right)^{\theta_\lambda^+} - \left(\frac{U}{L}\right)^{\theta_\lambda^-}}\hat{\hat{V}}_\pi(U) - \frac{\left(\frac{s_0}{U}\right)^{\theta_\lambda^+} - \left(\frac{s_0}{U}\right)^{\theta_\lambda^-}}{\left(\frac{L}{U}\right)^{\theta_\lambda^+} - \left(\frac{L}{U}\right)^{\theta_\lambda^-}}\hat{\hat{V}}_\pi(L).
\end{aligned} \tag{4.71}$$

Now we rewrite the ratios right in front of $\hat{\hat{V}}_\pi(U)$ and $\hat{\hat{V}}_\pi(L)$ in terms of infinite series. For example, taking geometric expansion of the first ratio, we have

$$\frac{1}{\left(\frac{U}{L}\right)^{\theta_\lambda^+} - \left(\frac{U}{L}\right)^{\theta_\lambda^-}} = \frac{\left(\frac{U}{L}\right)^{-\theta_\lambda^+}}{1 - \left(\frac{U}{L}\right)^{\theta_\lambda^- - \theta_\lambda^+}} = \left(\frac{U}{L}\right)^{-\theta_\lambda^+} \sum_{n=0}^{\infty} \left(\frac{U}{L}\right)^{n(\theta_\lambda^- - \theta_\lambda^+)}.$$

With some arrangements, the second term on the right-hand side of (4.71) can be rewritten as the difference of the two infinite sums

$$\sum_{n=0}^{\infty} \left(\frac{U^n}{L^n}\right)^{\theta_\lambda^- - \theta_\lambda^+} \left(\frac{s_0}{U}\right)^{\theta_\lambda^+} \hat{\hat{V}}_\pi(U), \tag{4.72}$$

and

$$\sum_{n=0}^{\infty} \left(\frac{s_0 U^n}{L^{n+1}}\right)^{\theta_\lambda^- - \theta_\lambda^+} \left(\frac{s_0}{U}\right)^{\theta_\lambda^+} \hat{\hat{V}}_\pi(U). \tag{4.73}$$

The argument used to obtain (4.7) is again considered and we have

$$\hat{\hat{V}}_\pi(s_1) = \left(\frac{s_1}{U}\right)^{\theta_\lambda^-} \hat{\hat{V}}_\pi(U), \quad s_1 > U,$$

or equivalently,

$$\hat{\hat{V}}_\pi(U) = \left(\frac{U}{s_1}\right)^{\theta_\lambda^-} \hat{\hat{V}}_\pi(s_1), \quad s_1 > U. \tag{4.74}$$

Then we apply (4.74) to substitute $\hat{\hat{V}}_\pi(U)$ into equations (4.72) and (4.73). We specifically choose $s_1 = \frac{U^{2n+2}}{s_0 L^{2n}}$. It is easy to verify that $s_1 > U$ since $L < s_0 < U$, and hence (4.72) becomes

$$\sum_{n=0}^{\infty} \left(\frac{U^{n+1}}{s_0 L^n}\right)^{\kappa} \hat{\hat{V}}_\pi\left(\frac{U^{2n+2}}{s_0 L^{2n}}\right),$$

where $\kappa = -(\theta_\lambda^+ + \theta_\lambda^-) = \frac{2\mu}{\sigma^2}$. Likewise, we choose $s_1 = \frac{s_0 U^{2n+2}}{L^{2n+2}}$ and (4.73) becomes

$$\sum_{n=0}^{\infty} \left(\frac{U^{n+1}}{L^{n+1}}\right)^{\kappa} \hat{\hat{V}}_\pi\left(\frac{s_0 U^{2n+2}}{L^{2n+2}}\right).$$

To evaluate the third term on the right-hand side of (4.71), we simply switch U and L. As a consequence,

$$\begin{aligned}
&\mathrm{E}_{s_0}\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(L < S_{\varepsilon_\lambda} < U, \tau_{U,L} > \varepsilon_\lambda)\right] \\
&= \hat{\hat{V}}_\pi(s_0) - \sum_{n=0}^{\infty} \left(\frac{U^{n+1}}{s_0 L^n}\right)^{\kappa} \hat{\hat{V}}_\pi\left(\frac{U^{2n+2}}{s_0 L^{2n}}\right) \\
&\qquad + \sum_{n=0}^{\infty} \left(\frac{U^{n+1}}{L^{n+1}}\right)^{\kappa} \hat{\hat{V}}_\pi\left(\frac{s_0 U^{2n+2}}{L^{2n+2}}\right) \\
&\qquad - \sum_{n=0}^{\infty} \left(\frac{L^{n+1}}{s_0 U^n}\right)^{\kappa} \hat{\hat{V}}_\pi\left(\frac{L^{2n+2}}{s_0 U^{2n}}\right) \\
&\qquad + \sum_{n=0}^{\infty} \left(\frac{L^{n+1}}{U^{n+1}}\right)^{\kappa} \hat{\hat{V}}_\pi\left(\frac{s_0 L^{2n+2}}{U^{2n+2}}\right).
\end{aligned}$$

Since κ does not depend on λ, the exponential random variable ε_λ in the equation above can be replaced by a fixed positive time T, and correspondingly, $\hat{\hat{V}}_\pi(s)$ becomes $\bar{V}_\pi(s)$. The pricing formula (4.69) follows after some rearrangements. □

Remark 4.5.1. In fact, the proof presented above gives rise to a Laplace transform approach to valuing the double barrier options with flat boundaries. Note that because ε_λ is independent of $\{S_t\}$,

$$\mathrm{E}_{s_0}\left[\pi(S_{\varepsilon_\lambda})\mathbb{1}(\tau_{U,L} < \varepsilon_\lambda)\right] = \int_0^\infty \mathrm{E}_{s_0}\left[\pi(S_t)\mathbb{1}(\tau_{U,L} < t)\right]\lambda e^{-\lambda t}\mathrm{d}t.$$

Hence, the expression given by the right-hand side of (4.70) divided by λ yields a closed-form solution of the Laplace transform with respect to the maturity time as long as we can explicitly calculate $\hat{V}_\pi(s) = \mathrm{E}_s\left[\pi(S_{\varepsilon_\lambda})\right]$. In particular, we only need to identify the distribution of S_{ε_λ} or $X_{\varepsilon_\lambda} = \ln \frac{S_{\varepsilon_\lambda}}{S_0}$. Let $f_{X_{\varepsilon_\lambda}}(x)$ be the density function of X_{ε_λ}, then

$$f_{X_{\varepsilon_\lambda}}(x) = \frac{\theta_\lambda^+\theta_\lambda^-}{\theta_\lambda^- - \theta_\lambda^+}\begin{cases} e^{-\theta_\lambda^+ x} & x > 0 \\ \\ e^{-\theta_\lambda^- x} & x < 0 \end{cases} \tag{4.75}$$

which is an two-sided exponential distribution. It follows that we are able to compute the Laplace transform and invert it numerically for any given payoff and fixed maturity time. One way to derive the density above is through the inversion of a bilateral Laplace transform. Specifically, the bilateral Laplace transform of X_{ε_λ} is determined as follows.

$$\begin{aligned} &\int_{-\infty}^\infty e^{-zx} f_{X_{\varepsilon_\lambda}}(x)\mathrm{d}x \\ &= \mathrm{E}\left[e^{-zX_{\varepsilon_\lambda}}\right] = \mathrm{E}\left[e^{\left(-\mu z + \frac{1}{2}\sigma^2 z^2\right)\varepsilon_\lambda}\right] \\ &= \frac{\lambda}{\lambda + \mu z - \frac{1}{2}\sigma^2 z^2} = \frac{\lambda}{-\frac{1}{2}\sigma^2(z+\theta_\lambda^+)(z+\theta_\lambda^-)} \\ &= \frac{\theta_\lambda^+\theta_\lambda^-}{(z+\theta_\lambda^+)(z+\theta_\lambda^-)} = \frac{\theta_\lambda^+\theta_\lambda^-}{\theta_\lambda^- - \theta_\lambda^+}\left(\frac{1}{z+\theta_\lambda^+} - \frac{1}{z+\theta_\lambda^-}\right), \end{aligned} \tag{4.76}$$

where $-\theta_\lambda^+ < z < -\theta_\lambda^-$ and $\theta_\lambda^+ > 0$ and $\theta_\lambda^- < 0$ are the roots of the quadratic equation $\frac{1}{2}\sigma^2\theta^2 + \mu\theta - \lambda = 0$. Hence, the distribution given by (4.75) can be derived by inverting (4.76).

Remark 4.5.2. See also Gerber, Shiu and Yang (2012) which recovered the expression of $f_{X_{\varepsilon_\lambda}}(x)$ through a discounted density approach.

Remark 4.5.3. Geman and Yor (1996) also derived a closed-form formula for a double knock-out call option with flat barriers.

CHAPTER 5

APPLICATIONS TO PRICING EXOTIC VARIATIONS OF BARRIER OPTIONS

5.1 Introduction

The objective of this chapter is to evaluate some popular variations of the standard barrier options treated in Chapter 4, including sequential barrier options, immediate rebate options, multi-asset barrier options and window barrier options. These variations are created as complements to traditional barrier options, by modifying the original asset model or barrier structure for the purpose to more closely accommodate to the investors' hedging or speculating needs. We shall show that we can manage to reduce the problem of each variation to the one already studied in Chapter 4, and our valuation approach can be easily applied to derive the prices of these variations. We still assume an arbitrary payoff function and allow the barriers to change exponentially in time. The pricing formulas for sequential barrier options and immediate rebate options already exist and will be reproduced using our new method, while we will pay closer attention to multi-asset barrier options and window barrier options, and derive several pricing formulas that are not yet available in the literature as far as we know.

The remainder of this chapter is organized as follows: Section 5.2 treats sequetial barrier options, Section 5.3 treats immediate rebate options, Section 5.4 treats multi-asset barrier options and Section 5.5 treats window barrier options.

5.2 Sequential barrier options

The analysis of our approach in Chapter 4 can be structured and readily applied to pricing sequential barrier options, where the barrier event is determined by a pre-specified sequential order of the breaching times $\tilde{\tau}_U$, $\tilde{\tau}_L$ and the maturity time T. Recall that $\tilde{\tau}_U$ and $\tilde{\tau}_L$ denote the first times the asset price process hits the boundaries $U_t = Ue^{\delta_1 t}$ and $L_t = Le^{\delta_2 t}$ respectively. Similar problem was solved by Sidenius (1998) and Li (1998) which derived the joint density functions given in Corollary 4.4.6. Kolkiewicz (2002) developed a systematic procedure for sequential barrier options by considering the density functions of related exit times. We follow a different path however, and use Proposition 4.4.1 and Lemma 4.4.3 to investigate some examples of sequential barrier options. To avoid complicated equations, we only study the path counting of each barrier provision.

(1) Upper-barrier knock-in options: This type of options comes into being if the upper barrier is breached before the lower one prior to maturity. The associated indicator function is given by

$$\mathbb{1}(\tilde{\tau}_U < \tilde{\tau}_L \wedge T) = \sum_{n=1}^{\infty} \mathbb{1}_{\mathcal{B}_{2n-1}} - \sum_{n=1}^{\infty} \mathbb{1}_{\mathcal{B}_{-2n}}. \tag{5.1}$$

(2) Lower-barrier knock-in options: This type of options comes into being if the lower barrier is breached before the upper one prior to maturity. The associated indicator function is given by

$$\mathbb{1}(\tilde{\tau}_L < \tilde{\tau}_U \wedge T) = \sum_{n=1}^{\infty} \mathbb{1}_{\mathcal{B}_{-(2n-1)}} - \sum_{n=1}^{\infty} \mathbb{1}_{\mathcal{B}_{2n}}. \tag{5.2}$$

(3) Upper-then-lower knock-in options: This type of options comes into being when the upper barrier is breached first and then the lower barrier is breached prior to maturity. The associated indicator function is given by

$$\mathbb{1}(\tilde{\tau}_U < \tilde{\tau}_L < T) = \sum_{n=1}^{\infty} \mathbb{1}_{\mathcal{B}_{2n}} - \sum_{n=2}^{\infty} \mathbb{1}_{\mathcal{B}_{-(2n-1)}}. \tag{5.3}$$

To obtain the identity above, observe the following identity

$$\mathbb{1}_{\mathcal{B}_{-1}} = \mathbb{1}(\tilde{\tau}_U < \tilde{\tau}_L < T) + \mathbb{1}(\tilde{\tau}_L < \tilde{\tau}_U \wedge T), \tag{5.4}$$

and then apply (5.2). The identity (5.4) can be understood in this way: the event $\mathcal{B}_{-1}$ is equivalent to $\tilde{\tau}_L < T$ and can be partitioned into two disjoint events depending on which barrier is breached first, which are given correspondingly on the right-hand side of (5.4).

(4) Lower-then-upper knock-in options: This type of options comes into being when the lower barrier is breached first and then the upper barrier is breached prior to maturity. The associated indicator function is given by

$$\mathbb{1}(\tilde{\tau}_L < \tilde{\tau}_U < T) = \sum_{n=1}^{\infty} \mathbb{1}_{\mathcal{B}_{-2n}} - \sum_{n=2}^{\infty} \mathbb{1}_{\mathcal{B}_{2n-1}}. \tag{5.5}$$

(5) Double-hit knock-in options: This type of option comes into being if both barriers are breached prior to maturity. It is the sum of upper-then-lower knock-in option and lower-then-upper knock-in option. The associated indicator function is given by

$$\mathbb{1}(\tilde{\tau}_L < T, \tilde{\tau}_U < T) = \sum_{n=-\infty}^{\infty} \mathbb{1}_{\mathcal{B}_{2n}} - \sum_{n=-\infty}^{\infty} \mathbb{1}_{\mathcal{B}_{2n-1}} + \mathbb{1}_{\mathcal{B}_1} + \mathbb{1}_{\mathcal{B}_{-1}} - 1. \tag{5.6}$$

To further find the values of these variations, one can directly use formulas (4.36) to (4.41) (the calculations are elemantary but quite lengthy).

It is worth mentioning that the sequential barrier options are discussed here in a broad sense: we treate any barrier option that is knocked in or out depending on the sequential order of the hitting times as a type of sequential barrier options (the double knock-out options analyzed in the last chapter can obviously be visualized as a type of sequential barrier options). In fact, the definition of sequential barrier options, as in Pfeffer (2001) and Section 7.11 of Buchen (2012) for example, is slightly different, and essentially refers to eight types of options: ui/di (up-and-in/down-and-in), ui/do (up-and-in/down-and-out), di/ui, di/uo, uo/di, uo/do, do/ui and do/uo options. In our context, we call these options the standard sequential barrier options. With an ui/di option, if the asset price process breaches the upper boundary U_t at the hitting time $\tilde{\tau}_U$ before the maturity time T, the holder is immediately given a down-and-in option with the remaining lifetime $T - \tilde{\tau}_U$ and the lower boundary L_t. By analogy, one can easily understand the concepts of the other seven types of sequential barrier options. Fortunately, we are still able to utilize the technique of path counting to evaluate these options. For instance, the indicators associated with ui/di and uo/di options can respectively be expressed as

$$\mathbb{1}\left(\tilde{\tau}_{U|L} < T\right) \quad \text{and} \quad \mathbb{1}\left(\tilde{\tau}_L < T < \tilde{\tau}_U\right), \tag{5.7}$$

where we define $\tilde{\tau}_{U|L} = \inf\{t > 0 | t > \tilde{\tau}_U, S_t = L_t\}$ as the first time the asset price process breaches the lower boundary after first breaching the upper boundary earlier. Readers should notice the difference between the ui/di options and the upper-then-lower knock-in options described by (5.3). Comparing their corresponding barrier provisions, we see that the ui/di options do not put any restrictions on the asset price process prior to first breaching the upper boundary, while the upper-then-lower knock-in options require the lower boundary never be breached first. According to Definition 4.4.1 and the identity (4.25) in Proposition 4.4.1, we obtain the following identities for the indicators in (5.7),

$$\mathbb{1}\left(\tilde{\tau}_{U|L} < T\right) = \mathbb{1}_{\mathcal{B}_2}, \tag{5.8}$$

$$\mathbb{1}\left(\tilde{\tau}_L < T < \tilde{\tau}_U\right) = \mathbb{1}\left(\tilde{\tau}_U > T\right) - \mathbb{1}\left(\tilde{\tau}_U \wedge \tilde{\tau}_L > T\right). \tag{5.9}$$

Note that the pricing formula for an ui/di option only has one term, and an uo/di option can simply be written as the difference between an up-and-out single-barrier option and a double-barrier knock-out option.

Table 5.1 below summarizes the eight types of standard sequential barrier options mentioned earlier in terms of the indicators associated with their barrier provisions. Similar to $\tilde{\tau}_{U|L}$, we also define $\tilde{\tau}_{L|U} = \inf\{t > 0 | t > \tilde{\tau}_L, S_t = U_t\}$ as the first time the asset price process breaches the upper boundary after breaching the lower boundary earlier.

Table 5.1: Standard sequential barrier options

Sequential order	Indicator
ui/di	$\mathbb{1}\left(\tilde{\tau}_{U\|L} < T\right)$
ui/do	$\mathbb{1}\left(\tilde{\tau}_U < T, \tilde{\tau}_{U\|L} > T\right)$
di/ui	$\mathbb{1}\left(\tilde{\tau}_{L\|U} < T\right)$
di/uo	$\mathbb{1}\left(\tilde{\tau}_L < T, \tilde{\tau}_{L\|U} > T\right)$
uo/di	$\mathbb{1}\left(\tilde{\tau}_L < T < \tilde{\tau}_U\right)$
uo/do	$\mathbb{1}\left(\tilde{\tau}_U \wedge \tilde{\tau}_L > T\right)$
do/ui	$\mathbb{1}\left(\tilde{\tau}_U < T < \tilde{\tau}_L\right)$
do/uo	$\mathbb{1}\left(\tilde{\tau}_U \wedge \tilde{\tau}_L > T\right)$

Several immediate results can be obtained from Table 5.1. First, we observe the following parity relations

$$V_{\mathrm{ui/di}} + V_{\mathrm{ui/do}} = V_{\mathrm{eui}}, \quad V_{\mathrm{di/ui}} + V_{\mathrm{di/uo}} = V_{\mathrm{edi}}, \tag{5.10}$$

$$V_{\mathrm{uo/di}} + V_{\mathrm{uo/do}} = V_{\mathrm{euo}}, \quad V_{\mathrm{do/ui}} + V_{\mathrm{do/uo}} = V_{\mathrm{edo}}, \tag{5.11}$$

where $V_{\mathrm{eui}}, V_{\mathrm{edi}}, V_{\mathrm{euo}}$ and V_{edo} denote the forward prices of those barrier options with single exponential boundary (either an upper boundary U_t or a lower boundary L_t). Second, uo/do and do/uo options are identical to double-barrier knock-out options. Therefore, it is only necessary to evaluate two of the eight types of sequential barrier options, which are ui/di and uo/di options. In fact, di/ui and do/ui options can be produced respectively from ui/di and uo/di options by switching the two barriers. Now let us multiply the payoff $\pi(S_T)$ and take expectation on both sides of (5.8) to derive the time-0 forward price of ui/di options. Specifically,

$$V_{\mathrm{ui/di}} = \mathrm{E}\left[\pi(S_T)\mathbb{1}_{\mathcal{B}_2}\right] = B_2^{\pi}(s_0),$$

where $B_2^{\pi}(s_0)$ is given by (4.30). To calculate $B_2^{\pi}(s_0)$, we use Lemma 4.4.3 and equation (4.41). Note that $B_{-1}^{\pi}(s_0)$ is in fact the price of single-barrier down-and-in options, then

$$\begin{aligned} B_2^{\pi}(s_0) &= \left(\frac{U}{s_0}\right)^{\kappa_1} B_{-1}^{\pi}\left(\frac{U^2}{s_0}\right) \\ &= \left(\frac{U}{s_0}\right)^{\kappa_1}\left[V_{\pi}^{-}\left(\frac{U^2}{s_0}, L_T\right) + \left(\frac{s_0 L}{U}\right)^{\kappa_2} V_{\pi}^{+}\left(\frac{s_0 L^2}{U^2}, L_T\right)\right]. \end{aligned}$$

The formula for $V_{\mathrm{di/ui}}$ can be obtained from the equation above by switching the two barriers. In particular, we have the following results.

Proposition 5.2.1. *The time-0 forward prices of ui/di and di/ui sequential barrier options with the upper boundary $U_t = Ue^{\delta_1 t}$ and the lower boundary $L_t = Le^{\delta_2 t}$ are respectively given by*

$$\begin{aligned} V_{\mathrm{ui/di}} &= \left(\frac{U}{s_0}\right)^{\kappa_1}\left[V_{\pi}^{-}\left(\frac{U^2}{s_0}, L_T\right) + \left(\frac{s_0 L}{U}\right)^{\kappa_2} V_{\pi}^{+}\left(\frac{s_0 L^2}{U^2}, L_T\right)\right], \\ V_{\mathrm{di/ui}} &= \left(\frac{L}{s_0}\right)^{\kappa_2}\left[V_{\pi}^{+}\left(\frac{L^2}{s_0}, U_T\right) + \left(\frac{s_0 U}{L}\right)^{\kappa_1} V_{\pi}^{-}\left(\frac{s_0 U^2}{L^2}, U_T\right)\right], \end{aligned}$$

where $\kappa_1 = \frac{2(\mu-\delta_1)}{\sigma^2}, \kappa_2 = \frac{2(\mu-\delta_2)}{\sigma^2}$ and $V_{\pi}^{\pm}(s,x) = \mathrm{E}_s\left[\pi(S_T)\mathbb{1}(S_T \gtrless x)\right]$.

Remark 5.2.1. To value ui/do and di/uo options using Proposition 5.2.1, we simply apply the parity relations (5.10). In addition, uo/di and do/ui options can be valued by noting that they can be viewed as the difference between single-barrier knock-out options and double-barrier knock-out options, which is also implied by the parity relations (5.11) (uo/do and do/uo options are equal to double-barrier knock-out options).

Remark 5.2.2. Analogous discussions about standard sequential barrier options can be found in Pfeffer (2001) and Section 7.11 of Buchen (2012). The former used density integrations, while the latter proposed the method of images approach for a general payoff function. However, both of them only considered flat barriers, which is a special case of what we analyze here.

5.3 Immediate rebate options

In some cases of knock-out options, the specification of rebate is allowed and the option holder receives a fixed rebate payment as compensation at the moment the barrier is breached prior to maturity. We study the price of one dollar paid at the moment of breaching given it occurs before maturity. The traditional method requires to identify the distributions of one-sided and two-sided exit times (see, for example, Jeanblanc, Yor and Chesney (2009) and Karatzas and Shreve (1998)). For the case where the barriers are flat, we use a martingale method introduced in Gerber and Shiu (1994b, 1996) to show that immediate rebate options can reduce to standard barrier options with certain adjusted payoff functions. For the case where the barriers are exponential functions of time, we carry out the valuation approach discussed in Chapter 4 to derive the density functions of corresponding exit times.

Now let us first consider the case where the barriers are flat. The following theorem deals with single flat barrier.

Theorem 5.3.1. *The time-0 price of one dollar paid at the moment of breaching the barrier B before time T is either given by*

$$\mathrm{E}_{s_0}\left[e^{-r\tau_B}\mathbb{1}(\tau_B < T)\right] = e^{-rT}\mathrm{E}_{s_0}\left[(S_T/B)^{\theta_r^+}\mathbb{1}(\tau_B < T)\right], \tag{5.12}$$

or

$$\mathrm{E}_{s_0}\left[e^{-r\tau_B}\mathbb{1}(\tau_B < T)\right] = e^{-rT}\mathrm{E}_{s_0}\left[(S_T/B)^{\theta_r^-}\mathbb{1}(\tau_B < T)\right], \tag{5.13}$$

where $\theta_r^+ > 0$ and $\theta_r^- < 0$ are the two roots of equation $\frac{\sigma^2}{2}\theta^2 + \mu\theta - r = 0$. Formula (5.12) *and formula* (5.13) *are identical.*

The right-hand sides of (5.12) and (5.13) represent the time-0 prices of knock-in options with payoff functions $\pi(s) = \left(\frac{s}{B}\right)^{\theta_r^+}$ and $\pi(s) = \left(\frac{s}{B}\right)^{\theta_r^-}$ respectively. Whether to choose θ_r^+ or θ_r^-, it is not difficult to verify that these two expectations can always be rewritten as

$$\begin{cases} e^{-rT}\mathrm{E}_{s_0}\left[\left(\left(\dfrac{S_T}{B}\right)^{\theta_r^+} + \left(\dfrac{S_T}{B}\right)^{\theta_r^-}\right)\mathbb{1}(S_T > B)\right] & s_0 < B \\ e^{-rT}\mathrm{E}_{s_0}\left[\left(\left(\dfrac{S_T}{B}\right)^{\theta_r^+} + \left(\dfrac{S_T}{B}\right)^{\theta_r^-}\right)\mathbb{1}(S_T < B)\right] & s_0 > B \end{cases}$$

Therefore, the right-hand sides of (5.12) and (5.13) indeed yield an identical pricing formula.

The following theorem deals with two flat barriers.

Theorem 5.3.2. *The time-0 prices of one dollar paid at the moment of first breaching the upper barrier U, and first breaching the lower barrier L before time T are respectively given by*

$$\mathrm{E}_{s_0}\left[e^{-r\tau_U}\mathbb{1}(\tau_U < \tau_L \wedge T)\right] = \frac{\frac{s_0^{\theta_r^+} - P^+}{L^{\theta_r^+}} - \frac{s_0^{\theta_r^-} - P^-}{L^{\theta_r^-}}}{\left(\frac{U}{L}\right)^{\theta_r^+} - \left(\frac{U}{L}\right)^{\theta_r^-}}, \tag{5.14}$$

and

$$\mathrm{E}_{s_0}\left[e^{-r\tau_L}\mathbb{1}(\tau_L < \tau_U \wedge T)\right] = \frac{\frac{s_0^{\theta_r^+} - P^+}{U^{\theta_r^+}} - \frac{s_0^{\theta_r^-} - P^-}{U^{\theta_r^-}}}{\left(\frac{L}{U}\right)^{\theta_r^+} - \left(\frac{L}{U}\right)^{\theta_r^-}}, \tag{5.15}$$

where θ_r^+ *and* θ_r^- *are the two roots of the equation* $\frac{\sigma^2}{2}\theta^2 + \mu\theta - r = 0$ *and*

$$P^{\pm} = e^{-rT}\mathrm{E}_{s_0}\left[S_T^{\theta_r^{\pm}}\mathbb{1}(\tau_U \wedge \tau_L > T)\right]$$

denote the time-0 prices of double knock-out options with $\pi(s) = s^{\theta_r^{\pm}}$.

Remark 5.3.1. The equations in Theorem 5.3.2 reduce to the ones in Theorem 5.3.1 when the upper barrier U tends to ∞ or the lower barrier L tends to 0. For example, we let $L \to 0$ on both sides of equation (5.14). Then we have $\tau_L \to \infty$ and

$$P^+ \to e^{-rT}\mathrm{E}_{s_0}\left[S_T^{\theta_r^+}\mathbb{1}(\tau_U > T)\right] \quad \text{and} \quad P^- \to e^{-rT}\mathrm{E}_{s_0}\left[S_T^{\theta_r^-}\mathbb{1}(\tau_U > T)\right].$$

Because $\theta_r^+ - \theta_r^- > 0$ and $P^- < \infty$, equation (5.14) reduces to

$$\mathrm{E}_{s_0}\left[e^{-r\tau_U}\mathbb{1}(\tau_U < T)\right] = \left(\frac{s_0}{U}\right)^{\theta_r^+} - e^{-rT}\mathrm{E}_{s_0}\left[\left(\frac{S_T}{U}\right)^{\theta_r^+}\mathbb{1}(\tau_U > T)\right].$$

Note that $\{e^{-rt}(S_t)^{\theta_r^+}\}$ is a martingale, then the equation above becomes

$$\begin{aligned}\mathrm{E}_{s_0}\left[e^{-r\tau_U}\mathbb{1}(\tau_U < T)\right] &= e^{-rT}\mathrm{E}_{s_0}\left[\left(\frac{S_T}{U}\right)^{\theta_r^+}\right] - e^{-rT}\mathrm{E}_{s_0}\left[\left(\frac{S_T}{U}\right)^{\theta_r^+}\mathbb{1}(\tau_U > T)\right] \\ &= e^{-rT}\mathrm{E}_{s_0}\left[\left(\frac{S_T}{U}\right)^{\theta_r^+}\mathbb{1}(\tau_U < T)\right],\end{aligned}$$

which yields equation (5.12).

Remark 5.3.2. Theorem 5.3.2 deals with a general situation where the options have unequal rebate payments, depending on whether the upper or lower barrier is breached first. This situation should be considered as the financial positions that cause the options to be nullified in the two cases above may be quite different.

Equations (5.14) and (5.15) show that the prices are written explicitly in terms of $P^{\pm}$, which can be calculated by formula (4.47) with the payoff functions $\pi(s) = s^{\theta_r^{\pm}}$. If we let T tend to ∞, then $P^{\pm}$ become zero, and equations (5.14) and (5.15) reduce to equations (4.1.8) and (4.1.9) respectively in Gerber and Shiu (1994b). See the proof in Section 4.5.3 for an application of this special case.

Theorem 5.3.2 is more difficult to prove than Theorem 5.3.1, so we only prove Theorem 5.3.2 using a martingale approach introduced in Gerber and Shiu (1994b).

Proof of Theorem 5.3.2. First note that

$$\{e^{-rt}(S_t)^{\theta_r^+}\} \quad \text{and} \quad \{e^{-rt}(S_t)^{\theta_r^-}\}$$

are two martingales. Define $\tau_{U,L} = \tau_U \wedge \tau_L$, the minimum of τ_U and τ_L. Applying the optional stopping theorem to the first martingale above for the bounded stopping time $\tau_{U,L} \wedge T$, we have

$$\mathrm{E}_{s_0}\left[e^{-r(\tau_{U,L}\wedge T)}(S_{\tau_{U,L}\wedge T})^{\theta_r^+}\right] = s_0^{\theta_r^+}.$$

Rearranging the equation above yields

$$\begin{aligned} &U^{\theta_r^+}\mathrm{E}_{s_0}\left[e^{-r\tau_U}\mathbb{1}(\tau_U < \tau_L \wedge T)\right] + L^{\theta_r^+}\mathrm{E}_{s_0}\left[e^{-r\tau_L}\mathbb{1}(\tau_L < \tau_U \wedge T)\right] \\ &+ \mathrm{E}_{s_0}\left[e^{-rT}(S_T)^{\theta_r^+}\mathbb{1}(\tau_U \wedge \tau_L > T)\right] = s_0^{\theta_r^+}. \end{aligned}$$

By the definition of P^+,

$$U^{\theta_r^+}\mathrm{E}_{s_0}\left[e^{-r\tau_U}\mathbb{1}(\tau_U < \tau_L \wedge T)\right] + L^{\theta_r^+}\mathrm{E}_{s_0}\left[e^{-r\tau_L}\mathbb{1}(\tau_L < \tau_U \wedge T)\right] = s_0^{\theta_r^+} - P^+.$$

Likewise, we can apply the optional stopping theorem to the second martingale above and obtain a second equation

$$U^{\theta_r^-}\mathrm{E}_{s_0}\left[e^{-r\tau_U}\mathbb{1}(\tau_U < \tau_L \wedge T)\right] + L^{\theta_r^-}\mathrm{E}_{s_0}\left[e^{-r\tau_L}\mathbb{1}(\tau_L < \tau_U \wedge T)\right] = s_0^{\theta_r^-} - P^-.$$

With this pair of equations, we can compute the two expectations by calculating

$$\begin{pmatrix} U^{\theta_r^+} & L^{\theta_r^+} \\ U^{\theta_r^-} & L^{\theta_r^-} \end{pmatrix}^{-1} \begin{pmatrix} s_0^{\theta_r^+} - P^+ \\ s_0^{\theta_r^-} - P^- \end{pmatrix}.$$

The proof is thus complete. □

Remark 5.3.3. A derivation of $\mathrm{E}_{s_0}\left[e^{-r\tau_B}\mathbb{1}(\tau_B < T)\right]$ can also be found, for example, in Poulsen (2006) which also applied the optional stopping theorem to the martingales $\{e^{-rt}(S_t)^{\theta_r^{\pm}}\}$.

Now let us consider the case where the barriers are exponential functions of time. As discussed earlier in Chapter 4, extending single flat barrier to single exponential barrier is merely a matter of changing the drift term of the asset price process. Hence, we only pay attention to immediate rebate options with respect to two exponential barriers. In particular, we need to calculate the following two expectations

$$\mathrm{E}_{s_0}\left[e^{-r\tilde{\tau}_U}\mathbb{1}(\tilde{\tau}_U < \tilde{\tau}_L \wedge T)\right] \quad \text{and} \quad \mathrm{E}_{s_0}\left[e^{-r\tilde{\tau}_L}\mathbb{1}(\tilde{\tau}_L < \tilde{\tau}_U \wedge T)\right],$$

where we have defined $\tilde{\tau}_U$ and $\tilde{\tau}_L$ as the first times the asset price process hits the upper barrier $U_t = Ue^{\delta_1 t}$ and the lower barrier $L_t = Le^{\delta_2 t}$ respectively.

Some related density functions of $\tilde{\tau}_U$ and $\tilde{\tau}_L$ will be identified, primarily based on the formulas given in Corollary 4.4.7. First, we observe

$$\mathrm{E}_{s_0}\left[e^{-r\tilde{\tau}_U}\mathbb{1}(\tilde{\tau}_U < \tilde{\tau}_L \wedge T)\right] = \int_0^T e^{-rt}\mathrm{Pr}_{s_0}\left(\tilde{\tau}_U \in \mathrm{d}t, \tilde{\tau}_U < \tilde{\tau}_L\right), \tag{5.16}$$

$$\mathrm{E}_{s_0}\left[e^{-r\tilde{\tau}_L}\mathbb{1}(\tilde{\tau}_L < \tilde{\tau}_U \wedge T)\right] = \int_0^T e^{-rt}\mathrm{Pr}_{s_0}\left(\tilde{\tau}_L \in \mathrm{d}t, \tilde{\tau}_L < \tilde{\tau}_U\right). \tag{5.17}$$

The integrals on the right-hand sides of (5.16) and (5.17) can be calculated if we find the two probabilities

$$\Pr_{s_0}(\tilde{\tau}_U < \tilde{\tau}_L \wedge t) \quad \text{and} \quad \Pr_{s_0}(\tilde{\tau}_L < \tilde{\tau}_U \wedge t)$$

and take derivatives with respect to t. Let us take the calculation of (5.16) as an example. According to Corollary 4.4.7,

$$\Pr_{s_0}(\tilde{\tau}_U < \tilde{\tau}_L \wedge t) = \sum_{n=1}^{\infty} \Bigg[\left(\frac{s_0 \beta^{n-1}}{L} \right)^{\gamma_{n-1}} \beta^{(n-1)\kappa_2} G_t \left(\ln \frac{s_0 \beta^{2n-1}}{L}, \mu - \delta_1 \right) - \left(\frac{L\beta^{n-1}}{s_0} \right)^{\gamma_{n-1}+\kappa_2} G_t \left(\ln \frac{L\beta^{2n-1}}{s_0}, \mu - \delta_1 \right) \Bigg], \tag{5.18}$$

where

$$G_t(x,y) = \Phi\left(\frac{x+yt}{\sigma\sqrt{t}} \right) + e^{-\frac{2xy}{\sigma^2}} \Phi\left(\frac{x-yt}{\sigma\sqrt{t}} \right).$$

It is easy to show that when taking derivative of the right-hand side of (5.18), the order of summation and differentiation can be interchanged. Then the problem reduces to the calculation of the integral

$$\int_0^T e^{-rt} \frac{\partial G_t(x,y)}{\partial t} \mathrm{d}t,$$

where one can easily verify

$$\frac{\partial G_t(x,y)}{\partial t} = -\frac{x}{\sigma\sqrt{t^3}} \phi\left(\frac{x+yt}{\sigma\sqrt{t}} \right)$$

with $\phi(\cdot)$ being the standard normal density. One can notice that $\frac{\partial G_t(x,y)}{\partial t}$ resembles the density function of inverse Gaussian distribution. Assuming $x < 0$, we have

$$\begin{aligned}
\int_0^T e^{-rt} \frac{\partial G_t(x,y)}{\partial t} \mathrm{d}t &= e^{\frac{\nu-y}{\sigma^2}x} \int_0^T -\frac{x}{\sigma\sqrt{t^3}} \phi\left(\frac{x+\nu t}{\sigma\sqrt{t}} \right) \mathrm{d}t \\
&= e^{\frac{\nu-y}{\sigma^2}x} \int_0^T \frac{\partial G_t(x,\nu)}{\partial t} \mathrm{d}t \\
&= e^{\frac{\nu-y}{\sigma^2}x} [G_T(x,\nu) - G_0(x,\nu)] \\
&= e^{\frac{\nu-y}{\sigma^2}x} G_T(x,\nu) \\
&= e^{\frac{\nu-y}{\sigma^2}x} \left[\Phi\left(\frac{x+\nu T}{\sigma\sqrt{T}} \right) + e^{-\frac{2x\nu}{\sigma^2}} \Phi\left(\frac{x-\nu T}{\sigma\sqrt{T}} \right) \right],
\end{aligned} \tag{5.19}$$

where $\nu = \sqrt{y^2 + 2r\sigma^2}$. Considering the equation (5.18), we need to let $x = \ln \frac{s_0 \beta^{2n-1}}{L}$ or $\ln \frac{L\beta^{2n-1}}{s_0}$, which is always negative given $n \geq 1$. Hence, (5.19) can be applied, and combining (5.19) together with (5.16) and (5.18) leads to the following theorem.

Theorem 5.3.3. *The time-0 prices of one dollar paid at the moment of first breaching the upper barrier $U_t = Ue^{\delta_1 t}$, and first breaching the lower barrier $L_t = Le^{\delta_2 t}$ before time T are respectively given by*

$$\begin{aligned}
&\mathrm{E}_{s_0}\left[e^{-r\tilde{\tau}_U}\mathbb{1}(\tilde{\tau}_U < \tilde{\tau}_L \wedge T)\right] \\
&= \sum_{n=0}^{\infty}\left[\left(\frac{s_0\beta^n}{L}\right)^{\gamma_n}\beta^{n\kappa_2}H_1\left(\ln\frac{s_0\beta^{2n+1}}{L}\right) - \left(\frac{L\beta^n}{s_0}\right)^{\gamma_n+\kappa_2}H_1\left(\ln\frac{L\beta^{2n+1}}{s_0}\right)\right],
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{E}_{s_0}\left[e^{-r\tilde{\tau}_L}\mathbb{1}(\tilde{\tau}_L < \tilde{\tau}_U \wedge T)\right] \\
&= \sum_{n=0}^{\infty}\left[\left(\frac{L\beta^{n-1}}{s_0}\right)^{\gamma_n}\beta^{-n\kappa_1}H_2\left(-\ln\frac{L\beta^{2n}}{s_0}\right) - \left(\frac{s_0\beta^n}{U}\right)^{\gamma_n-\kappa_1}H_2\left(-\ln\frac{s_0\beta^{2n+1}}{U}\right)\right],
\end{aligned}$$

where for $k = 1, 2$,

$$H_k(x) = \exp\left(\frac{\nu_k - \mu + \delta_k}{\sigma^2}x\right)\Phi\left(\frac{x + \nu_k T}{\sigma\sqrt{T}}\right) + \exp\left(\frac{\delta_k - \nu_k - \mu}{\sigma^2}x\right)\Phi\left(\frac{x - \nu_k T}{\sigma\sqrt{T}}\right)$$

with $\nu_k = \sqrt{(\mu - \delta_k)^2 + 2r\sigma^2}$ and $\Phi(\cdot)$ denoting the standard normal distribution function. We have also defined $\beta = \frac{L}{U}, \gamma_n = n(\kappa_2 - \kappa_1), \kappa_1 = \frac{2(\mu-\delta_1)}{\sigma^2}$ and $\kappa_2 = \frac{2(\mu-\delta_2)}{\sigma^2}$.

Remark 5.3.4. The pricing of immediate rebate options restricted by two exponential barriers was also investigated in Kolkiewicz (2002) through the definitions of two random times

$$\tilde{\tau}^+_{U,L} = \begin{cases}\tilde{\tau}_U & \tilde{\tau}_U < \tilde{\tau}_L \\ \infty & \tilde{\tau}_U > \tilde{\tau}_L\end{cases} \quad \text{and} \quad \tilde{\tau}^-_{U,L} = \begin{cases}\tilde{\tau}_L & \tilde{\tau}_L < \tilde{\tau}_U \\ \infty & \tilde{\tau}_L > \tilde{\tau}_U\end{cases}$$

The distributions of these two random times were derived by a different method. In fact, for some fixed $t > 0$, it is not difficult to see

$$\begin{aligned}
\Pr_{s_0}\left(\tilde{\tau}^+_{U,L} \in \mathrm{d}t\right) &= \Pr_{s_0}\left(\tilde{\tau}_U \in \mathrm{d}t, \tilde{\tau}_U < \tilde{\tau}_L\right), \\
\Pr_{s_0}\left(\tilde{\tau}^-_{U,L} \in \mathrm{d}t\right) &= \Pr_{s_0}\left(\tilde{\tau}_L \in \mathrm{d}t, \tilde{\tau}_L < \tilde{\tau}_U\right).
\end{aligned}$$

Hence, the formulation in Kolkiewicz (2002) is essentially the same as ours.

Remark 5.3.5. Our approach can easily be generalized to the case where the rebate payments also depend on hitting times $\tilde{\tau}_U$ and $\tilde{\tau}_L$. In particular, we can evaluate

$$\mathrm{E}_{s_0}\left[e^{-r\tilde{\tau}_U} R_{\tilde{\tau}_U} \mathbb{1}(\tilde{\tau}_U < \tilde{\tau}_L \wedge T)\right] \quad \text{and} \quad \mathrm{E}_{s_0}\left[e^{-r\tilde{\tau}_L} R_{\tilde{\tau}_L} \mathbb{1}(\tilde{\tau}_L < \tilde{\tau}_U \wedge T)\right],$$

where R_t is some rebate payment function.

5.4 Multi-asset barrier options

The basic version of multi-asset barrier options was first studied by Heynen and Kat (1994a) which referred to such type of options as “outside barrier options” because the option features an “outside barrier” determined by an “outside asset” which is different from and correlated with the asset governing the original payoff. Similar discussions can also be found in Carr (1995) and Lee (2004) under the same bivariate model assumption as in Heynen and Kat (1994a). Kwok, Wu and Yu (1998) made an extension to multi-asset model by finding the Green function of a related partial differential equation. The problem of two-sided outside barriers was investigated in Wong and Kwok (2003) and they developed the splitting direction technique to yield a systematic valuation procedure by deriving the related joint density function.

The remainder of this section will develop a new unified formulation of multi-asset barrier options that handles both one-sided and two-sided barriers. We still assume an arbitrary payoff function and allow the barriers to vary exponentially in time. Using the well-known Cholesky decomposition, we shall show that the price of multi-asset barrier options can be easily derived from their one-asset counterparts without much further effort. Some numerical examples will be provided at the end. In addition, we shall extend our results to price a double knock-out option where the barriers are stochastic and modeled by geometric Brownian motions.

5.4.1 Multi-asset model and the representation formulas

Consider m underlying assets with time-t price vector $\boldsymbol{S}_t = (S_{1t}, \cdots, S_{mt})'$. The process of the i-th asset is modeled as

$$S_{it} = S_{i0} \exp\left(X_{it}\right), \quad i = 1, 2, \ldots, m, \quad t \geq 0,$$

where we assume

$$X_{it} = \mu_i t + \sigma_i Z_{it}$$

with $\{Z_{it}\}$ being a standard Brownian motion, $1 \leq i \leq m$. Denote by ρ_{ij} the correlation coefficient between $\mathrm{d}Z_{it}$ and $\mathrm{d}Z_{jt}$, $i \neq j$. We let the payoff function be $h : (0, \infty)^m \to (0, \infty)$ and without loss of generality, use the first individual asset as the outside asset. The time-0 forward price of multi-asset barrier options can be generally expressed as

$$\mathrm{E}_{\boldsymbol{s}}\left[h(\boldsymbol{S}_T)\mathbb{1}_{\mathcal{B}(S_{1t}, 0 \leq t \leq T)}\right], \tag{5.20}$$

where $\mathrm{E}_{\boldsymbol{s}}[\cdot]$ means the expectation is calculated given $\boldsymbol{S}_0 = \boldsymbol{s}$ and we denote $\mathcal{B}\left(S_{1t}, 0 \leq t \leq T\right)$ as some event regarding the sample path of $\{S_{1t}\}$ over time interval $[0, T]$.

The major complication involved in the derivation of (5.20) comes from the correlation between the outside asset $\{S_{1t}\}$ and the additional source of randomness $(S_{2T}, \cdots, S_{mT})'$ in the payoff. We use the Cholesky decomposition to isolate the outside asset price variable from other random variables. Define the drift vector $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_m)'$ and the diffusion matrix $\boldsymbol{\Sigma} = (\Sigma_{ij})_{m\times m}$ where $\Sigma_{ii} = \sigma_i^2$ and $\Sigma_{ij} = \rho_{ij}\sigma_i\sigma_j, i \neq j$. It is a well-known result that $\boldsymbol{X}_t = (X_{1t}, \cdots, X_{mt})'$ has the form

$$\boldsymbol{X}_t = \boldsymbol{\mu}t + \boldsymbol{P}\boldsymbol{W}_t, \tag{5.21}$$

where $\boldsymbol{P}$ is lower triangular such that $\boldsymbol{P}\boldsymbol{P}' = \boldsymbol{\Sigma}$ and $\boldsymbol{W}_t = (W_{1t}, \cdots, W_{mt})'$ is an m-dimensional standard Brownian motion which means that $\{W_{it}\}, 1 \leq i \leq m$, are all standard Brownian motions and mutually independent. Let $\boldsymbol{P} = (p_{ij})_{m\times m}$ where $p_{ij} = 0, i < j$. Then equation (5.21) yields m individual equations

$$\begin{aligned} S_{1t} &= S_{10} \exp\left(\mu_1 t + p_{11} W_{1t}\right), \\ S_{it} &= S_{i0} \exp\left(\mu_i t + \sum_{j=1}^{i} p_{ij} W_{jt}\right), \quad i = 2, \ldots, m. \end{aligned}$$

Therefore, we can rewrite $S_{it}, 2 \leq i \leq m$ as

$$S_{it} = (S_{1t})^{\frac{p_{i1}}{p_{11}}} \hat{S}_{it}, \tag{5.22}$$

where

$$\hat{S}_{it} = \hat{S}_{i0} \exp\left(\left(\mu_i - \frac{p_{i1}}{p_{11}}\mu_1\right) t + \sum_{j=2}^{i} p_{ij} W_{jt}\right), \tag{5.23}$$

and $\left\{\hat{S}_{it}\right\}$ is independent of $\{S_{1t}\}$, $2 \leq i \leq m$. Therefore, we can isolate the outside asset S_{1t} from other source of uncertainty, and the expectation (5.20) can be evaluated conditional on the joint distribution of $\left(\hat{S}_{2T}, \cdots, \hat{S}_{mT}\right)'$, which can reduce the problem of pricing multi-asset barrier options to that of pricing barrier options consisting of only one asset S_{1t}. In particular, we obtain the following result.

Lemma 5.4.1. *Let $q(x_2, \cdots, x_m)$ be the joint density function of $\left(\hat{S}_{2T}, \cdots, \hat{S}_{mT}\right)'$ defined by (5.23). Then the expectation (5.20) has the integral representation*

$$\int \cdots \int_{x_i > 0} \mathrm{E}\left[\hat{h}\left(S_{1T}; x_2, \cdots, x_m\right) \mathbb{1}_{\mathcal{B}(S_{1t}, 0 \le t \le T)}\right] q(x_2, \cdots, x_m) \mathrm{d}x_2 \cdots \mathrm{d}x_m, \quad (5.24)$$

where we define

$$\hat{h}(s; x_2, \cdots, x_m) = h\left(s, s^{\hat{\rho}_2} x_2, \cdots, s^{\hat{\rho}_m} x_m\right)$$

and $\hat{\rho}_i = \frac{\rho_{i1}\sigma_i}{\sigma_1}, i = 2, \ldots, m$.

Proof of Lemma 5.4.1. It is only necessary to calculate the entries in the first column of matrix $\boldsymbol{P}$. The classical algorithm for the Cholesky decomposition implies that $p_{11} = \sqrt{\Sigma_{11}} = \sigma_1$ and $p_{i1} = \frac{\Sigma_{i1}}{p_{11}} = \rho_{i1}\sigma_i, i = 2, \ldots, m$. Hence, $\hat{\rho}_i = \frac{p_{i1}}{p_{11}} = \frac{\rho_{i1}\sigma_i}{\sigma_1}$. Then the expression (5.24) follows immediately from the fact that $\{S_{1t}; 0 \le t \le T\}$ is independent of $\left(\hat{S}_{2T}, \cdots, \hat{S}_{mT}\right)'$. $\square$

Formula (5.24) provides a systematic methodology for pricing multi-asset barrier options. One can observe that the expectation in the integrand of (5.24), when fixing the dummy variables $(x_2, \cdots, x_m)'$, is the price of standard barrier options with one asset (the outside asset), which has been studied thoroughly in Chapter 4. Some may find (5.24) quite complicated as it requires high dimensional integration and every entry of $\boldsymbol{P}$ has to be determined. However, since the expectation in the integrand of (5.24) can be written as the combination of the prices of plain vanilla options, we will show that we do not need to know the formula for $q(x_2, \cdots, x_m)$, and the explicit calculation of the $(m-1)$-dimensional integral is not necessary, either. Based on the expression given in Lemma 5.4.1, we can handle multi-asset barrier options with a variety of barrier provisions.

The following two theorems present closed-form solutions of the prices of multi-asset barrier options with single flat barrier and double exponential barriers.

Theorem 5.4.2. *Let the initial asset price vector be $\boldsymbol{s}_0 = (s_1, \cdots, s_m)'$. The time-0 forward prices of up-and-in, up-and-out, down-and-in and down-and-out multi-asset barrier options with a flat barrier B are respectively given by*

$$C_{\text{ui}} = C_h^+(\boldsymbol{s}_0, B) + \left(\frac{B}{s_1}\right)^{\hat{\kappa}} C_h^-(\hat{\boldsymbol{s}}_0, B), \quad s_1 < B, \tag{5.25}$$

$$C_{\text{uo}} = C_h^-(\boldsymbol{s}_0, B) - \left(\frac{B}{s_1}\right)^{\hat{\kappa}} C_h^-(\hat{\boldsymbol{s}}_0, B), \quad s_1 < B, \tag{5.26}$$

$$C_{\text{di}} = C_h^-(\boldsymbol{s}_0, B) + \left(\frac{B}{s_1}\right)^{\hat{\kappa}} C_h^+(\hat{\boldsymbol{s}}_0, B), \quad s_1 > B, \tag{5.27}$$

$$C_{\text{do}} = C_h^+(\boldsymbol{s}_0, B) - \left(\frac{B}{s_1}\right)^{\hat{\kappa}} C_h^+(\hat{\boldsymbol{s}}_0, B), \quad s_1 > B, \tag{5.28}$$

where $C_h^{\pm}(\boldsymbol{s}, x) = \mathrm{E}_{\boldsymbol{s}}\left[h(\boldsymbol{S}_T)\mathbb{1}\left(S_{1T} \gtrless x\right)\right]$ are the forward prices of some binary options, $\hat{\kappa} = \frac{2\mu_1}{\sigma_1^2}$ and

$$\hat{\boldsymbol{s}}_0 = \begin{pmatrix} \frac{B^2}{s_1} & s_2\left(\frac{B}{s_1}\right)^{2\hat{\rho}_2} & \cdots & s_m\left(\frac{B}{s_1}\right)^{2\hat{\rho}_m} \end{pmatrix}'$$

with $\hat{\rho}_i = \frac{\rho_{i1}\sigma_i}{\sigma_1}, i = 2, \ldots, m$.

Theorem 5.4.3. *Let the initial asset price vector be $\boldsymbol{s}_0 = (s_1, \cdots, s_m)'$ with $L < s_1 < U$. The time-0 forward price of double knock-out multi-asset barrier options with the exponential boundaries $U_t = Ue^{\delta_1 t}$ and $L_t = Le^{\delta_2 t}$ is given by the doubly infinite sum*

$$C_{\text{edko}} = \sum_{n=-\infty}^{\infty} \left(\frac{s_1\beta^n}{L}\right)^{\hat{\gamma}_n} \beta^{n\hat{\kappa}_2} \left[C_h^*\left(\boldsymbol{s}_0^{(n)}\right) - \left(\frac{L}{s_1\beta^{2n}}\right)^{\hat{\kappa}_2} C_h^*\left(\hat{\boldsymbol{s}}_0^{(n)}\right)\right], \tag{5.29}$$

where $\beta = \frac{L}{U}$, $\hat{\gamma}_n = n(\hat{\kappa}_2 - \hat{\kappa}_1), \hat{\kappa}_1 = \frac{2(\mu_1-\delta_1)}{\sigma_1^2}$, $\hat{\kappa}_2 = \frac{2(\mu_1-\delta_2)}{\sigma_1^2}$ and

$$C_h^*(\boldsymbol{s}) = \mathrm{E}_{\boldsymbol{s}}\left[h(\boldsymbol{S}_T)\mathbb{1}\left(L_T < S_{1T} < U_T\right)\right] \tag{5.30}$$

denotes the forward price of a binary option and

$$\boldsymbol{s}_0^{(n)} = \begin{pmatrix} s_1\beta^{2n} & s_2\beta^{2n\hat{\rho}_2} & \cdots & s_m\beta^{2n\hat{\rho}_m} \end{pmatrix}',$$

$$\hat{\boldsymbol{s}}_0^{(n)} = \left(\frac{L^2}{s_1\beta^{2n}} \quad s_2\left(\frac{L}{s_1\beta^n}\right)^{2\hat{\rho}_2} \quad \cdots \quad s_m\left(\frac{L}{s_1\beta^n}\right)^{2\hat{\rho}_m} \right)',$$

with $\hat{\rho}_i = \frac{\rho_{i1}\sigma_i}{\sigma_1}, i = 2, \ldots, m.$

Theorems 5.4.2 and 5.4.3 can be regarded as extensions of Theorem 4.2.1 and 4.4.4, respectively. For the sake of brevity, we only provide the proof of (5.25) and leave it to readers to derive the remaining results in an analogous manner. It is worth noting that our approach, unlike identifying related joint density functions as in Kwok, Wu and Yu (1998) and Wong and Kwok (2003), yields simpler and more general formula by obviating the need for complicated integrations of density functions. We write the prices of multi-asset barrier options as the combination of $C_h^{\pm}$'s or C_h^{*}'s, the prices of binary options (all-or-nothing options), which can be numerically computed. For a commonly used payoff function h, the expressions of $C_h^{\pm}$ and C_h^{*} usually involve sequences of multivariate normal distribution functions. As a check, we let $m = 2$ and the payoff function $h(s_1, s_2) = (s_2 - K)^+$ in our setup; then formula (5.25) reproduces the result derived in Carr (1995) and our method is far less complicated. Similarly, one can also let the payoff function $h(\boldsymbol{s}) = (\max(s_2, \cdots, s_m) - K)^+$ and $\delta_1 = \delta_2 = 0$ in Theorem 5.4.3 to reproduce the pricing formula derived in Wong and Kwok (2003).

Remark 5.4.1. In the analysis of multi-asset barrier options, if the asset that controls the barrier provision is functionally independent of the payoff, this asset is called an external barrier variable; otherwise, this asset is called an internal barrier variable. The articles we mentioned earlier at the beginning of this section about pricing multi-asset barrier options all dealt with external barrier variables, while our formulation slightly admits an internal variable since the barrier variable s_1 also governs the payoff function $h(\boldsymbol{s})$. Unfortunately, our approach presented here fails for a more general internal variable, for example, that is determined by a geometric basket of assets. The discussion of this case can be found in Skipper (2007) and Section 10.9 of Buchen (2012), which assumed that the barrier variable takes the form of a power function $\boldsymbol{s}^{\boldsymbol{x}} = \prod_{i=1}^{m} s_i^{x_i}$ for some powers x_i's. They derived the image solution for single-barrier options based on a generalized binary pricing formula, but nevertheless did not consider two-sided exponential boundaries.

Proof of (5.25). Define the hitting time $\tau_B^{(1)} = \inf\{t > 0 | S_{1t} = B\}$. Then the time-0 forward price of an up-and-in multi-asset barrier option is expressed as

$$\mathrm{E}_{\boldsymbol{s}_0}\left[h(\boldsymbol{S}_T)\mathbb{1}\left(\tau_B^{(1)} < T\right)\right].$$

Fixing the dummy variables $x_2, \cdots, x_m$, we apply Theorem 4.2.1 by considering a payoff function $\hat{h}(s; x_2, \ldots, x_m)$. Write $\hat{h}(s; x_2, \cdots, x_m)$ as $\hat{h}(s)$ for short; then Theorem 4.2.1 implies

$$\begin{aligned}\mathrm{E}_{s_1}\left[\hat{h}\left(S_{1T}\right)\mathbb{1}\left(\tau_B^{(1)} < T\right)\right] &= \mathrm{E}_{s_1}\left[\hat{h}\left(S_{1T}\right)\mathbb{1}\left(S_{1T} > B\right)\right] \\ &\quad + \left(\frac{B}{s_1}\right)^{\hat{\kappa}} \mathrm{E}_{\frac{B^2}{s_1}}\left[\hat{h}\left(S_{1T}\right)\mathbb{1}\left(S_{1T} < B\right)\right],\end{aligned}$$

where $\hat{\kappa} = \frac{2\mu_1}{\sigma_1^2}$. In Lemma 5.4.1, we let $\mathcal{B}\left(S_{1t}, 0 \le t \le T\right) = \{\tau_B^{(1)} < T\}$ and substitute the right-hand side of the equation above into (5.24). Formula (5.25) then follows by some simple algebraic calculations. Now let us explain how to obtain the modified initial value vector $\hat{\boldsymbol{s}}_0$. Given $S_{i0} = s_i, i = 1, 2, \ldots, m$, (5.22) yields

$$\hat{S}_{i0} = \frac{S_{i0}}{\left(S_{10}\right)^{\hat{\rho}_i}} = \frac{s_i}{\left(s_1\right)^{\hat{\rho}_i}}, \quad i = 2, \ldots, m.$$

If S_{10} becomes $\frac{B^2}{s_1}$, we use (5.22) again. For $i \geq 2$, we have

$$S_{i0} = (S_{10})^{\hat{\rho}_i}\, \hat{S}_{i0} = \left(\frac{B^2}{s_1}\right)^{\hat{\rho}_i} \frac{s_i}{(s_1)^{\hat{\rho}_i}} = s_i \left(\frac{B}{s_1}\right)^{2\hat{\rho}_i}.$$

Thus the proof is complete. $\square$

Example 5.4.1 (C_h^* for a rainbow call option)**.** For certain forms of $h(\boldsymbol{s})$, we can derive closed-form solutions of C_h^* given by (5.30), in which case the formulas in Theorems 5.4.2 and 5.4.3 become explicit. In order to numerically implement our results in the next section, we study a particular example of a rainbow call option written on a geometric basket of assets. In particular, we let the payoff function $h(\boldsymbol{s}) = (\boldsymbol{s}^{\boldsymbol{w}} - K)^+$ where $\boldsymbol{s}^{\boldsymbol{w}}$ is defined as $\boldsymbol{s}^{\boldsymbol{w}} = \prod_{i=1}^m s_i^{w_i}$ and obtain the following proposition. The proof is given in Section 5.6.1.

Proposition 5.4.4. *Let* $\boldsymbol{s} = (s_1, \cdots, s_m)'$ *and* $h(\boldsymbol{S}_T) = ({\boldsymbol{S}_T}^{\boldsymbol{w}} - K)^+$ *with a strike price* K*. Then*

$$\begin{aligned} C_h^*(\boldsymbol{s}) = \boldsymbol{s}^{\boldsymbol{w}} \exp\left(\boldsymbol{w}'\boldsymbol{\mu}T + \frac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}T\right) &\left[\Psi\left(\ln\frac{U_T}{s_1}\right) - \Psi\left(\ln\frac{L_T}{s_1}\right)\right] \\ - K &\left[\hat{\Psi}\left(\ln\frac{U_T}{s_1}\right) - \hat{\Psi}\left(\ln\frac{L_T}{s_1}\right)\right], \end{aligned} \tag{5.31}$$

where

$$\Psi(z) = \Phi_2\left(\frac{z - \mu_1 T - \boldsymbol{e}'\boldsymbol{\Sigma}\boldsymbol{w}T}{\sigma_1\sqrt{T}}, \frac{\ln\frac{\boldsymbol{s}^{\boldsymbol{w}}}{K} + \boldsymbol{w}'\boldsymbol{\mu}T + \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}T}{\sqrt{\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}T}}; -\frac{\boldsymbol{e}'\boldsymbol{\Sigma}\boldsymbol{w}}{\sigma_1\sqrt{\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}}}\right),$$

$$\hat{\Psi}(z) = \Phi_2\left(\frac{z - \mu_1 T}{\sigma_1\sqrt{T}}, \frac{\ln\frac{\boldsymbol{s}^{\boldsymbol{w}}}{K} + \boldsymbol{w}'\boldsymbol{\mu}T}{\sqrt{\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}T}}; -\frac{\boldsymbol{e}'\boldsymbol{\Sigma}\boldsymbol{w}}{\sigma_1\sqrt{\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}}}\right),$$

$\boldsymbol{s}^{\boldsymbol{w}} = \prod_{i=1}^m s_i^{w_i}$ *and* $\Phi_2(\cdot, \cdot\,; \rho)$ *is the joint distribution function of bivariate standard normal with correlation* ρ*. Here,* $\boldsymbol{\mu}$ *and* $\boldsymbol{\Sigma}$ *are the drift vector and the diffusion matrix of* $\boldsymbol{X}_T$ *respectively and* $\boldsymbol{w} = (w_1, w_2, \cdots, w_m)', \boldsymbol{e} = (1, 0, \cdots, 0)'$.

5.4.2 Numerical examples

We perform numerical valuation of the formulas in Theorem 5.4.3 and Proposition 5.4.4 for a special case where $\boldsymbol{w} = (0, 1, 0, \cdots, 0)'$; then the payoff function $h(\boldsymbol{s}) = (s_2 - K)^+$. We calculate the option prices given varying levels of barriers, volatilities and correlations. The common parameter values are: $s_1 = s_2 = 1000$, $K = 1000$, $r = 0.05$, $\sigma_2 = 0.3$, $T = 0.5$, $\delta_1 = 0.1$ and $\delta_2 = -0.1$. We also consider different choices of σ_1 and ρ_{12}. The numerical results are provided in Table 5.2. To obtain the no-arbitrage time-0 prices, we replace μ_i by $r - \frac{1}{2}\sigma_i^2, i = 1, 2$, and multiply the discount factor e^{-rT} in the corresponding formula.

Table 5.2: Double knock-out call with an external variable

σ_1/σ_2	L/U	$\rho_{12} = -0.2$	$\rho_{12} = 0$	$\rho_{12} = 0.2$	$\rho_{12} = 1$
0.2/0.3	$0/\infty$	96.35	96.35	96.35	96.35
	400/1600	96.34	96.31	96.27	95.93
	500/1500	96.27	96.16	95.97	94.54
	600/1400	95.92	95.44	94.72	89.57
	700/1300	93.65	92.24	90.14	75.16
0.3/0.3	$0/\infty$	96.35	96.35	96.35	96.35
	400/1600	95.59	94.81	93.68	85.88
	500/1500	94.38	92.85	90.74	76.57
	600/1400	90.33	88.03	84.71	61.48
	700/1300	76.96	75.03	71.47	40.54
0.4/0.3	$0/\infty$	96.35	96.35	96.35	96.35
	400/1600	92.53	90.04	86.77	65.70
	500/1500	88.01	85.00	80.87	53.04
	600/1400	76.33	74.01	70.08	37.86
	700/1300	52.92	52.27	49.98	21.75

Our implementation requires the computation of some bivariate normal distributions. We point out that for certain payoff functions, multivariate normal distributions will be used, leading to computational inefficiency. For example, when $h(\boldsymbol{s}) = (\max(s_2, \cdots, s_m) - K)^+$, which was considered in Wong and Kwok (2003), the pricing formula involves m-dimensional normal probabilities, which are never easy to compute numerically for a large value of m. When analyzing our examples, we employed the function "pmvnorm" in the R package "mvtnorm", which computes multivariate normal probabilities with arbitrary correlation matrices. This program specifies an algorithm proposed in Genz (1992), and this algorithm performs a sequence of initial transformations to transform the original integral into an integral over a unit hyper-cube, which can be handled efficiently using either Monte-Carlo or subregion adaptive numerical integration. This moderately remedies the computational issue caused by high dimensionality.

Through the implementation, we note that the doubly infinite series in (5.29) converge extremely rapidly and only a few terms are required to achieve numerical accuracy. The option prices are observed to increase as the barrier interval grows wider or the volatility of the outside asset becomes smaller. This justifies our intuition that the value of knock-out options rises as it is less likely for the barriers to be breached. The impact of the correlation coefficient is however undetermined. The overall trend according to the numbers in Table 5.2 appears to be that the options are worth less as the two assets become more positively correlated, but it is not the case for other parameter values. Figure 5.1 illustrates the complicated relations between the option prices and the correlations. It is also worthwhile to point out that when $\rho_{12} = 1$ and $\sigma_1 = \sigma_2$, formula (5.29) reduces to the known formula (4.34) under the one-asset model. Therefore, we should expect with no surprise that the numbers in the last column of Table 5.2 when $\sigma_1 = \sigma_2 = 0.3$ are identical to those in Table 4.1 when $\sigma = 0.3$, $\delta_1 = 0.1$ and $\delta_2 = -0.1$. Interestingly, the option prices are seen to be all identical when $L = 0$ and $U = \infty$. Because in these two cases, the barrier options reduce to vanilla call options whose values only depend on the performance of $\{S_{2t}\}$. Hence, the option values will have nothing to do with either the correlation or the value of the outside asset.

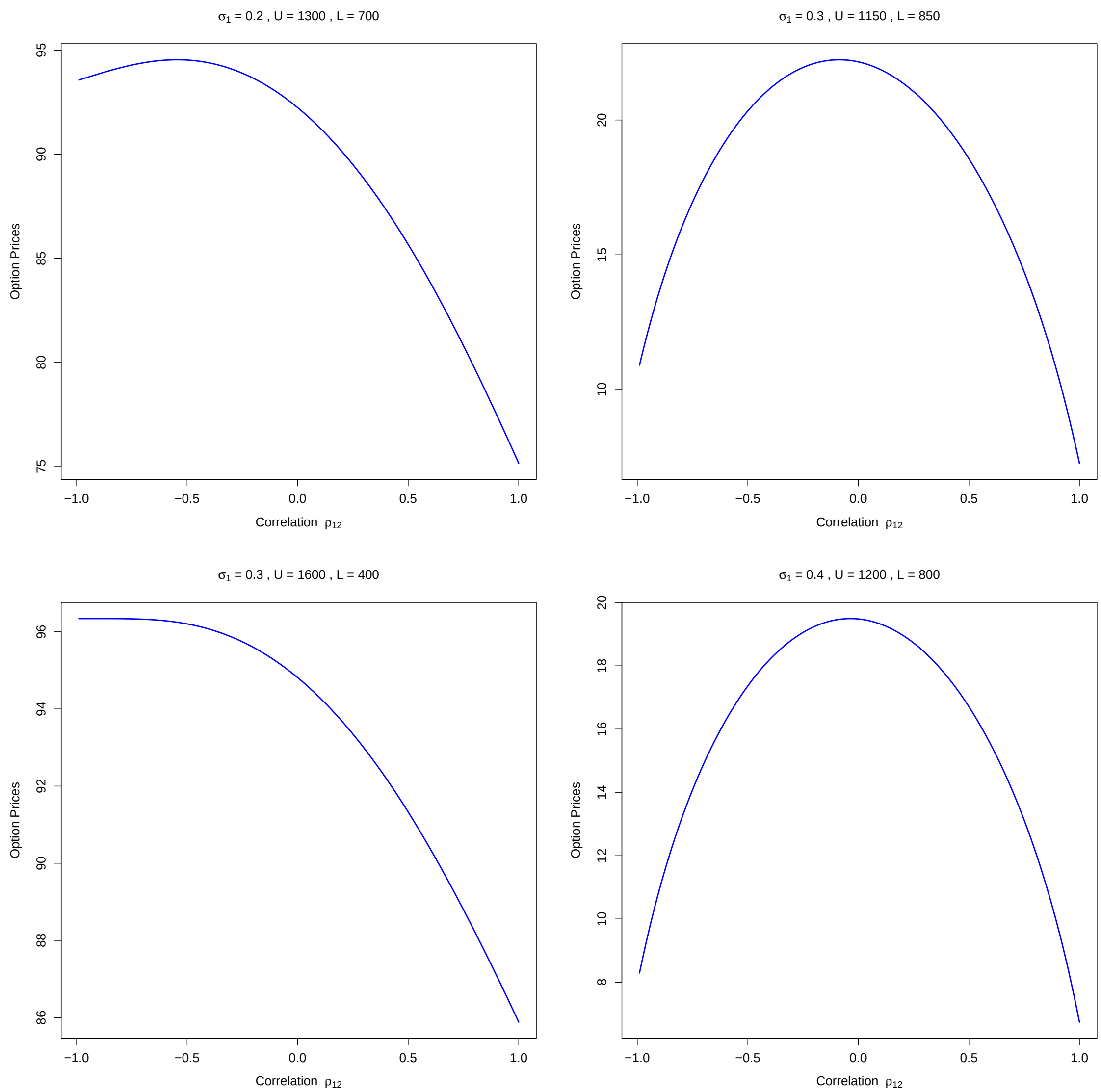

Figure 5.1: Multi-asset double knock-out call price vs. correlation

5.4.3 Extension to two-sided stochastic barrier

The availability of pricing formulas within the multi-asset framework allows us to make an extension to a more flexible case where the time-varying barriers are stochastic and also driven by geometric Brownian motions. Let us express our point by studying a double knock-out option.

We still follow the model setting given at the beginning of Section 5.4.1: the payoff function is $h : (0,\infty)^m \to (0,\infty)$ and the asset price process $\boldsymbol{S}_t = (S_{1t}, \cdots, S_{mt})'$ satisfies the usual conditions. We define a stochastic upper boundary $U_t = Ue^{X_{1t}^B}$ and a stochastic lower boundary $L_t = Le^{X_{2t}^B}$ where

$$X_{1t}^B = \mu_1^B t + \sigma_B Z_t^B, \quad t \geq 0,$$
$$X_{2t}^B = \mu_2^B t + \sigma_B Z_t^B, \quad t \geq 0,$$

with $\{Z_t^B\}$ being a standard Brownian motion. Hence, the two boundaries share the same uncertainty and volatility, and we assume the correlation between $\mathrm{d}Z_{it}$ and $\mathrm{d}Z_t^B$ is denoted by ρ_i^B, $i = 1, 2, \ldots, m$. Then the time-0 forward price of a double knock-out option restricted by these two stochastic boundaries is given by

$$\mathrm{E}_{\boldsymbol{s}}\left[h(\boldsymbol{S}_T)\mathbb{1}\left(Le^{X_{2t}^B} < S_{1t} < Ue^{X_{1t}^B}, 0 \leq t \leq T\right)\right], \quad \boldsymbol{s} = (s_1, s_2, \cdots, s_m)'. \tag{5.32}$$

As usual, we assume $L < s_1 < U$, and also $Le^{\mu_2^B T} < Ue^{\mu_1^B T}$ to avoid the situation where the two boundaries intersect at some point before time T. We can easily absorb the uncertainty component introduced by the boundaries into the barrier variable by noting the following obvious identity:

$$\left\{Le^{X_{2t}^B} < S_{1t} < Ue^{X_{1t}^B}, 0 \leq t \leq T\right\} = \left\{Le^{(\mu_2^B - \mu_1^B)t} < s_1 e^{X_{1t} - X_{1t}^B} < U, 0 \leq t \leq T\right\}.$$

Alternatively, we can also rewrite the left-hand side of the identity above as

$$\left\{L < s_1 e^{X_{1t} - X_{2t}^B} < Ue^{(\mu_1^B - \mu_2^B)t}, 0 \leq t \leq T\right\}.$$

Therefore, the problem reduces to the one we already solved in the multi-asset framework, where the process of the barrier variable is given by $\{X_{1t}-X_{1t}^B\}$ and the curvatures of the upper and lower boundaries are respectively 0 and $\mu_2^B-\mu_1^B$. Now it only becomes necessary to identify the distribution of the multivariate process $(X_{1t}-X_{1t}^B, X_{1t}, \cdots, X_{mt})'$.

The drift vector and diffusion matrix of $(X_{1t}^B, X_{1t}, \cdots, X_{mt})'$ are respectively given by

$$\begin{pmatrix} \mu_1^B \\ \boldsymbol{\mu} \end{pmatrix}_{(m+1)\times 1} \quad \text{and} \quad \begin{pmatrix} \sigma_B^2 & \boldsymbol{\sigma}'_B \\ \boldsymbol{\sigma}_B & \boldsymbol{\Sigma} \end{pmatrix}_{(m+1)\times(m+1)},$$

where $\boldsymbol{\sigma}_B = \left(\rho_1^B\sigma_1\sigma_B, \rho_2^B\sigma_2\sigma_B, \ldots, \rho_m^B\sigma_m\sigma_B\right)'$, and $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are respectively the drift vector and the diffusion matrix of $(X_{1t}, X_{2t}, \cdots, X_{mt})'$. Therefore, by a simple transformation, one can easily write down the drift vector and the diffusion matrix of $(X_{1t}-X_{1t}^B, X_{1t}, \cdots, X_{mt})'$ as

$$\begin{pmatrix} -1 & \boldsymbol{e}' \\ \boldsymbol{0} & \boldsymbol{I}_m \end{pmatrix}\begin{pmatrix} \mu_1^B \\ \boldsymbol{\mu} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & \boldsymbol{e}' \\ \boldsymbol{0} & \boldsymbol{I}_m \end{pmatrix}\begin{pmatrix} \sigma_B^2 & \boldsymbol{\sigma}'_B \\ \boldsymbol{\sigma}_B & \boldsymbol{\Sigma} \end{pmatrix}\begin{pmatrix} -1 & \boldsymbol{e}' \\ \boldsymbol{0} & \boldsymbol{I}_m \end{pmatrix}',$$

where $\boldsymbol{0} = (0, 0, \cdots, 0)'$, $\boldsymbol{e} = (1, 0, \cdots, 0)'$ and $\boldsymbol{I}_m$ denotes an $m\times m$ identity matrix. Now one is able to readily apply the pricing formula provided in Theorem 5.4.3 to calculate the expectation (5.32).

5.5 Window barrier options

The purpose of this section is to discuss the valuation of a class of barrier options where the underlying asset price is monitored only during a fraction of the options' life. Before mentioning window barrier options, let us first introduce their regular form, which is called partial barrier options. There are two types of partial barrier options. One is forward-starting barrier options where the monitoring period starts at a specified date strictly after the options initiate. The other is early-ending barrier options where the monitoring period terminates at a specified date strictly before the options expire. With the feature of partial monitoring window, partial barrier options can be employed to limit the risk of the barriers to be knocked in or out, and have been extensively traded in the foreign exchanges or the over-the-counter markets, to replace or supplement traditional barrier options. Heynen and Kat (1994b) and Carr (1995) derived pricing formulas of partial barrier options in terms of bivariate normal probabilities. The application of static hedging technique to valuing partial barrier options can be found in Carr and Chou (2002).

Window barrier options incorporate both the forward-starting and early-ending monitorings and thus offer more flexible structure, as opposed to partial barrier options. Investors who hold window barrier options can enjoy a more customized hedging or investing experience by carefully choosing the location of monitoring period according to how they evaluate the financial markets. Guillaume (2003) studied window double knock-out options with flat barriers as well as a more exotic case where single and double barriers are mixed during multiple disjoint monitoring periods.

We will apply the results obtained in Chapter 4 to value window double knock-out options with exponential boundaries and arbitrary payoff functions. By repetitive conditioning, we will show that window barrier options can be viewed as compound options written on certain standard barrier options. Assuming a single segment of monitoring window with a fixed starting time t_1 and a fixed ending time t_2, we express the pricing formula in terms of trivariate normal distribution functions. The following begins with the case of forward-starting window and then extends to the case of forward-starting and early-ending window. At the end, some numerical examples and related discussion are provided.

5.5.1 Forward-starting monitoring window

We assume that the barriers are not visible until some pre-specified time point t_1 and stay active in the remaining lifetime of the contract. Therefore, this type of options behaves as plain vanilla options prior to time t_1 and then becomes identical to a standard double-barrier option during the monitoring period from t_1 to T. We can express the time-0 forward price $W_\pi^{(1)}$ of the forward-starting double knock-out options as the following expectation,

$$W_\pi^{(1)}(s_0, t_1, T) = \mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}\left(\max_{t_1 \le t \le T}(S_t - U_t) < 0, \min_{t_1 \le t \le T}(S_t - L_t) > 0\right)\right]. \quad (5.33)$$

Note that in this case, s_0 is not necessarily between L and U if $t_1 \neq 0$. By conditioning on $\mathcal{F}_{t_1}$, we can rewrite (5.33) as

$$W_\pi^{(1)}(s_0, t_1, T) = \mathrm{E}_{s_0}\left[\mathbb{1}\left(L_{t_1} < S_{t_1} < U_{t_1}\right) V_\pi(S_{t_1}, T - t_1; L_{t_1}, U_{t_1})\right], \quad (5.34)$$

where we let $V_\pi(s, T - t_1; L_{t_1}, U_{t_1})$ denote the time-0 forward price of the standard double knock-out options; π is the payoff function, s is the initial asset price, $T - t_1$ is the maturity time, L_{t_1} and U_{t_1} are the initial levels of the two barriers. Equation (5.34) shows that this window barrier option can be viewed as a compound option: this compound option gives the holder the right to purchase a standard barrier option at time t_1 at the cost of V_π if $L_{t_1} < S_{t_1} < U_{t_1}$. The explicit formula for V_π can be derived from the pricing formula (4.34). Then we express $W_\pi^{(1)}$ by the integration

$$W_\pi^{(1)}(s_0, t_1, T) = \int_{L_{t_1}}^{U_{t_1}} V_\pi(s, T - t_1; L_{t_1}, U_{t_1}) f_{S_{t_1}}(s) \mathrm{d}s, \quad (5.35)$$

where $f_{S_{t_1}}(s)$ denotes the probability density function of S_{t_1} given $S_0 = s_0$. In fact, we are able to give an analytical solution of $W_\pi^{(1)}$. But to save space and focus on the more general case, let us move on to the next section where we combine forward-starting and early-ending windows.

5.5.2 Forward-starting and early-ending monitoring window

We assume that monitoring of the double barriers starts at time t_1 and ends at time t_2 where $0 < t_1 < t_2 < T$. The time-0 forward price $W_\pi^{(2)}$ of this type of window barrier options can be expressed as

$$W_\pi^{(2)}(s_0, t_1, t_2, T) = \mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}\left(\max_{t_1 \le t \le t_2}(S_t - U_t) < 0, \min_{t_1 \le t \le t_2}(S_t - L_t) > 0\right)\right]. \tag{5.36}$$

The case of forward-starting and early-ending window can immediately reduce to the case of only forward-starting window. In particular, we have

$$W_\pi^{(2)}(s_0, t_1, t_2, T) = W_\chi^{(1)}(s_0, t_1, t_2) \tag{5.37}$$

$$= \mathrm{E}_{s_0}\left[\mathbb{1}\left(L_{t_1} < S_{t_1} < U_{t_1}\right) V_\chi(S_{t_1}, t_2 - t_1; L_{t_1}, U_{t_1})\right], \tag{5.38}$$

where $\chi(s)$ is defined as

$$\chi(s) = \mathrm{E}\left[\pi(S_T) | \mathcal{F}_{t_2}, S_{t_2} = s\right]. \tag{5.39}$$

Equation (5.37) is derived by conditioning on $\mathcal{F}_{t_2}$ of the expectation in (5.36), and (5.38) follows immediately from (5.34). Similar to (5.35), $W_\pi^{(2)}$ can also be computed by numerical integration as long as we know the formula for V_χ, which is derived from Theorem 4.4.4 by replacing T by $t_2 - t_1$, $\pi(s)$ by $\chi(s)$, L by L_{t_1} and U by U_{t_1}. Specifically, one can easily show that

$$\begin{aligned} V_\chi(s, t_2 - t_1; L_{t_1}, U_{t_1}) = &\sum_{n=-\infty}^{\infty}\left(\frac{s\underline{\beta}^n}{L_{t_1}}\right)^{\gamma_n}\underline{\beta}^{n\kappa_2} P_\chi\left(s\underline{\beta}^{2n}, t_2 - t_1, L_{t_2}, U_{t_2}\right) \\ &- \sum_{n=-\infty}^{\infty}\left(\frac{s\underline{\beta}^n}{L_{t_1}}\right)^{\gamma_n - \kappa_2} P_\chi\left(\frac{L_{t_1}^2}{s\underline{\beta}^{2n}}, t_2 - t_1, L_{t_2}, U_{t_2}\right), \end{aligned} \tag{5.40}$$

where for the ease of analysis, we define $\underline{\beta} = \frac{L_{t_1}}{U_{t_1}}$ and

$$P_\chi(s, t, x, y) = \mathrm{E}_s\left[\chi(S_t)\mathbb{1}(x < S_t < y)\right].$$

Based on the definition of $\chi(s)$ by (5.39), we evaluate $P_\chi\left(s, t_2 - t_1, L_{t_2}, U_{t_2}\right)$ in the following manner.

$$
\begin{aligned}
P_\chi\left(s, t_2-t_1, L_{t_2}, U_{t_2}\right) &= \mathrm{E}_s\left[\chi(S_{t_2-t_1})\mathbb{1}\left(L_{t_2} < S_{t_2-t_1} < U_{t_2}\right)\right] \\
&= \mathrm{E}\left[\chi(S_{t_2})\mathbb{1}\left(L_{t_2} < S_{t_2} < U_{t_2}\right)|S_{t_1}=s\right] \\
&= \mathrm{E}\left[\mathrm{E}\left[\pi(S_T)|\mathcal{F}_{t_2}\right]\mathbb{1}\left(L_{t_2} < S_{t_2} < U_{t_2}\right)|S_{t_1}=s\right] \\
&= \mathrm{E}\left[\pi(S_T)\mathbb{1}\left(L_{t_2} < S_{t_2} < U_{t_2}\right)|S_{t_1}=s\right]. \qquad (5.41)
\end{aligned}
$$

When the payoff function $\pi(s)$ is of call or put type, we can explicitly write P_χ in terms of bivariate normal probabilities.

We pursue a closed-form solution of $W_\pi^{(2)}$, and this can be achieved by combining (5.38), (5.40) and (5.41). In particular, we obtain the following theorem.

Theorem 5.5.1. *Let s_0 be the initial asset value. The two exponential boundaries $U_t = Ue^{\delta_1 t}$ and $L_t = Le^{\delta_2 t}$ are monitored over the time interval from t_1 to t_2 where $0 < t_1 < t_2 < T$. The time-0 forward price $W_\pi^{(2)}$ of the window double knock-out options is given by the doubly infinite sum*

$$
W_\pi^{(2)}(s_0, t_1, t_2, T) = \sum_{n=-\infty}^{\infty}\left\{\mathrm{E}_{s_0}\left[\pi\left(\underline{\beta}^{2n}S_T\right)I_n\right] - \mathrm{E}_{s_0}\left[\pi\left(\frac{L_{t_1}^2}{\underline{\beta}^{2n}}\frac{S_T}{S_{t_1}^2}\right)J_n\right]\right\}, \qquad (5.42)
$$

where $\underline{\beta} = \frac{L_{t_1}}{U_{t_1}}$,

$$
\begin{aligned}
I_n &= \left(\frac{S_{t_1}\underline{\beta}^n}{L_{t_1}}\right)^{\gamma_n}\underline{\beta}^{n\kappa_2}\mathbb{1}\left(L_{t_1} < S_{t_1} < U_{t_1}, L_{t_2} < \underline{\beta}^{2n}S_{t_2} < U_{t_2}\right), \\
J_n &= \left(\frac{S_{t_1}\underline{\beta}^n}{L_{t_1}}\right)^{\gamma_n-\kappa_2}\mathbb{1}\left(L_{t_1} < S_{t_1} < U_{t_1}, L_{t_2} < \frac{L_{t_1}^{2n}}{\underline{\beta}^{2n}}\frac{S_{t_2}}{S_{t_1}^2} < U_{t_2}\right),
\end{aligned}
$$

and $\gamma_n = n(\kappa_2-\kappa_1)$, $\kappa_1 = \frac{2(\mu-\delta_1)}{\sigma^2}$ and $\kappa_2 = \frac{2(\mu-\delta_2)}{\sigma^2}$.

Theorem 5.5.1 provides a representation formula for $W_\pi^{(2)}$ in terms of the prices of options whose payoffs only depend on the asset values at time points $t=0$, t_1, t_2 and T. For the formula to be well-defined, t_1 must be strictly greater than 0 and

t_2 must be strictly less than T. However, one can still let t_2 tend to T to recover a forward-starting monitoring window and let t_1 tend to 0 to recover an early-ending monitoring window. The known formula (4.34) for the standard double knock-out options can be reproduced by letting t_1 and t_2 tend to 0 and T respectively at the same time.

Proof of Theorem 5.5.1. Formula (5.42) can be derived by substituting (5.40) and (5.41) into (5.38). The order of summation and expectation can be interchanged because every term involved is positive. Let us consider the n-th summand in the first doubly infinte sum of (5.40) and denote it by $Q_n(s)$. By equation (5.41),

$$\begin{aligned} Q_n(s) &= \left(\frac{s\underline{\beta}^n}{L_{t_1}}\right)^{\gamma_n} \underline{\beta}^{n\kappa_2} \mathrm{E}\left[\pi(S_T)\mathbb{1}\left(L_{t_2} < S_{t_2} < U_{t_2}\right) \middle| S_{t_1} = s\underline{\beta}^{2n}\right] \\ &= \left(\frac{s\underline{\beta}^n}{L_{t_1}}\right)^{\gamma_n} \underline{\beta}^{n\kappa_2} \mathrm{E}\left[\pi\left(\underline{\beta}^{2n} S_T\right)\mathbb{1}\left(L_{t_2} < \underline{\beta}^{2n} S_{t_2} < U_{t_2}\right) \middle| S_{t_1} = s\right], \end{aligned}$$

and hence, conditional on $\mathcal{F}_{t_1}$, we have

$$Q_n(S_{t_1}) = \mathrm{E}\left[\left(\frac{S_{t_1}\underline{\beta}^n}{L_{t_1}}\right)^{\gamma_n} \underline{\beta}^{n\kappa_2} \pi\left(\underline{\beta}^{2n} S_T\right)\mathbb{1}\left(L_{t_2} < \underline{\beta}^{2n} S_{t_2} < U_{t_2}\right) \middle| \mathcal{F}_{t_1}\right].$$

From the inspection of (5.38), we need to compute

$$\mathrm{E}_{s_0}\left[\mathbb{1}\left(L_{t_1} < S_{t_1} < U_{t_1}\right) Q_n(S_{t_1})\right].$$

Using the fact that $\mathbb{1}\left(L_{t_1} < S_{t_1} < U_{t_1}\right)$ is $\mathcal{F}_{t_1}$-measurable as well as using the law of iterated expectations lead to the first expectation in the summations of (5.42). The second expectation can be computed in an analogous manner. $\square$

Now let us apply Theorem 5.5.1 to find the explicit solution for a window double knock-out call option.

Corollary 5.5.2. *When $\pi(s) = (s-K)^+$ for a strike price K, then the doubly infinite series given by (5.42) can be expressed as*

$$\begin{aligned}
&W_\pi^{(2)}(s_0, t_1, t_2, T) \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{\underline{\beta}^n}{L_{t_1}}\right)^{\gamma_n+\kappa_2} L_{t_1}^{\kappa_2} \left\{\underline{\beta}^{2n}\mathcal{E}(\gamma_n,1)\mathcal{I}_{1n} - K\mathcal{E}(\gamma_n,0)\mathcal{I}_{2n}\right\} \\
&\quad - \sum_{n=-\infty}^{\infty} \left(\frac{\underline{\beta}^n}{L_{t_1}}\right)^{\gamma_n-\kappa_2} \frac{1}{\underline{\beta}^{2n}} \left\{L_{t_1}^2\mathcal{E}(\gamma_n-\kappa_2-2,1)\mathcal{I}_{3n} - \underline{\beta}^{2n}K\mathcal{E}(\gamma_n-\kappa_2,0)\mathcal{I}_{4n}\right\},
\end{aligned}$$

where we have defined $\underline{\beta} = \frac{L_{t_1}}{U_{t_1}}$, $\gamma_n = n(\kappa_2-\kappa_1)$, $\kappa_1 = \frac{2(\mu-\delta_1)}{\sigma^2}$, and $\kappa_2 = \frac{2(\mu-\delta_2)}{\sigma^2}$. We also define

$$\mathcal{E}(x,y) = s_0^{x+y}\exp\left(\left(\mu x + \frac{\sigma^2}{2}x^2 + \sigma^2 xy\right)t_1 + \left(\mu y + \frac{\sigma^2}{2}y^2\right)T\right)$$

and

$$\begin{aligned}
\mathcal{I}_{1n} &= \Theta_1^+(\alpha_2,\alpha_4,\alpha_5) - \Theta_1^+(\alpha_2,\alpha_3,\alpha_5) - \Theta_1^+(\alpha_1,\alpha_4,\alpha_5) + \Theta_1^+(\alpha_1,\alpha_3,\alpha_5), \\
\mathcal{I}_{2n} &= \Theta_2^+(\alpha_2,\alpha_4,\alpha_5) - \Theta_2^+(\alpha_2,\alpha_3,\alpha_5) - \Theta_2^+(\alpha_1,\alpha_4,\alpha_5) + \Theta_2^+(\alpha_1,\alpha_3,\alpha_5), \\
\mathcal{I}_{3n} &= \Theta_3^-(\alpha_2,\alpha_7,\alpha_8) - \Theta_3^-(\alpha_2,\alpha_6,\alpha_8) - \Theta_3^-(\alpha_1,\alpha_7,\alpha_8) + \Theta_3^-(\alpha_1,\alpha_6,\alpha_8), \\
\mathcal{I}_{4n} &= \Theta_4^-(\alpha_2,\alpha_7,\alpha_8) - \Theta_4^-(\alpha_2,\alpha_6,\alpha_8) - \Theta_4^-(\alpha_1,\alpha_7,\alpha_8) + \Theta_4^-(\alpha_1,\alpha_6,\alpha_8),
\end{aligned}$$

where

$$\begin{aligned}
\Theta_2^\epsilon(x,y,z) &= \Phi_3\Bigg(\frac{x-\mu t_1-\gamma_n\sigma^2 t_1}{\sigma\sqrt{t_1}}, \frac{y-\mu t_2-\gamma_n\sigma^2 t_1}{\sigma\sqrt{t_2}}, \frac{z+\mu T+\gamma_n\sigma^2 t_1}{\sigma\sqrt{T}}; \\
&\qquad\qquad \epsilon\sqrt{\frac{t_1}{t_2}}, -\epsilon\sqrt{\frac{t_1}{T}}, -\sqrt{\frac{t_1}{t_2}}\Bigg), \\
\Theta_1^\epsilon(x,y,z) &= \Theta_2^\epsilon\left(x-\sigma^2 t_1, y-\sigma^2 t_2, z+\sigma^2 T\right), \\
\Theta_4^\epsilon(x,y,z) &= \Theta_2^\epsilon\left(x+\kappa_2\sigma^2 t_1, y+2\mu t_1+(2\gamma_n-\kappa_2)\sigma^2 t_1, z-2\mu t_1-(2\gamma_n-\kappa_2)\sigma^2 t_1\right), \\
\Theta_3^\epsilon(x,y,z) &= \Theta_4^\epsilon\left(x+\sigma^2 t_1, y-\sigma^2 t_2, z+\sigma^2 T\right),
\end{aligned}$$

with $\epsilon = 1$ *or* -1 *and* $\Phi_3(\cdot,\cdot,\cdot\,;\rho_{12},\rho_{13},\rho_{23})$ *being the distribution function of trivariate standard normal with correlations* ρ_{12}, ρ_{13} *and* ρ_{23}, *and*

$$\alpha_1 = \ln\frac{L_{t_1}}{s_0},\quad \alpha_2 = \ln\frac{U_{t_1}}{s_0},\quad \alpha_3 = \ln\frac{L_{t_2}}{s_0\underline{\beta}^{2n}},\quad \alpha_4 = \ln\frac{U_{t_2}}{s_0\underline{\beta}^{2n}},$$
$$\alpha_5 = \ln\frac{s_0\underline{\beta}^{2n}}{K},\quad \alpha_6 = \ln\frac{s_0L_{t_2}\underline{\beta}^{2n}}{L_{t_1}^2},\quad \alpha_7 = \ln\frac{s_0U_{t_2}\underline{\beta}^{2n}}{L_{t_1}^2},\quad \alpha_8 = \ln\frac{L_{t_1}^2}{s_0K\underline{\beta}^{2n}}.$$

Proof of Corollary 5.5.2. Let $\pi(s) = (s-K)^+$ in Theorem 5.5.1 and define $X_t = \ln\frac{S_t}{S_0}$. After some simple calculations, we have

$$\begin{aligned} W_\pi^{(2)}(s_0,t_1,t_2,T) = &\sum_{n=-\infty}^{\infty}\left(\frac{\underline{\beta}^n}{L_{t_1}}\right)^{\gamma_n+\kappa_2} L_{t_1}^{\kappa_2}\left(\underline{\beta}^{2n}\mathcal{J}_{1n} - K\mathcal{J}_{2n}\right) \\ &- \sum_{n=-\infty}^{\infty}\left(\frac{\underline{\beta}^n}{L_{t_1}}\right)^{\gamma_n-\kappa_2} \frac{1}{\underline{\beta}^{2n}}\left(L_{t_1}^2\mathcal{J}_{3n} - \underline{\beta}^{2n}K\mathcal{J}_{4n}\right), \end{aligned}$$

where

$$\begin{aligned} \mathcal{J}_{1n} &= \mathrm{E}\left[S_{t_1}^{\gamma_n}S_T\mathbb{1}\left(\alpha_1 < X_{t_1} < \alpha_2, \alpha_3 < X_{t_2} < \alpha_4, -X_T < \alpha_5\right)\right], \\ \mathcal{J}_{2n} &= \mathrm{E}\left[S_{t_1}^{\gamma_n}\mathbb{1}\left(\alpha_1 < X_{t_1} < \alpha_2, \alpha_3 < X_{t_2} < \alpha_4, -X_T < \alpha_5\right)\right], \\ \mathcal{J}_{3n} &= \mathrm{E}\left[S_{t_1}^{\gamma_n-\kappa_2-2}S_T\mathbb{1}\left(\alpha_1 < X_{t_1} < \alpha_2, \alpha_6 < X_{t_2} - 2X_{t_1} < \alpha_7, 2X_{t_1} - X_T < \alpha_8\right)\right], \\ \mathcal{J}_{4n} &= \mathrm{E}\left[S_{t_1}^{\gamma_n-\kappa_2}\mathbb{1}\left(\alpha_1 < X_{t_1} < \alpha_2, \alpha_6 < X_{t_2} - 2X_{t_1} < \alpha_7, 2X_{t_1} - X_T < \alpha_8\right)\right]. \end{aligned}$$

Then it is sufficient to show that

$$\begin{aligned} \mathcal{J}_{1n} &= \mathcal{E}(\gamma_n, 1)\mathcal{I}_{1n}, & \mathcal{J}_{2n} &= \mathcal{E}(\gamma_n, 0)\mathcal{I}_{2n}, \\ \mathcal{J}_{3n} &= \mathcal{E}(\gamma_n - \kappa_2 - 2, 1)\mathcal{I}_{3n}, & \mathcal{J}_{4n} &= \mathcal{E}(\gamma_n - \kappa_2, 0)\mathcal{I}_{4n}. \end{aligned}$$

Note that by the definition of $\mathcal{E}(x,y)$, we have $\mathcal{E}(x,y) = \mathrm{E}_{s_0}\left[S_{t_1}^x S_T^y\right]$. Applying the Esscher transform factorization (2.8), we can rewrite each of the expectation above as the product of $\mathcal{E}(x,y)$ and a trivariate normal probability under some transformed measure with certain choices of x and y. Let us take $\mathcal{J}_{3n}$ as an example.

$$\begin{aligned}\mathcal{J}_{3n} &= \mathcal{E}(\gamma_n - \kappa_2 - 2, 1) \\ &\qquad \times \Pr\left(\alpha_1 < X_{t_1} < \alpha_2, \alpha_6 < X_{t_2} - 2X_{t_1} < \alpha_7, 2X_{t_1} - X_T < \alpha_8; \boldsymbol{h}\right),\end{aligned}$$

where $\boldsymbol{h} = (\gamma_n - \kappa_2 - 2, 0, 1)'$. We need to show that the probability in the expression above is equal to $\mathcal{I}_{3n}$. In fact, this two-sided probability can be easily rewritten in terms of four one-sided probabilities, and it is only necessary to identify the mean vector and the covariance matrix of the trivariate normal vector $(X_{t_1}, X_{t_2} - 2X_{t_1}, 2X_{t_1} - X_T)'$ under the transformed measure with index $\boldsymbol{h}$. It is easy to verify that the mean vector and the covariance matrix of the triplet $(X_{t_1}, X_{t_2}, X_T)'$ under the original measure are respectively given by

$$\boldsymbol{m} = \begin{pmatrix} \mu t_1 \\ \mu t_2 \\ \mu T \end{pmatrix} \quad \text{and} \quad \boldsymbol{Q} = \begin{pmatrix} \sigma^2 t_1 & \sigma^2 t_1 & \sigma^2 t_1 \\ \sigma^2 t_1 & \sigma^2 t_2 & \sigma^2 t_2 \\ \sigma^2 t_1 & \sigma^2 t_2 & \sigma^2 T \end{pmatrix}.$$

Because of the transformation

$$\begin{pmatrix} X_{t_1} \\ X_{t_2} - 2X_{t_1} \\ 2X_{t_1} - X_T \end{pmatrix} = \boldsymbol{R} \begin{pmatrix} X_{t_1} \\ X_{t_2} \\ X_T \end{pmatrix} \quad \text{with} \quad \boldsymbol{R} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix},$$

the mean vector and the covariance matrix of $(X_{t_1}, X_{t_2} - 2X_{t_1}, 2X_{t_1} - X_T)'$ under the transformed measure with index $\boldsymbol{h}$ can be respectively expressed as

$$\boldsymbol{R}(\boldsymbol{m} + \boldsymbol{Q}\boldsymbol{h}) \quad \text{and} \quad \boldsymbol{R}\boldsymbol{Q}\boldsymbol{R}'.$$

We standardize the trivariate normal vector and the result follows immediately. The other three quantities $\mathcal{J}_{1n}$, $\mathcal{J}_{2n}$ and $\mathcal{J}_{4n}$ can be computed in a similar fashion. $\square$

5.5.3 Numerical examples

We perform the numerical valuation of the formula in Corollary 5.5.2. We examine various choices of barrier levels, curvature rates and monitoring periods. The common parameter values are: $s_0 = 1000$, $K = 1000$, $r = 0.05$, $\sigma = 0.3$, $t_2 = 0.4$ and $T = 0.5$. The left end-point t_1 of the monitoring window has values 0.1, 0.2 or 0.3. To obtain the time-0 arbitrage-free price, we let $\mu = r - \frac{1}{2}\sigma^2$ and multiply the discount factor e^{-rT} in the formula. The related trivariate normal probabilities are computed using the function "pmvnorm" in the R package "mvtnorm" which employs an algorithm proposed by Genz (1992). The results are provided in Table 5.3 where we also compare them with the prices of standard double knock-out call options (SDKOCall), in which case $t_1 = 0$ and $t_2 = T = 0.5$.

Table 5.3: Window double knock-out call vs. standard double knock-out call

δ_1/δ_2	L/U	$t_1 = 0.1$	$t_1 = 0.2$	$t_1 = 0.3$	SDKOCall
0.1/-0.1	$0/\infty$	96.35	96.35	96.35	96.35
	400/1600	91.19	91.20	91.41	85.88
	500/1500	84.88	84.95	85.54	76.57
	600/1400	72.92	73.18	74.63	61.48
	700/1300	53.56	54.40	57.19	40.54
0/0	$0/\infty$	96.35	96.35	96.35	96.35
	400/1600	88.20	88.22	88.50	80.06
	500/1500	79.61	79.68	80.40	67.88
	600/1400	64.89	65.15	66.71	50.23
	700/1300	43.77	44.53	47.16	28.90
-0.1/0.1	$0/\infty$	96.35	96.35	96.35	96.35
	400/1600	84.00	84.02	84.37	72.22
	500/1500	72.85	72.93	73.74	57.30
	600/1400	55.65	55.91	57.49	38.10
	700/1300	33.93	34.56	36.88	18.22

The option values are observed to decrease with the narrowing of the barrier interval or the monitoring window, as a consequence of the increasing likelihood for the options to expire worthless upon breaching the barriers. Two extreme cases are also investigated. One is $L = 0$ and $U = \infty$, and the result reduces to \$96.35, the price of vanilla call options regardless of the monitoring period. The other is $t_1 = 0$ and $t_2 = 0.5$. In this case, the barriers are visible during the entire lifetime of the contract and the results become the prices of standard double knock-out call (SDKOCall), which are listed in the last column of Table 5.3 as benchmarks. These numbers are copied directly from Table 4.1, not computed using the formula given by Corollary 5.5.2, because when $t_1 = 0$ and $t_2 = 0.5$, the formula is invalid for implementation as the trivariate normal probabilities reduce to univariate normal probabilities.

5.6 Appendix

5.6.1 Proof of Proposition 5.4.4

Proof. First we can verify that given $\boldsymbol{s} = (s_1, s_2, \cdots, s_m)'$,

$$\begin{aligned} C_h^*(\boldsymbol{s}) &= \mathrm{E}\left[\left(\prod_{i=1}^{m}(S_{iT})^{w_i} - K\right)^+ \mathbb{1}\left(L_T < S_{1T} < U_T\right)\right] \\ &= \boldsymbol{s}^{\boldsymbol{w}}\mathrm{E}\left[\exp\left(\sum_{i=1}^{m} w_i X_{iT}\right) \mathbb{1}\left(\ln\frac{L_T}{s_1} < X_{1T} < \ln\frac{U_T}{s_1}, \sum_{i=1}^{m} w_i X_{iT} > \ln\frac{K}{\boldsymbol{s}^{\boldsymbol{w}}}\right)\right] \\ &\quad - K\mathrm{Pr}\left(\ln\frac{L_T}{s_1} < X_{1T} < \ln\frac{U_T}{s_1}, \sum_{i=1}^{m} w_i X_{iT} > \ln\frac{K}{\boldsymbol{s}^{\boldsymbol{w}}}\right). \end{aligned} \tag{5.43}$$

Applying the Esscher transform factorization (2.8), we further express the last expectation in (5.43) as

$$\mathrm{E}\left[\exp\left(\sum_{i=1}^{m} w_i X_{iT}\right)\right] \mathrm{Pr}\left(\ln\frac{L_T}{s_1} < X_{1T} < \ln\frac{U_T}{s_1}, -\sum_{i=1}^{m} w_i X_{iT} < \ln\frac{\boldsymbol{s}^{\boldsymbol{w}}}{K}; \boldsymbol{w}\right),$$

where $\boldsymbol{w}$ after the semicolon indicates that the probability is calculated under the transformed measure with index $\boldsymbol{w}$. Note that $\sum_{i=1}^m w_i X_{iT} = \boldsymbol{w}'\boldsymbol{X}_T$ follows normal distribution with mean $\boldsymbol{w}'\boldsymbol{\mu}T$ and variance $\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}T$. Hence,

$$\mathrm{E}\left[\exp\left(\sum_{i=1}^m w_i X_{iT}\right)\right] = \exp\left(\boldsymbol{w}'\boldsymbol{\mu}T + \frac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}T\right).$$

Now it is only necessary to identify the distribution of the vector

$$\left(X_{1T}, -\sum_{i=1}^m w_i X_{iT}\right)'$$

under the new measure. Note that

$$\left(X_{1T}, -\sum_{i=1}^m w_i X_{iT}\right)' = \boldsymbol{\nu}'\boldsymbol{X}_T,$$

where $\boldsymbol{\nu} = (\boldsymbol{e}, -\boldsymbol{w})$ is an $m \times 2$ matrix with $\boldsymbol{e} = (1, 0, \cdots, 0)'$. Then the random vector $\boldsymbol{\nu}'\boldsymbol{X}_T$ has a bivariate normal distribution. As a result of the discussion at the end of Example 2.5.1, one can show the mean vector and the covariance matrix under the transformed measure are respectively given by

$$\boldsymbol{\nu}'(\boldsymbol{\mu} + \boldsymbol{\Sigma}\boldsymbol{w})T = \begin{pmatrix} \mu_1 T + \boldsymbol{e}'\boldsymbol{\Sigma}\boldsymbol{w}T \\ -\boldsymbol{w}'\boldsymbol{\mu}T - \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}T \end{pmatrix}_{2\times 1}$$

and

$$\boldsymbol{\nu}'\boldsymbol{\Sigma}\boldsymbol{\nu}T = \begin{pmatrix} \sigma_1^2 T & -\boldsymbol{e}'\boldsymbol{\Sigma}\boldsymbol{w}T \\ -\boldsymbol{e}'\boldsymbol{\Sigma}\boldsymbol{w}T & \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}T \end{pmatrix}_{2\times 2}.$$

On the left-hand sides of the identities above, the product of the vector (matrix) and the scalar T is taken entry-wise. Now we can compute (5.43) by standardizing the normal vectors and the desired result follows. $\square$

CHAPTER 6

PRICING BARRIER OPTIONS IN JUMP-DIFFUSION MODELS

6.1 Introduction

Jump-diffusion processes have been extensively considered in insurance and finance as a standard modeling tool. For actuaries, the use of pure jump processes can be traced back to the fundamental work of Filip Lundberg in collective risk theory, where the aggregate claims were modeled by compound Poisson processes. See, for example, Gerber (1970, 1972) and Dufresne and Gerber (1991b), where a Brownian motion was added to the compound Poisson process. Jump diffusions are also widely used to model an insurer's liabilities. For example, Cummins (1988) and Duan and Yu (2005) evaluated the risk-based premiums of insurance guaranty funds in jump diffusions. It is analytically convenient if the distribution of individual claim size is a mixture (or a combination) of exponential distributions, and quantities such as ruin probability and expected discounted dividends until ruin can be determined in explicit forms. See, for example, Dufresne and Gerber (1988, 1989, 1991a,b), Chan (1990), Gerber and Shiu (1998a, 2005), Chan, Gerber and Shiu (2006), Gerber, Shiu and Smith (2006) and Avanzi and Gerber (2008).

Jump diffusions also find their popularity in finance. Jump-diffusion models are able to capture dramatic changes in the underlyings, and naturally exhibit some significant empirical facts such as leptokurtosis and implied volatility smiles, which the traditional BS framework cannot account for. Merton (1976) modeled stock price movements by adding an independent compound Poisson process to a Brownian motion and assuming the jump magnitude is normally distributed. To obtain particular tractability for path-dependent options, Kou (2002) proposed a jump-diffusion model where the jump magnitude follows an asymmetric double exponential distribution. To generalize the model in Kou (2002), Cai, Chen and Wan (2009) and Cai and Kou (2011) respectively considered a mixture and a combination of exponential distributions as the distribution of the jump magnitude. Besides analytical convenience, the benefit of assuming combinations of exponential distributions for jump sizes is that they can be used to approximate the distribution of any jump size in the sense of weak convergence. See Cai and Kou (2011) for some numerical examples where combinations of exponential distributions were used to estimate heavy-tail distributions including Gamma, Pareto and Weibull. For models based on general Lévy processes, one can see Carr et al. (2003), Cont and Tankov (2004) and reference therein.

The volatilities of asset returns estimated from empirical data are basically stochastic and clustered; such phenomena cannot be captured under diffusions with deterministic volatilities. Therefore, models incorporating stochastic volatilities were proposed to fit clustering effects and long-term behavior. See, for example, Heston (1993), Bates (1996), Duffie, Pan and Singleton (2000), Carr et al. (2003) and Alòs, Chen and Rheinländer (2016).

The difficulty in pricing barrier options under jump-diffusion models mainly comes from the fact that the boundary crossing can be realized by either touching the boundary or jumping across the boundary, which should be treated separately. One method is to establish a partial integro-differential equation that the option value satisfies and solve it numerically. Some techniques such as variational methods and extrapolation have been developed by, for example, Feng, Linetsky and Marcozzi (2004) and Feng and Linetsky (2008), to efficiently solve this type of equations.

The other method is to identify the joint distribution of the jump-diffusion process at maturity and its exit times through Laplace transform inversion. In general, this Laplace transform is difficult, if not impossible, to derive analytically. However, for particular cases where the distribution of the jump magnitude is a mixture or combination of exponential distributions, the Laplace transforms associated with the one-sided and two-sided exit times can be obtained explicitly in fairly simple forms, partially because of the memoryless property. See, for example, Kou and Wang (2004), Kou, Petrella and Wang (2005), Cai, Chen and Wan (2009) and Cai and Kou (2011). See also Gerber, Shiu and Yang (2013) who obtained corresponding Laplace transforms based on the Wiener-Hopf factorization. A recent study can be found in Alòs, Chen and Rheinländer (2016) which valued barrier options under stochastic volatility models based on a general self-duality.

In the approach pioneered by Kou and his co-authors, it is not as easy to get the Laplace transform for curved boundaries as for flat boundaries, and double exponential distribution has to be assumed for the jump size to obtain closed-form solutions. Our purpose is to derive pricing formulas for knock-out options with exponential boundaries, arbitrary payoffs, and more flexible jump distributions. In our model, the underlying asset price process $\{S_t\}$ is modeled by the dynamic

$$S_t = S_0 \exp\left(\mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right), \quad t \geq 0, \tag{6.1}$$

where $\{W_t\}$ is a standard Brownian motion, $\{N_t\}$ is a Poisson process with intensity λ and $\{Y_i\}_{i\geq 1}$ are jump sizes and are independent and identically distributed (i.i.d.) with a common density function $f_Y(y)$. We further assume $\{W_t\}$, $\{N_t\}$ and $\{Y_i\}_{i\geq 1}$ are mutually independent. The sum part $\left\{\sum_{i=1}^{N_t} Y_i\right\}$ is usually called a compound Poisson process.

Our work can be viewed as an extention of Shao and Wang (2012) who studied the distributions of one-sided and two-sided exit times of $\{\ln(S_t/S_0)\}$ with respect to linear boundaries (and more general non-linear boundaries). We shall extend to derive the joint distributions of $\ln(S_T/S_0)$ and the exit times where T is the maturity time. In fact, as in Shao and Wang (2012), our pricing formulas are also valid for more general counting processes and jump sizes as long as the joint distribution of $(Y_1, Y_2, \ldots, Y_n)$ is available for all $n \geq 1$. However, for risk-neutral valuation purpose, a (homogeneous) Poisson process and i.i.d. jump sizes should be assumed in our discussion. We also point out that we will only calculate the (discounted) expected values under the physical measure. Therefore, our formula is valid in no-arbitrage pricing only if the asset price process has the same jump-diffusion form under the selected risk-neutral measure, which holds if the distribution of the jump size belongs to the exponential family (See Corollary 1 and the remark after it in Kou (2002)). This definitely includes the well-known models proposed in Merton (1976) and Kou (2002).

The remainder of this chapter is organized as follows. Section 6.2 briefly reviews the basic set-up for option pricing in jump-diffusion models (such as models in Merton (1976) and Kou (2002)), preparing for risk-neutral valuation and the numerical analysis thereafter. Section 6.3 studies up-and-out options with single exponential boundary, and Section 6.4 studies double knock-out options with exponential boundaries. Section 6.5 describes how to implement our pricing formulas and some numerical examples are provided. Section 6.6 talks about an immediate application of our method to pricing a step double knock-out option with a set of piecewise exponential boundaries in the BS economy.

6.2 The setting for risk-neutral pricing

We follow the basic setting for option pricing given in Kou (2002, 2007). The market is incomplete when jumps are incorporated into asset prices and thus the risk-neutral measure is not unique. Within an equilibrium framework, Kou (2002) demonstrated that a risk-neutral measure can be chosen under which the asset price process takes the same jump-diffusion form as in (6.1), given that the distribution of $\{Y_i\}_{i\geq 1}$ belongs to the exponential family. In our discussion, we assume the underlying asset does not pay any dividends. We will take a risk-neutral measure as given and assume under that measure,

$$S_t = S_0 \exp\left(\mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right), \quad t \geq 0, \tag{6.2}$$

where the drift is expressed as

$$\mu = r - \frac{\sigma^2}{2} - \left(\mathrm{E}\left[e^{Y_1}\right] - 1\right)\lambda$$

with r being the constant risk-free interest rate. Note that the discounted asset price process $\{e^{-rt}S_t\}$ with the given parameter μ is a martingale.

In Merton (1976), $\{Y_i\}_{i\geq 1}$ follow i.i.d. normal distributions and

$$f_Y(y) = \frac{1}{\sqrt{v}}\phi\left(\frac{y-m}{v}\right),$$

where $\phi(\cdot)$ is the density function of standard normal, m is the mean and $v > 0$ is the standard deviation. The drift under the risk-neutral measure is given by

$$\mu = r - \frac{\sigma^2}{2} - \left(e^{m+\frac{v^2}{2}} - 1\right)\lambda.$$

In Kou (2002), $\{Y_i\}_{i\geq 1}$ follow i.i.d. two-sided exponential distributions and

$$f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y}\mathbb{1}(y \geq 0) + (1-p)\cdot \eta_2 e^{\eta_2 y}\mathbb{1}(y < 0),$$

where $0 \leq p \leq 1$, $\eta_1 > 1$ and $\eta_2 > 0$. The condition $\eta_1 > 1$ guarantees the existence of $\mathrm{E}[e^{Y_1}]$. The drift under the risk-neutral measure is given by

$$\mu = r - \frac{\sigma^2}{2} - \left(\frac{p\eta_1}{\eta_1 - 1} + \frac{(1-p)\eta_2}{\eta_2 + 1} - 1\right)\lambda.$$

We remark that a variety of other jump size distributions can also be incorporated into the model (6.2) such as Bernoulli distribution, Gamma distribution, a mixture (or a combination) of exponential distributions and so on.

Here we also want to reconsider the risk-neutral valuation from a viewpoint of Esscher transforms. When the market is incomplete, the method of Esscher transforms gives a general and unambiguous solution by leading to a unique Esscher risk-neutral measure, which can be determined by solving the equation (2.9) where the Lévy process is given by

$$\ln \frac{S_t}{S_0} = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i.$$

One can easily show that the moment-generating functions of this Lévy process under the physical measure and the new measure with index a^* are respectively written as

$$\mathrm{E}\left[\left(\frac{S_t}{S_0}\right)^z\right] = \exp\left\{\left(\mu z + \frac{\sigma^2 z^2}{2} + \left(\mathrm{E}\left[e^{zY_1}\right] - 1\right)\lambda\right) t\right\},$$

and

$$\mathrm{E}\left[\left(\frac{S_t}{S_0}\right)^z ; a^*\right] = \exp\left\{\left(\left(\mu + a^*\sigma^2\right) z + \frac{\sigma^2 z^2}{2} + \left(\mathrm{E}\left[e^{zY_1}; a^*\right] - 1\right)\lambda \mathrm{E}\left[e^{a^* Y_1}\right]\right) t\right\},$$

from which we can conclude that $\{S_t\}$ takes the same jump-diffusion form under the physical measure and the risk-neutral measure if Y_1 is in the same distribution family under these two measures. This essentially means the jump size distribution belongs to the exponential family, as evidenced by Corollary 1 in Kou (2002).

6.3 Single-barrier options

The objective of this section is to derive an explicit formula for the price of up-and-out options with a time-varying boundary $B_t = Be^{\delta t}$. In particular, we are interested in calculating the expectation

$$\mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}\left(S_t < Be^{\delta t}, 0 \le t \le T\right)\right], \quad s_0 < B. \tag{6.3}$$

We can rewrite the asset price as

$$S_t = S_0 \exp\left(X_t + \sum_{i=1}^{N_t} Y_i\right),$$

where $\{X_t\}$ is a Brownian motion defined by

$$X_t = \mu t + \sigma W_t, \quad t \ge 0. \tag{6.4}$$

By variable transformations and change of drift, one can easily show that the expectation (6.3) is equivalent to

$$\mathrm{E}\left[\psi\left(X_T + \sum_{i=1}^{N_T} Y_i\right)\mathbb{1}\left(X_t + \sum_{i=1}^{N_t} Y_i < b, 0 \le t \le T\right)\right]. \tag{6.5}$$

for some function $\psi(x)$ and some $b > 0$. We can recover (6.3) by letting $\psi(x) = \pi\left(s_0 e^{\delta T + x}\right)$, $b = \ln \frac{B}{s_0}$, and replacing the drift term μ by $\mu - \delta$ in (6.5). Therefore, we shall focus on the evaluation of (6.5) in this section thereafter. Shao and Wang (2012) calculated the boundary crossing probability, which is equal to expectation (6.5) with $\psi(x) = 1$. We will show that the method in Shao and Wang (2012) can be carried out to calculate (6.5) for an arbitrary function $\psi(x)$.

Notice that we can easily rewrite the knock-out event as

$$\left\{X_t + \sum_{i=1}^{N_t} Y_i < b, 0 \le t \le T\right\} = \left\{X_t < b - \sum_{i=1}^{N_t} Y_i, 0 \le t \le T\right\}. \tag{6.6}$$

In light of (6.6), the problem of the jump diffusion delimited by the fixed boundary can be translated into the problem of a linear Brownian motion delimited by a stochastic boundary which is a step function in time and moves according to the Poisson process $\{N_t\}$ (For example, see Figure 6.1 for an illustration). In particular, we partition the time interval $[0, T]$ with arrival times of the Poisson process up to time T and piecewise evaluate the event given by (6.6).

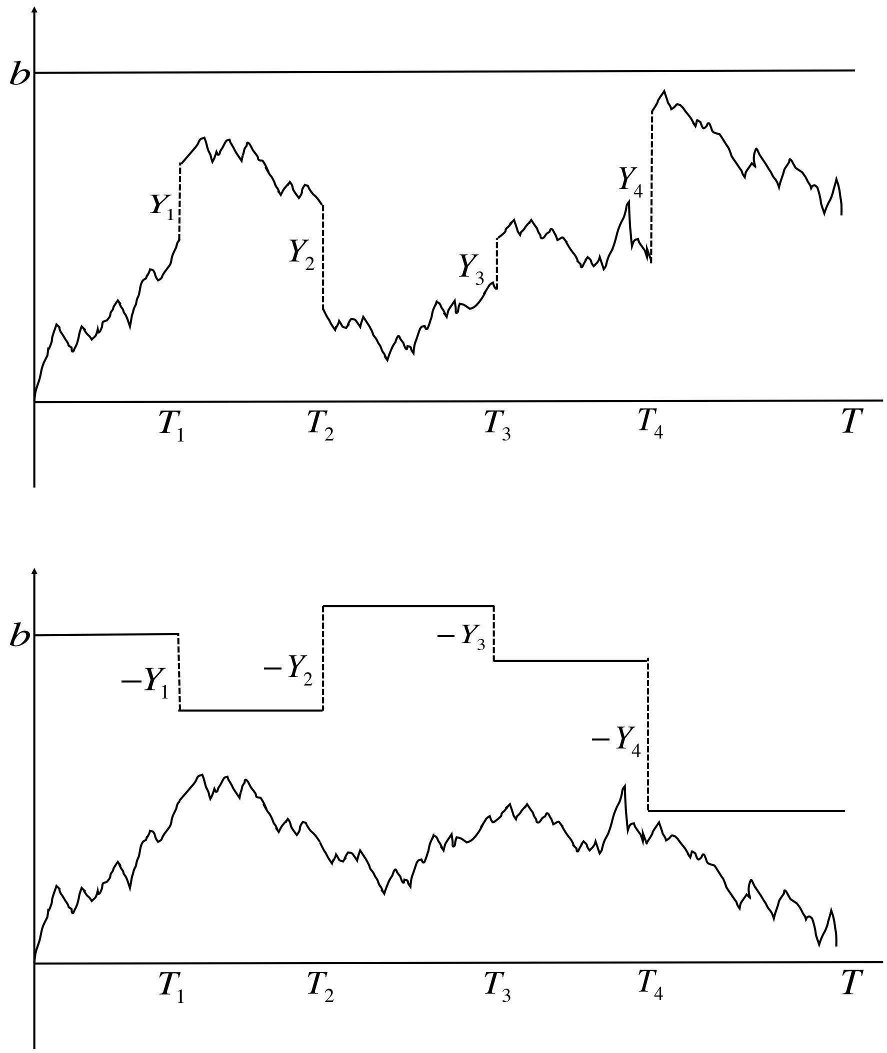

Figure 6.1: The jump-diffusion process with an upper boundary b given $N_T = 4$

For $n \geq 1$, let T_n denote the arrival time of the n-th jump for the Poisson point process $\{N_t\}$. In particular, we define

$$T_0 = 0, \quad T_{n+1} = \inf\left\{t > T_n | N_{t^-} \neq N_t\right\}, \quad n \geq 0.$$

It is a well-known result that the joint distribution of $(T_1, T_2, \cdots, T_n)$ conditional on $N_T = n$ is the same as the distribution of the order statistics of n independent uniform random variables in $(0, T)$. Let $F_n(t_1, t_2, \cdots, t_n)$ denote the corresponding density function. Then

$$F_n(t_1, t_2, \cdots, t_n) = \frac{n!}{T^n}, \quad 0 < t_1 < t_2 < \cdots < t_n < T. \tag{6.7}$$

For notional convenience, we define the cumulative jump size $\hat{Y}_k = \sum_{i=1}^{k} Y_i$, $k \geq 0$, with the convention $\sum_{i\in\emptyset} = 0$. By conditioning on N_T, we can express (6.5) as

$$\begin{aligned}
&\mathrm{E}\left[\psi\left(X_T + \hat{Y}_{N_T}\right)\mathbb{1}\left(X_t + \hat{Y}_{N_t} < b, 0 \leq t \leq T\right)\right] \\
&= \sum_{n=0}^{\infty} \Pr(N_T = n)\mathrm{E}\left[\psi\left(X_T + \hat{Y}_n\right)\mathbb{1}\left(X_t + \hat{Y}_{N_t} < b, 0 \leq t \leq T\right)\middle| N_T = n\right].
\end{aligned} \tag{6.8}$$

The last expectation can be further evaluated by conditioning on the first n arrival times. Based on the reasoning right after (6.6), we have

$$\begin{aligned}
&\mathrm{E}\left[\psi\left(X_T + \hat{Y}_n\right)\mathbb{1}\left(X_t + \hat{Y}_{N_t} < b, 0 \leq t \leq T\right)\middle| N_T = n\right] \\
&= \mathrm{E}\Bigg[\psi\left(X_T + \hat{Y}_n\right)\prod_{k=0}^{n-1}\mathbb{1}\left(X_t < b - \hat{Y}_k, T_k \leq t < T_{k+1}\right) \\
&\qquad\qquad \times \mathbb{1}\left(X_t < b - \hat{Y}_n, T_n \leq t \leq T\right)\Bigg| N_T = n\Bigg].
\end{aligned} \tag{6.9}$$

Here we use the convention $\prod_{k\in\emptyset} = 1$. Because $\{X_t\}$, $\{\hat{Y}_k\}$ and $\{N_t\}$ are mutually independent, and because we know the joint density functions of $(Y_1, Y_2, \cdots, Y_n)$ and $(T_1, T_2, \cdots, T_n)$ given $N_T = n$, it is only necessary to calculate the expectation (6.9) for every non-negative n given $T_k = t_k$ and $Y_k = y_k$ for fixed t_k and y_k, $0 \leq k \leq n$, which essentially corresponds to the value of an up-and-out option restricted by an $(n+1)$-period step boundary in the BS framework. Borrowing an idea from Shao and Wang (2012), we shall show inductively that this $(n+1)$-period step boundary can reduce to a single-period step boundary, which has been studied thoroughly in Chapter 4.

Let us explain how to deal with (6.9) for $n = 0$ and $n = 1$. When $n = 0$, because we assume $\{N_T\}$ is independent of $\{X_t\}$, we can easily reduce (6.9) to

$$\mathrm{E}\left[\psi\left(X_T\right)\mathbb{1}\left(X_t < b, 0 \leq t \leq T\right)\right],$$

which is recognized as the value of an up-and-out option in the BS framework. When $n = 1$, given the density function $F_1(t_1)$ defined by (6.7) and the density function $f_Y(y)$, we can express (6.9) as

$$\int_0^T F_1(t_1)\mathrm{d}t_1 \int_{\mathbb{R}} f_Y(y_1)\mathrm{d}y_1 \mathrm{E}\big[\psi\left(X_T + y_1\right) \mathbb{1}\left(X_t < b, 0 \le t < t_1\right) \\ \times \mathbb{1}\left(X_t < b - y_1, t_1 \le t \le T\right)\big]. \tag{6.10}$$

Note that $\{X_t, t > t_1\}$ and $\{X_t, t < t_1\}$ are independent given $X_{t_1} = x_1$ due to the independent increments property of Brownian motions. Therefore, conditional on $X_{t_1} = x_1$, the expectation in (6.10) is evaluated in the following manner.

$$\begin{aligned}
&\mathrm{E}\left[\psi\left(X_T + y_1\right) \mathbb{1}\left(X_t < b, 0 \le t < t_1\right) \mathbb{1}\left(X_t < b - y_1, t_1 \le t \le T\right)\right] \\
&= \int_{-\infty}^{b \wedge (b - y_1)} \phi_{t_1}(x_1) \mathrm{Pr}(X_t < b, 0 \le t \le t_1 | X_{t_1} = x_1) \\
&\qquad \times \mathrm{E}\left[\psi\left(X_T + y_1\right) \mathbb{1}\left(X_t < b - y_1, t_1 \le t \le T\right) | X_{t_1} = x_1\right] \mathrm{d}x_1 \\
&= \int_{-\infty}^{b \wedge (b - y_1)} \phi_{t_1}(x_1) \mathrm{Pr}(X_t < b, 0 \le t \le t_1 | X_{t_1} = x_1) \\
&\qquad \times \mathrm{E}\left[\psi\left(X_{T - t_1} + x_1 + y_1\right) \mathbb{1}\left(X_t < b - x_1 - y_1, 0 \le t \le T - t_1\right)\right] \mathrm{d}x_1,
\end{aligned}$$

where $\phi_{t_1}(x_1)$ denotes the density function of X_{t_1} and the last step utilizes the independent and stationary increments property of Brownian motions. In the last integral, the conditional probability and the expectation have closed-form formulas, which will be given in Lemma 6.3.1. In fact, the conditional probability is already mentioned in Example 2.2.2, and the expectation corresponds to the value of an up-and-out option in the BS framework with modified payoff function, barrier level and maturity time. For $n \ge 2$, we can evaluate (6.9) in a similar fashion. A general proof will be given right after Theorem (6.3.2).

Lemma 6.3.1. *For $b > 0$, define two auxiliary functions*

$$\mathcal{H}_1(b, x, T) = \Pr(X_t < b, 0 \leq t < T | X_T = x), \quad x < b,$$

$$\mathcal{H}_2^{\psi}(b, y, T) = \mathrm{E}\left[\psi(X_T + y)\mathbb{1}\left(X_t < b, 0 \leq t \leq T\right)\right], \quad y \in \mathbb{R}.$$

Then we have

$$\mathcal{H}_1(b, x, T) = 1 - e^{-\frac{2b(b-x)}{\sigma^2 T}}, \tag{6.11}$$

$$\begin{aligned}\mathcal{H}_2^{\psi}(b, y, T) = {} & \mathrm{E}[\psi(X_T + y)\mathbb{1}\left(X_T < b\right)] \\ & - e^{\frac{2\mu b}{\sigma^2}}\mathrm{E}\left[\psi(X_T + y + 2b)\mathbb{1}\left(X_T < -b\right)\right]. \end{aligned}\tag{6.12}$$

Proof of Lemma 6.3.1. Formula (6.11) is an immediate result of (2.4). Formula (6.12) can be obtained from (4.12) where the payoff function $\pi(s) = \psi(\ln(s/s_0) + y)$ and the barrier $B = s_0 e^b$. Note that (6.11) can also be derived from (6.12) where $y = 0$ and $\psi(z) = \mathbb{1}(z \in \mathrm{d}x)$. □

Theorem 6.3.2. *For a function $\psi(x)$ and $b > 0$, we have*

$$\begin{aligned}
&\mathrm{E}\left[\psi\left(X_T + \hat{Y}_{N_T}\right)\mathbb{1}\left(X_t + \hat{Y}_{N_t} < b, 0 \leq t \leq T\right)\right] \\
&= \sum_{n=0}^{\infty}\left\{\lambda^n e^{-\lambda T}\int\cdots\int_{\mathcal{S}_T^n}\mathrm{d}t_1\ldots\mathrm{d}t_n\int\cdots\int_{\mathbb{R}^n}\mathrm{d}y_1\ldots\mathrm{d}y_n\int_{-\infty}^{a_1}\mathrm{d}x_1\cdots\int_{-\infty}^{a_n}\mathrm{d}x_n\right. \\
&\qquad\times\prod_{k=1}^{n}\Big[f_Y(y_k)\phi_{t_k - t_{k-1}}(x_k)\mathcal{H}_1\left(b - z_{k-1}, x_k, t_k - t_{k-1}\right)\Big] \\
&\qquad\qquad\left.\times\,\mathcal{H}_2^{\psi}\left(b - z_n, z_n, T - t_n\right)\right\},
\end{aligned}\tag{6.13}$$

where $\mathcal{S}_T^n = \{(t_1, t_2, \cdots, t_n) | 0 < t_1 < t_2 < \cdots < t_n < T\}$, $f_Y(y)$ is the density function of Y_1 and $\phi_t(x)$ is the density function of X_t. The expressions of $\mathcal{H}_1$ and $\mathcal{H}_2^{\psi}$ are given by (6.11) and (6.12) respectively. For every $k \geq 1$, we define $z_k = \sum_{i=1}^{k}(x_i + y_i)$ with $z_0 = 0$ and $a_k = (b - z_{k-1}) \wedge (b - z_{k-1} - y_k)$. Here we use *the conventions* $\sum_{i\in\emptyset}=0$ *and* $\prod_{i\in\emptyset}=1$.

Proof of Theorem 6.3.2. Let us sketch the proof. From (6.8) and (6.9), we have

$$\begin{aligned}
&\mathrm{E}\left[\psi\left(X_T+\hat{Y}_{N_T}\right)\mathbb{1}\left(X_t+\hat{Y}_{N_t}<b, 0\le t\le T\right)\right]\\
&=\sum_{n=0}^{\infty}\Pr(N_T=n)\int\cdots\int_{\mathcal{S}_T^n}F_n\mathrm{d}t_1\cdots\mathrm{d}t_n\int\cdots\int_{\mathbb{R}^n}\prod_{k=1}^{n}f_Y(y_k)\mathrm{d}y_1\cdots\mathrm{d}y_n\\
&\qquad\times\mathrm{E}\left[\psi\left(X_T+\hat{y}_n\right)\prod_{k=0}^{n}\mathbb{1}\left(X_t<b-\hat{y}_k, t_k\le t<t_{k+1}\right)\right],
\end{aligned}\tag{6.14}$$

where $F_n=F_n(t_1,t_2,\cdots,t_n)$ is the joint density function of the first n arrival times conditional on $N_T=n$, which is given by (6.7), $f_Y(y)$ is the density function of Y_1, and for fixed $n\ge 0$, we define $t_0=0$ and $t_{n+1}=T$. Note that by defining $t_{n+1}=T$, we can absorb the last indicator in (6.9) into the product. The expectation in (6.14) only involves Brownian motion $\{X_t\}$ and can be evaluated inductively in the following way. Conditional on $X_{t_1}=x_1$, the expectation in (6.14) is written as

$$\int_{-\infty}^{a_1}\phi_{t_1}(x_1)\mathrm{E}\left[\psi\left(X_T+\hat{y}_n\right)\prod_{k=0}^{n}\mathbb{1}\left(X_t<b-\hat{y}_k, t_k\le t<t_{k+1}\right)\middle| X_{t_1}=x_1\right]\mathrm{d}x_1,$$

where $a_1=b\wedge(b-y_1)$ and $\phi_{t_1}(x_1)$ denotes the density function of X_{t_1}. The upper limit a_1 can be derived from the inspection of the first two indicators in (6.14) with $k=0,1$. By the independent increments property of Brownian motions, one can easily show $\{X_t, t>t_1\}$ and $\{X_t, t<t_1\}$ are independent given $X_{t_1}=x_1$. Therefore, the conditional expectation above can be expressed as

$$\begin{aligned}
&\Pr\left(X_t<b, 0\le t<t_1|X_{t_1}=x_1\right)\\
&\times\mathrm{E}\left[\psi\left(X_T+\hat{y}_n\right)\prod_{k=1}^{n}\mathbb{1}\left(X_t<b-\hat{y}_k, t_k\le t<t_{k+1}\right)\middle| X_{t_1}=x_1\right]\\
&=\mathcal{H}_1(b,x_1,t_1)\\
&\quad\times\mathrm{E}\left[\psi\left(X_{T-t_1}+x_1+\hat{y}_n\right)\prod_{k=1}^{n}\mathbb{1}\left(X_t<b-x_1-\hat{y}_k, t_k-t_1\le t<t_{k+1}-t_1\right)\right],
\end{aligned}$$

where $\mathcal{H}_1(b, x_1, t_1)$ is defined in Lemma 6.3.1, and we remove the condition $X_{t_1} = x_1$ by using the stationary increments property of $\{X_t\}$. We treat the last expectation in a similar manner by conditioning on $X_{t_2-t_1} = x_2$ and noting that $\{X_t, t > t_2-t_1\}$ and $\{X_t, t < t_2 - t_1\}$ are independent given $X_{t_2-t_1} = x_2$. In particular, we have

$$
\begin{aligned}
&\mathrm{E}\left[\psi\left(X_{T-t_1} + x_1 + \hat{y}_n\right)\prod_{k=1}^{n}\mathbb{1}\left(X_t < b - x_1 - \hat{y}_k, t_k - t_1 \le t < t_{k+1} - t_1\right)\right] \\
&= \int_{-\infty}^{a_2} \phi_{t_2-t_1}(x_2)\Pr\left(X_t < b - x_1 - \hat{y}_1, 0 \le t < t_2 - t_1 | X_{t_2-t_1} = x_2\right) \\
&\qquad \times \mathrm{E}\left[\psi\left(X_{T-t_2} + \hat{x}_2 + \hat{y}_n\right)\prod_{k=2}^{n}\mathbb{1}\left(X_t < b - \hat{x}_2 - \hat{y}_k, t_k - t_2 \le t < t_{k+1} - t_2\right)\right] \mathrm{d}x_2, \\
&= \int_{-\infty}^{a_2} \phi_{t_2-t_1}(x_2)\mathcal{H}_1(b - \hat{x}_1 - \hat{y}_1, x_2, t_2 - t_1) \\
&\qquad \times \mathrm{E}\left[\psi\left(X_{T-t_2} + \hat{x}_2 + \hat{y}_n\right)\prod_{k=2}^{n}\mathbb{1}\left(X_t < b - \hat{x}_2 - \hat{y}_k, t_k - t_2 \le t < t_{k+1} - t_2\right)\right] \mathrm{d}x_2,
\end{aligned}
$$

where $a_2 = (b - \hat{x}_1 - \hat{y}_1) \wedge (b - \hat{x}_1 - \hat{y}_2)$. Again, the last expectation can be computed similarly conditional on $X_{t_3-t_2} = x_3$. We carry out this procedure for n times and the desired result follows given formula (6.12) for $\mathcal{H}_2^{\psi}$. □

Remark 6.3.1. In formula (6.13), $\{Y_i\}_{i\ge 1}$ in fact are not necessarily i.i.d. random variables. The formula is valid as long as we know the joint density of $(Y_1, Y_2, \cdots, Y_n)$ for all $n \ge 1$. However, when the i.i.d. assumption is violated, the setting for our risk-neutral valuation requires further investigation.

The formula presented in Theorem 6.3.2 is an infinite sum and should be truncated in practical computation. Because the probability for a large number of jumps to occur before time T tends to be very small, one should expect that not many terms of the infinite sum is required for good convergence. To measure the truncation error, we assume that the function $\psi(x)$ is bounded: $\psi(x) \le C^{\psi}$ when

$x < b$ for some constant C^ψ. We express the right-hand side of (6.14) as

$$\sum_{n=0}^{\infty} \Pr(N_T = n) A_n = \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} e^{-\lambda T} A_n,$$

where A_n represents the multiple integral on the right-hand side of (6.14). It is not difficult to see that $A_n \leq C^\psi$. Hence, for a positive integer M,

$$\sum_{n>M} \frac{(\lambda T)^n}{n!} e^{-\lambda T} A_n \leq C^\psi \sum_{n>M} \frac{(\lambda T)^n}{n!} e^{-\lambda T},$$

where the truncated infinite sum

$$\sum_{n>M} \frac{(\lambda T)^n}{n!} e^{-\lambda T} = \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} e^{-\lambda T} \times \frac{(\lambda T)^{M+1} n!}{(n+M+1)!} \leq \frac{(\lambda T)^{M+1}}{(M+1)!}.$$

Then it follows that for a pre-specified $\epsilon > 0$, we are able to find a large enough M such that $C^\psi \frac{(\lambda T)^{M+1}}{(M+1)!} < \epsilon$. Then we can truncate the infinite sum at $M+1$ finite terms and the truncation error is at most ϵ, which is not affected by the distribution of the jump sizes $\{Y_i\}_{i\geq 1}$. Note that when $x < b$,

$$(s_0 e^x - K)^+ \leq (s_0 e^b - K)^+, \quad (K - s_0 e^x)^+ \leq K.$$

Hence, our analysis above applies to both call and put options as special cases.

Remark 6.3.2. Our method can also be used to value knock-out options with a downstream barrier. For $b < 0$, the expectation

$$\mathrm{E}\left[\psi\left(X_T + \sum_{i=1}^{N_T} Y_i\right) \mathbb{1}\left(X_t + \sum_{i=1}^{N_t} Y_i > b, 0 \leq t \leq T\right)\right]$$

can be evaluated in a similar way as in Theorem 6.3.2. When $\psi(x) = 1$, one may interpret the expectation above as the survival function of the time-until-ruin random variable where the surplus process starts at an initial capital $-b > 0$. In this regard, $\{Y_i\}_{i\geq 1}$ usually only take negative values in order to model individual claim sizes.

6.4 Double-barrier options

In this section, we shall consider the valuation of a double knock-out option where the underlying asset price S_t is modeled by (6.2) and is restricted by two exponential boundaries. Our formulation here is the same as in Section 4.4 where the BS framework is assumed: the upper boundary is $U_t = Ue^{\delta_1 t}$ and the lower boundary is $L_t = Le^{\delta_2 t}$ with $Ue^{\delta_1 T} > Le^{\delta_2 T}$. The last inequality ensures that the two boundaries do not intersect before the maturity time T. We want to calculate the following expectation.

$$\mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}(L_t < S_t < U_t, 0 \le t \le T)\right], \quad L < s_0 < U. \tag{6.15}$$

Through the change of variables $\psi(x) = \pi(s_0 e^x)$, $u = \ln\frac{U}{s_0}$ and $l = \ln\frac{L}{s_0}$, we can easily rewrite the expectation (6.15) as

$$\mathrm{E}\left[\psi\left(X_T + \hat{Y}_{N_T}\right)\mathbb{1}\left(l_t < X_t + \hat{Y}_{N_t} < u_t, 0 \le t \le T\right)\right], \quad l < 0 < u, \tag{6.16}$$

where we have defined $\hat{Y}_k = \sum_{i=1}^{k} Y_i, k \ge 0$, as the cumulative jump size and $u_t = u + \delta_1 t$, $l_t = l + \delta_2 t$ are two linear boundaries. The inequality $Ue^{\delta_1 T} > Le^{\delta_2 T}$ is equivalent to $u_T > l_T$, which guarantees that the two linear boundaries do not intersect before time T.

Conditioning on the number of jumps before maturity, we have

$$\begin{aligned}
&\mathrm{E}\left[\psi\left(X_T + \hat{Y}_{N_T}\right)\mathbb{1}\left(l_t < X_t + \hat{Y}_{N_t} < u_t, 0 \le t \le T\right)\right] \\
&= \sum_{n=0}^{\infty}\Pr(N_T = n)\mathrm{E}\left[\psi\left(X_T + \hat{Y}_n\right)\mathbb{1}\left(l_t < X_t + \hat{Y}_{N_t} < u_t, 0 \le t \le T\right)\middle| N_T = n\right].
\end{aligned} \tag{6.17}$$

The expectation on the right-hand side of (6.17) is evaluated conditional on the first n arrival times of the Poisson process $\{N_t\}$. By the same reasoning for the case of one-sided boundary, we can show that for $n \geq 0$,

$$\begin{aligned}
&\mathrm{E}\left[\psi\left(X_T+\hat{Y}_n\right)\mathbb{1}\left(l_t<X_t+\hat{Y}_{N_t}<u_t, 0\leq t\leq T\right)\Big| N_T=n\right]\\
&=\mathrm{E}\left[\psi\left(X_T+\hat{Y}_n\right)\prod_{k=0}^{n-1}\mathbb{1}\left(l_t-\hat{Y}_k<X_t<u_t-\hat{Y}_k, T_k\leq t<T_{k+1}\right)\right.\\
&\qquad\qquad\left.\times\,\mathbb{1}\left(l_t-\hat{Y}_n<X_t<u_t-\hat{Y}_n, T_n\leq t\leq T\right)\Big| N_T=n\right],
\end{aligned} \tag{6.18}$$

where we use the convention $\prod_{k\in\emptyset}=1$. Again, the remaining work is to calculate the last expectation for every non-negative n given $T_k=t_k$ and $Y_k=y_k$, $0\leq k\leq n$, which essentially corresponds to the value of a double-knock out option with a two-sided $(n+1)$-period step boundary in the BS framework, and this $(n+1)$-period step boundary, through inductive reasoning, can reduce to a two-sided single-period boundary, which has been discussed in Chapter 4 as one of our main results.

The explicit formula for (6.16) is derived based on the following results in the BS model.

Lemma 6.4.1. *Let $u_t=u+\delta_1 t$ and $l_t=l+\delta_2 t$ with $l<0<u$ and $l_T<u_T$ for some time horizon $T>0$. Define two auxiliary functions*

$$\mathcal{L}_1(u,l,x,T)=\Pr\left(l_t<X_t<u_t, 0\leq t<T|X_T=x\right), \quad l_T<x<u_T,$$
$$\mathcal{L}_2^{\psi}(u,l,y,T)=\mathrm{E}\left[\psi(X_T+y)\mathbb{1}(l_t<X_t<u_t,0\leq t\leq T)\right], \quad y\in\mathbb{R}.$$

Then we have

$$\begin{aligned}
&\mathcal{L}_1(u,l,x,T)\\
&=\sum_{m=-\infty}^{\infty}\mathcal{K}_m(u,l)\exp\left\{\frac{2m(l-u)(x-\mu T-m(l-u))}{\sigma^2 T}\right\}\\
&\quad-\sum_{m=-\infty}^{\infty}\hat{\mathcal{K}}_m(u,l)\exp\left\{\frac{2(mu-(m-1)l)(x-\mu T-mu+(m-1)l)}{\sigma^2 T}\right\},
\end{aligned} \tag{6.19}$$

and

$$\begin{aligned}
&\mathcal{L}_2^{\psi}(u,l,y,T)\\
&= \sum_{m=-\infty}^{\infty} \mathcal{K}_m(u,l)\mathrm{E}\big[\psi\left(X_T + y + 2m(l-u)\right)\\
&\qquad\qquad\qquad \times \mathbb{1}\left(l_T < X_T + 2m(l-u) < u_T\right)\big]\\
&\quad - \sum_{m=-\infty}^{\infty} \hat{\mathcal{K}}_m(u,l)\mathrm{E}\big[\psi\left(X_T + y + 2mu - (2m-2)l\right)\\
&\qquad\qquad\qquad \times \mathbb{1}\left(l_T < X_T + 2mu - (2m-2)l < u_T\right)\big],
\end{aligned} \tag{6.20}$$

where

$$\mathcal{K}_m(u,l) = \exp\left\{((m-1)l - mu)\gamma_m + m(l-u)\kappa_2\right\},$$

$$\hat{\mathcal{K}}_m(u,l) = \exp\left\{((m-1)l - mu)(\gamma_m - \kappa_2)\right\},$$

and $\gamma_m = m(\kappa_2 - \kappa_1)$, $\kappa_1 = \frac{2(\mu-\delta_1)}{\sigma^2}$ *and* $\kappa_2 = \frac{2(\mu-\delta_2)}{\sigma^2}$.

Proof of Lemma 6.4.1. It is not difficult to see that $\mathcal{L}_2^{\psi}$ essentially corresponds to the value of a double knock-out option with two exponential boundaries, and we can immediately recover its formula from (4.34) in Theorem 4.4.4 by letting $\pi(s) = \psi(\ln(s/s_0) + y)$, $U = s_0 e^u$ and $L = s_0 e^l$. To derive formula (6.19), first note that

$$\Pr\left(l_t < X_t < u_t, 0 \le t < T | X_T = x\right) = \frac{\Pr\left(X_T \in \mathrm{d}x \ \&\ l_t < X_t < u_t, 0 \le t < T\right)}{\Pr\left(X_T \in \mathrm{d}x\right)},$$

and then apply the formula for $\mathcal{L}_2^{\psi}$ with $y = 0$ and $\psi(z) = \mathbb{1}(z \in \mathrm{d}x)$. □

In terms of $\mathcal{L}_1$ and $\mathcal{L}_2^{\psi}$, the following result presents an explicit formula for the expectation (6.16) as an infinite sum of multiple integrals.

Theorem 6.4.2. *Let $u_t = u + \delta_1 t$ and $l_t = l + \delta_2 t$ with $l < 0 < u$ and $l_T < u_T$ for a time horizon $T > 0$. For a function $\psi(x)$, we have*

$$
\begin{aligned}
&\mathrm{E}\left[\psi\left(X_T + \hat{Y}_{N_T}\right) \mathbb{1}\left(l_t < X_t + \hat{Y}_{N_t} < u_t, 0 \le t \le T\right)\right] \\
&= \sum_{n=0}^{\infty} \Bigg\{ \lambda^n e^{-\lambda T} \int \cdots \int_{\mathcal{S}_T^n} \mathrm{d}t_1 \cdots \mathrm{d}t_n \int \cdots \int_{\mathbb{R}^n} \mathrm{d}y_1 \ldots \mathrm{d}y_n \int_{c_1}^{d_1} \mathrm{d}x_1 \cdots \int_{c_n}^{d_n} \mathrm{d}x_n \\
&\qquad \times \prod_{k=1}^{n} \left[f_Y(y_k) \phi_{t_k - t_{k-1}}(x_k) \mathcal{L}_1\left(u_{t_{k-1}} - z_{k-1}, l_{t_{k-1}} - z_{k-1}, x_k, t_k - t_{k-1}\right)\right] \\
&\qquad \times \mathcal{L}_2^{\psi}\left(u_{t_n} - z_n, l_{t_n} - z_n, z_n, T - t_n\right) \Bigg\},
\end{aligned}
\tag{6.21}
$$

where $\mathcal{S}_T^n = \{(t_1, t_2, \cdots, t_n) | 0 < t_1 < t_2 < \cdots < t_n < T\}$, $f_Y(y)$ is the density function of Y_1 and $\phi_t(x)$ is the density function of X_t. The expressions of $\mathcal{L}_1$ and $\mathcal{L}_2^{\psi}$ are given by (6.19) and (6.20) respectively. For every $k \ge 1$, we define $z_k = \sum_{i=1}^{k}(x_i + y_i)$, $c_k = (l_{t_k} - z_{k-1}) \vee (l_{t_k} - z_{k-1} - y_k)$ and $d_k = (u_{t_k} - z_{k-1}) \wedge (u_{t_k} - z_{k-1} - y_k)$. Here we use the conventions $\sum_{i\in\emptyset} = 0$ and $\prod_{i\in\emptyset} = 1$.

Proof of Theorem 6.4.2. We shall only give an outline. Define $t_0 = 0$ and $t_{n+1} = T$ for every $n \ge 0$. It follows from (6.17) and (6.18) that we only need to calculate

$$
\mathrm{E}\left[\psi\left(X_T + \hat{y}_n\right) \prod_{k=0}^{n} \mathbb{1}\left(l_t < X_t + \hat{y}_k < u_t, t_k \le t < t_{k+1}\right)\right], \tag{6.22}
$$

for fixed y_i and t_i, $1 \le i \le n$. Conditioning on $X_{t_1} = x_1$, we can express (6.22) as

$$
\begin{aligned}
&\int_{c_1}^{d_1} \phi_{t_1}(x_1) \Pr\left(l_t < X_t < u_t, 0 \le t < t_1 | X_{t_1} = x_1\right) \\
&\quad \times \mathrm{E}\left[\psi\left(X_T + \hat{y}_n\right) \prod_{k=1}^{n} \mathbb{1}\left(l_t < X_t + \hat{y}_k < u_t, t_k \le t < t_{k+1}\right) \middle| X_{t_1} = x_1\right] \mathrm{d}x_1 \\
&= \int_{c_1}^{d_1} \phi_{t_1}(x_1) \mathcal{L}_1(u, l, x_1, t_1) \\
&\qquad \times \mathrm{E}\Bigg[\psi\left(X_{T-t_1} + x_1 + \hat{y}_n\right) \prod_{k=1}^{n} \mathbb{1}\Big(l_{t_1} + \delta_2 t < X_t + x_1 + \hat{y}_k < u_{t_1} + \delta_1 t, \\
&\qquad\qquad t_k - t_1 \le t < t_{k+1} - t_1\Big)\Bigg] \mathrm{d}x_1,
\end{aligned}
$$

where the limits $c_1 = l_{t_1} \vee (l_{t_1} - \hat{y}_1)$ and $d_1 = u_{t_1} \wedge (u_{t_1} - \hat{y}_1)$ follow from the inspection of the first two indicators in (6.22) with $k = 0, 1$. The formula for $\mathcal{L}_1$ is given by (6.19). We can continue to evaluate the expectation above in a similar way by conditioning on $X_{t_2 - t_1} = x_2$. Repeating this inductive reasoning for n times yields the formula (6.21). $\square$

In practical implementation of formula (6.21), the infinite sum should be truncated to finite terms. In addition, the functions $\mathcal{L}_1$ and $\mathcal{L}_2^{\psi}$ are expressed as doubly infinite sums and should also be truncated. From the discussion in Section 4.4.3 about the convergence of the double-barrier option pricing formula, one can conclude that the formula for $\mathcal{L}_2^{\psi}$ will be rapidly convergent as long as $\psi(x)$ is bounded over the interval (l_T, u_T). Since $\mathcal{L}_1$ is derived from $\mathcal{L}_2^{\psi}$ with $\psi(z) = \mathbb{1}(z \in \mathrm{d}x)$, its formula is also convergent. In the numerical analysis, only a few terms are needed for accurate estimates of $\mathcal{L}_1$ and $\mathcal{L}_2^{\psi}$, and the computation is very time-efficient. Similar to the discussion right after Remark 6.3.1, the overall truncation error can also be estimated.

6.5 Numerical examples based on Monte Carlo simulations

In this section, we will describe how to numerically implement the formulas (6.13) and (6.21). Some numerical examples will be given in the jump-diffusion models where the jump size follows normal distribution, double exponential distribution and Gamma distribution.

Note that $\mathcal{H}_1$ and $\mathcal{H}_2^{\psi}$ in (6.13) and $\mathcal{L}_1$ and $\mathcal{L}_2^{\psi}$ in (6.21) are all expectations (the probabilities can be viewed as the expectations of indicator functions), so we can regard the infinite sums in (6.13) and (6.21) as expectations with respect to the sources of randomness $\{N_T\}$, $\{T_i, 1 \leq i \leq N_T\}$, $\{X_{T_i}, 1 \leq i \leq N_T\}$ and

$\{Y_i, 1 \le i \le N_T\}$. It indicates that we can compute the pricing formulas using Monte Carlo simulations. The procedure is as follows.

(1) Generate the total number of jumps n from Poisson distribution with parameter λT.

(2) Generate n consecutive arrival times t_i, $1 \le i \le n$. This can be achieved by first simulating n i.i.d. uniform random variables in $[0, T]$ and ranking them in the ascending order.

(3) Generate the jump sizes $y_i, 1 \le i \le n$, from the predetermined distribution with density function $f_Y(y)$.

(4) Generate independent random numbers $x_i, 1 \le i \le n$, from normal distribution with mean $\mu(t_i - t_{i-1})$ and variance $\sigma^2(t_i - t_{i-1})$. Note that we do not directly simulate a multivariate normal vector $(X_{t_1}, X_{t_2-t_1}, \cdots, X_{t_n-t_{n-1}})$ by the virtue of the multiple integrals with respect to x_i given in (6.13) and (6.21).

(5) Calculate the integrands in (6.13) and (6.21). When $n = 0$ in step (1), one can skip (2) to (4) and directly calculate $\mathcal{H}_2^{\psi}(b, 0, T)$ and $\mathcal{L}_2^{\psi}(u, l, 0, T)$ using formulas (6.12) and (6.20).

(6) Repeat steps (1) to (5) for a large enough number of times and compute the Monte Carlo estimator and its standard error.

We in particular follow the procedure above to price an up-and-out call option with an exponential boundary $B_t = Be^{\delta t}$ within a variety of parameter choices. To apply formula (6.13), we need to let $\psi(x) = (s_0 e^{x+\delta T} - K)^+$ where K is the strike price and $b = \ln \frac{B}{s_0}$, and also replace μ by $\mu - \delta$. We further assume the strike K is below the terminal barrier level $Be^{\delta T}$ to avoid the trivial case. Recall that in Merton's model, the drift is given by

$$\mu = r - \frac{\sigma^2}{2} - \left(e^{m+\frac{v^2}{2}} - 1\right)\lambda,$$

and in Kou's model, the drift is given by

$$\mu = r - \frac{\sigma^2}{2} - \left(\frac{p\eta_1}{\eta_1 - 1} + \frac{(1-p)\eta_2}{\eta_2 + 1} - 1\right)\lambda.$$

The third model we consider in our numerical analysis is a Gamma distribution with the density function

$$f_Y(y) = \frac{\gamma^k y^{k-1} e^{-\gamma y}}{(k-1)!}, \quad \gamma > 1.$$

Then the drift is given by

$$\mu = r - \frac{\sigma^2}{2} - \left(\left(\frac{\gamma}{\gamma - 1}\right)^k - 1\right)\lambda.$$

The commom parameter values are: $s_0 = 1000$, $K = 1100$, $B = 1300$, $T = 1$, $r = 0.05$, $\sigma = 0.2$. We perform $N = 5 \times 10^6$ simulations. For Merton's model, we consider $m = -0.1, v = 0.15$, $m = 0.1, v = 0.1$, or $m = 0.2, v = 0.1$. For Kou's model, we consider $p = 0.6$ and $\eta_1 = \eta_2 = 20$, $\eta_1 = \eta_2 = 30$ or $\eta_1 = \eta_2 = 40$. For the Gamma distribution model, we consider $k = 2, \gamma = 40$, $k = 3, \gamma = 30$ or $k = 5, \gamma = 40$. We also allow the curvature δ of the barrier and the intensity λ of the Poisson process to vary to examine their impacts on the option values. The simulation results are given in the following tables. Inside the brackets are the standard errors of the Monte Carlo estimators.

In our numerical examples, the average computational time is about 15 minutes, much faster than the crude Monte Carlo method. The latter requires us to simulate the entire sample path of the price process before maturity time, which is achieved by discretization. One can then reduce the corresponding discretization error by shortening the time step in the path generation, but this will substantially increase the computational time. On the other hand, however, our algorithm is not as time-efficient as those proposed in, for example, Kou and Wang (2004), Kou, Petrella and Wang (2005), Feng and Linetsky (2008) and Cai and Kou (2011). The advantage of our model is that it allows a more general assumption for the jump size distribution and the barriers can also depend on time through exponential functions. Therefore, we achieve a good balance between model complexity and solution tractability.

Table 6.1: Up-and-out call with an exponential boundary when the jump size follows normal distrbution

δ	λ	$m=-0.1, v=0.15$	$m=0.1, v=0.1$	$m=0.2, v=0.1$
0.1	1	$24.8980(1.003\times10^{-2})$	$18.7074(1.128\times10^{-2})$	$12.1652(1.110\times10^{-2})$
	2	$21.7000(1.246\times10^{-2})$	$14.4307(1.286\times10^{-2})$	$7.6757(1.095\times10^{-2})$
	3	$18.1979(1.329\times10^{-2})$	$11.6204(1.298\times10^{-2})$	$5.4275(1.007\times10^{-2})$
0	1	$8.7885(3.631\times10^{-3})$	$6.8680(4.331\times10^{-3})$	$4.2393(4.211\times10^{-3})$
	2	$7.1761(4.439\times10^{-3})$	$4.9969(4.989\times10^{-3})$	$2.4254(4.198\times10^{-3})$
	3	$5.7939(4.748\times10^{-3})$	$3.8491(5.072\times10^{-3})$	$1.6236(3.868\times10^{-3})$
-0.1	1	$0.8098(3.791\times10^{-4})$	$0.6623(5.284\times10^{-4})$	$0.3983(5.268\times10^{-4})$
	2	$0.6442(4.750\times10^{-4})$	$0.4607(6.254\times10^{-4})$	$0.2052(5.430\times10^{-4})$
	3	$0.5223(5.244\times10^{-4})$	$0.3405(6.520\times10^{-4})$	$0.1298(5.133\times10^{-4})$

Table 6.2: Up-and-out call with an exponential boundary when the jump size follows double exponential distribution

δ	λ	$\eta_1=\eta_2=20$	$\eta_1=\eta_2=30$	$\eta_1=\eta_2=40$
0.1	1	$24.7507(1.025\times10^{-2})$	$25.5186(1.010\times10^{-2})$	$25.8019(1.002\times10^{-2})$
	2	$23.4421(1.310\times10^{-2})$	$24.8942(1.309\times10^{-2})$	$25.4500(1.304\times10^{-2})$
	3	$22.2491(1.465\times10^{-2})$	$24.3056(1.483\times10^{-2})$	$25.1179(1.486\times10^{-2})$
0	1	$9.3967(3.956\times10^{-3})$	$9.7780(3.907\times10^{-3})$	$9.9324(3.872\times10^{-3})$
	2	$8.7164(5.125\times10^{-3})$	$9.4212(5.153\times10^{-3})$	$9.7215(5.152\times10^{-3})$
	3	$8.1260(1.786\times10^{-3})$	$9.0911(5.918\times10^{-3})$	$9.5154(5.949\times10^{-3})$
-0.1	1	$0.9142(4.560\times10^{-4})$	$0.9563(4.463\times10^{-4})$	$0.9742(4.377\times10^{-4})$
	2	$0.8388(6.050\times10^{-4})$	$0.9135(6.024\times10^{-4})$	$0.9467(5.949\times10^{-4})$
	3	$0.7722(6.927\times10^{-4})$	$0.8744(7.035\times10^{-4})$	$0.9221(7.026\times10^{-4})$

Table 6.3: Up-and-out call with an exponential boundary when the jump size follows Gamma distribution

δ	λ	$k=2,\gamma=40$	$k=3,\gamma=30$	$k=5,\gamma=40$
0.1	1	24.6936(1.210×10^{-2})	18.0269(1.743×10^{-2})	13.5670(2.146×10^{-2})
	2	22.2322(1.421×10^{-2})	14.6423(1.819×10^{-2})	10.9810(2.255×10^{-2})
	3	20.1494(1.521×10^{-2})	12.4098(1.696×10^{-2})	9.7343(2.011×10^{-2})
0	1	9.3058(4.850×10^{-3})	6.7811(6.094×10^{-3})	5.2861(6.863×10^{-3})
	2	8.5459(5.841×10^{-3})	5.5728(6.487×10^{-3})	4.2283(7.043×10^{-3})
	3	7.7903(6.336×10^{-3})	4.5923(6.159×10^{-3})	3.5218(6.206×10^{-3})
-0.1	1	0.8926(6.193×10^{-4})	0.6520(7.392×10^{-4})	0.5259(7.905×10^{-4})
	2	0.8411(7.593×10^{-4})	0.5502(8.197×10^{-4})	0.4219(8.407×10^{-4})
	3	0.7765(8.344×10^{-4})	0.4493(8.085×10^{-4})	0.3382(7.789×10^{-4})

The expected jump magnitude is equal to m in Merton's model, equal to $\frac{p}{\eta_1} - \frac{1-p}{\eta_2}$ in Kou's model and equal to $\frac{k}{\gamma}$ in the Gamma distribution model. Certain patterns can be observed from the numbers in Tables 6.1, 6.2 and 6.3: the option values tend to decrease as the upstream barrier drops, the jumps become frequent, or the expected jump magnitude grows. These trends are consistent with our financial intuition that a knock-out option loses its value when there is a greater chance for the barrier to be breached.

6.6 The BS model revisited: double knock-out options with piecewise exponential boundaries

A step-barrier option has a barrier that is a step function in time. Guillaume (2010) derived a closed-form solution for a 2-period step double knock-out option and proposed a Monte Carlo algorithm to recursively compute an n-period double knock-out option based the 2-period counterpart. Moreover, the step-barrier can also be a set of piecewise exponential functions in time, which case was considered in Kunitomo and Ikeda (1992) to estimate more general curved boundaries, but nevertheless no explicit formula was given. In this section, we shall present an immediate application to pricing an $(n+1)$-period step-barrier option restricted by two piecewise exponentially time-varying boundaries.

Let us return to the BS framework and let the underlying asset price $S_t = S_0 e^{X_t}$ with $X_t = \mu t + \sigma W_t$. For a fixed $n \geq 1$, we form an $(n+1)$-period time partition $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$, and based on this partition we define the time-varying boundaries U_t and L_t by

$$U_t = \sum_{k=0}^{n} U^{(k)} e^{\delta_{1,k} t} \mathbb{1}(t_k \leq t < t_{k+1}),$$
$$L_t = \sum_{k=0}^{n} L^{(k)} e^{\delta_{2,k} t} \mathbb{1}(t_k \leq t < t_{k+1}).$$

Then U_t and L_t are piecewise exponential functions in time. In some cases, it is possible to approximate a general curved boundary using piecewise functions like U_t and L_t. For a double knock-out option delineated by U_t and L_t, we are interested in calculating the expectation

$$\begin{aligned}
&\mathrm{E}_{s_0}\left[\pi(S_T)\mathbb{1}\left(L_t < S_t < U_t, 0 \leq t \leq T\right)\right] \\
&= \mathrm{E}_{s_0}\left[\pi(S_T)\prod_{k=0}^{n}\mathbb{1}\left(L^{(k)}e^{\delta_{2,k}t} < S_t < U^{(k)}e^{\delta_{1,k}t}, t_k \leq t < t_{k+1}\right)\right],
\end{aligned}$$

with $L^{(0)} < s_0 < U^{(0)}$. Of course, we always assume that the two boundaries never intersect before time T. Define two linear functions $u_t^{(k)} = u^{(k)} + \delta_{1,k} t$ and $l_t^{(k)} = l^{(k)} + \delta_{2,k} t$ where $u^{(k)} = \ln \frac{U^{(k)}}{s_0}$ and $l^{(k)} = \ln \frac{L^{(k)}}{s_0}$ for $0 \leq k \leq n$. Then the expectation above is equivalent to

$$\mathrm{E}\left[\psi(X_T)\prod_{k=0}^{n}\mathbb{1}\left(l_t^{(k)} < X_t < u_t^{(k)}, t_k \leq t < t_{k+1}\right)\right] \tag{6.23}$$

with $\psi(x) = \pi(s_0 e^x)$. The expectation (6.23) can be evaluated by the inductive method used in the proof of Theorem 6.4.2. In particular, we obtain the following.

Theorem 6.6.1. *Let* $u_t^{(k)} = u^{(k)} + \delta_{1,k}t$ *and* $l_t^{(k)} = l^{(k)} + \delta_{2,k}t$, $k = 0, 1, \ldots, n$, *with* $l^{(0)} < 0 < u^{(0)}$. *For a function* $\psi(x)$, *we rewrite the two auxiliary functions defined in Lemma 6.4.1 as* $\mathcal{L}_1(u, l, x, T; \delta_1, \delta_2)$ *and* $\mathcal{L}_2^{\psi}(u, l, y, T; \delta_1, \delta_2)$ *to indicate their dependence on the two curvatures* δ_1 *and* δ_2. *Then we have*

$$\begin{aligned}
&\mathrm{E}\left[\psi(X_T)\prod_{k=0}^{n}\mathbb{1}\left(l_t^{(k)} < X_t < u_t^{(k)}, t_k \le t < t_{k+1}\right)\right] \\
&= \int_{e_1}^{f_1}\mathrm{d}x_1\cdots\int_{e_n}^{f_n}\mathrm{d}x_n \\
&\quad \times \prod_{k=1}^{n}\left[\phi_{t_k - t_{k-1}}(x_k)\mathcal{L}_1\left(u_{t_{k-1}}^{(k-1)} - \hat{x}_{k-1}, l_{t_{k-1}}^{(k-1)} - \hat{x}_{k-1}, x_k, t_k - t_{k-1}; \delta_{1,k-1}, \delta_{2,k-1}\right)\right] \\
&\quad\quad \times \mathcal{L}_2^{\psi}\left(u_{t_n}^{(n)} - \hat{x}_n, l_{t_n}^{(n)} - \hat{x}_n, \hat{x}_n, T - t_n; \delta_{1,n}, \delta_{2,n}\right),
\end{aligned} \tag{6.24}$$

where $\hat{x}_k = \sum_{i=1}^{k} x_i$ *with* $\hat{x}_0 = 0$, $e_k = \left(l_{t_k}^{(k-1)} - \hat{x}_{k-1}\right) \vee \left(l_{t_k}^{(k)} - \hat{x}_{k-1}\right)$, $f_k = \left(u_{t_k}^{(k-1)} - \hat{x}_{k-1}\right) \wedge \left(u_{t_k}^{(k)} - \hat{x}_{k-1}\right)$ *for* $1 \le k \le n$, *and* $\phi_t(x)$ *denotes the density function of* X_t.

We omit the proof of Theorem 6.6.1 because it is merely an intermediate step when we prove Theorem 6.4.2.

Similar to the procedure described in Section 6.5, we can numerically implement the formula (6.24) using Monte Carlo simulations. We generate n independent normal random variables $x_k, k = 1, 2, \ldots, n$, where x_k has mean $\mu(t_k - t_{k-1})$ and variance $\sigma^2(t_k - t_{k-1})$ with $t_0 = 0$ and compute the integrand in (6.24); then we repeat this procedure for a large enough number of times and compute the Monte Carlo estimator and its standard error.

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