FROM VALUING EQUITY-LINKED DEATH BENEFITS
TO PRICING AMERICAN OPTIONS

by

Zhenhao Zhou

A thesis submitted in partial fulfillment
of the requirements for the Doctor of Philosophy
degree in Statistics in the
Graduate College of
The University of Iowa

May 2017

Thesis Supervisor: Professor Elias S.W. Shiu

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Zhenhao Zhou

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Statistics at the May 2017 graduation.

Thesis Committee:

Elias S.W. Shiu, Thesis Supervisor

Palle Jorgensen

Ambrose Lo

Nariankadu Shyamalkumar

Qihe Tang

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to Prof. Elias Shiu, my advisor, for his continuous support of my Ph.D. study and research, for his writing guidance, patience, and immense knowledge. Without him, this entire academic journey and the completion of this dissertation would have not been possible. I want to express my deepest appreciation for his invaluable advice, not only on research-related work, but also on life and experience in general. I have learned so much from him.

I would also like to thank my other thesis committee members, Professors Palle Jorgensen, Ambrose Lo, Nariankadu Shyamalkumar, Qihe Tang. They have generously given their time and expertise to better my understanding of this area.

My sincere thanks also go to Margie Ebert, Tammy Siegel and Dena Miller. They have provided me with great help in my life and work which will be an enormous fortune to me.

I gratefully acknowledge the financial support of the Society of Actuaries Center of Actuarial Excellence Research Grant and the Graduate College Summer Fellowship. In addition, I would like to thank Mr. and Mrs. Robert Taylor for the Taylor Award in Actuarial Stochastics.

Last but not least, I would like to thank my wonderful family: my parents, my parents-in-law and my wife. Without their endless support and love, this journey and this dissertation would not have been possible to finish.

ABSTRACT

Motivated by the Guaranteed Minimum Death Benefits (GMDB) in variable annuities, we are interested in valuing equity-linked options whose expiry date is the time of death of the policyholder. Because the time-until-death distribution can be approximated by linear combinations of exponential distributions or mixtures of Erlang distributions, the analysis can be reduced to the case where the time-until-death distribution is exponential or Erlang.

We present two probability methods to price American options with an exponential expiry date. Both methods give the same results. An American option with Erlang expiry date can be seen as an extension of the exponential expiry date case. We calculate its price as the sum of the price of the corresponding European option and the early exercise premium. Because the optimal exercise boundary takes the form of a staircase, the pricing formula is a triple sum. We determine the optimal exercise boundary recursively by imposing the "smooth pasting" condition. The examples of the put option, the exchange option, and the maximum option are provided to illustrate how the methods work.

Another issue related to variable annuities is the surrender behavior of the policyholders. To model this behavior, we suggest using barrier options. We generalize the reflection principle and use it to derive explicit formulas for outside barrier options, double barrier options with constant barriers, and double barrier options with time varying exponential barriers.

Finally, we provide a method to approximate the distribution of the time-until-death random variable by combinations of exponential distributions or mixtures of Erlang distributions. Compared to directly fitting the distributions, my method has two advantages: 1) It is more robust to the initial guess. 2) It is more likely to obtain the global minimizer.

PUBLIC ABSTRACT

Most variable annuities are essentially an equity investment fund embedded with options or guarantees. These options or guarantees provide the policyholders downside protection plus some chance of upside gains. To provide the protection, insurance companies may buy put options. However, due to the uncertainty of the time of payment, options with a random expiration date need to be considered.

In this thesis, we consider the valuation problem of American options with exponentially distributed or Erlang distributed expiration date. With such expiration dates, analytic pricing formulas can be obtained. Compared to the European option, an American option allows its owner to exercise the option at any time prior to the expiration date. Therefore, we can calculate the price of an American option as the sum of the price of the corresponding European option and the early exercise premium. To determine the optimal exercise boundary, we equate the exercise value with the option price at the exercise boundary.

The surrender behavior of the policyholders is another issue related to variable annuities. If policyholders choose to surrender the contract, they give up the protection, stop paying fees to the insurance company and receive a surrender value. Hence, the insurance company may choose to buy up-and-out put options instead of regular put options to provide the protection, because barrier options are cheaper. We derive explicit formulas for valuing various barrier options, including outside barrier options and double barrier options.

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CHAPTER 1

INTRODUCTION

1.1 Objective of the thesis

The variable annuity was created in 1952 by TIAA (Teachers Insurance and Annuity Association) as a method of investing for retirement of teachers and professors. The background to create this financial product was in reaction to rising inflation and lengthening life expectancies, and a dramatic expansion of the education sector with the G.I. Bill. Initially, variable annuities were mainly mutual funds and index funds.

In the 1980's and 1990's, volatilities the stock market caused some people to lose faith in investing in stocks. Insurance companies made variable annuity products more attractive by offering downside protections with minimum benefit guarantees called riders. Most riders are one of the following four types: Guaranteed Minimum Death Benefit (GMDB), Guaranteed Minimum Income Benefit (GMIB), Guaranteed Minimum Accumulation Benefit (GMAB), and Guaranteed Minimum Withdrawal Benefit (GMWB). Most US variable annuity products have at least one rider, and some popular products have a combination of riders.

These riders are generally embedded options. Since the time of death or withdrawal is uncertain, options with random expiry dates need to be considered. A key goal of this thesis is to value options with random expiry dates. Let $T(x)$ denote the time-until-death random variable for a life aged x. For $t > 0$, let $S(t)$ denote the value of a stock or a fund at time t. We are interested in evaluating expectations of the form

$$\mathrm{E}\left[e^{-rT(x)} b\left(S(t),\, 0 \leq t \leq T(x)\right)\right] \tag{1.1.1}$$

where the expectation is taken with respect to an appropriate probability distribution, r is a force of interest and $b\left(S\left(t\right),\, 0 \leq t \leq T(x)\right)$ is the payoff. Note that the payoff can be dependent on the history of the stock price up to the time of death $T(x)$. Examples of $b\left(S\left(t\right),\, 0 \leq t \leq T(x)\right)$ include

$$\text{Call option:} \quad \left[S\left(T(x)\right) - K\right]_{+}$$

$$\text{High water mark :} \quad \underset{0 \leq t \leq T(x)}{\text{Maximum}}\ S(t)$$

$$\text{Up-and-out put option:} \quad I\big(\underset{0 \leq t \leq T(x)}{\text{Maximum}}\, S(t) < B\big)\left[K - S\left(T(x)\right)\right]_{+}$$

We know that the density function of the positive random variable $T(x)$ can be approximated by combinations (no restriction on the signs of the coefficients) of exponential density functions,

$$f_{T(x)}(t) \approx \sum_{j} \alpha_j f_{\tau_j}(t) = \sum_{j} \alpha_j \lambda_j e^{-\lambda_j t}$$

or by mixtures (positive coefficients only) of *Erlang* distributions

$$f_{T(x)}(t) \approx \sum_{j} \beta_j f_{Y_j}(t) = \sum_{j} \beta_j e^{-\lambda t} \frac{\lambda^{r_j} t^{r_j - 1}}{(r_j - 1)!}$$

Then, under the assumption that $T(x)$ is independent of the stock price process $\{S(t)\}$, the problem of evaluating the expectations in 1.1.1 can be approximated by evaluating

$$\sum_{j} \alpha_j \mathrm{E}\left[e^{-r\tau_j} b\left(S(t),\, 0 \leq t \leq \tau_j\right)\right]$$

where τ_j are exponential random variables independent of $\{S(t)\}$, or by evaluating

$$\sum_{j} \beta_j \mathrm{E}\left[e^{-rY_j} b\left(S(t),\, 0 \leq t \leq Y_j\right)\right]$$

where Y_j are *Erlang* random variables independent of $\{S(t)\}$.

Valuing American options with an Erlang distributed expiry date has another function. We can use the result to approximate the price of an American option with a fixed expiry date by setting the mean of the Erlang distribution to be the expiry date and letting the variance to be small.

Another issue related to the pricing riders is policyholder's behavior. Policyholder's behavior includes policy lapses and surrenders, transfers between investment funds, and annuitization. I shall mainly discuss the lapse behavior in this thesis. If policyholders choose to lapse their contracts, they give up the underlying protection, stop paying fees to the insurer and receive a surrender value. Lapses can be divided into two types: deterministic and dynamic lapses. Deterministic lapses are due to unforeseen events in the policyholder's life and are generally seen as diversifiable. On the other hand, dynamic lapses result from an investment decision change due to the evolution of markets. Therefore, they are very difficult to diversify. In general, lapse rates are negatively related to internal rates of return, such as high guaranteed minimum crediting rates, and positively related to external rates of return, such as market interest rates or stock returns. Especially, when the guarantee is deeply out-of-the-money, the policyholder has a strong incentive to lapse the contract and choose an alternative investment product. This is simply because the policyholder is paying high fees for a guarantee that is very unlikely to be triggered in the future.

Suppose that a policyholder invests in a guaranteed minimum death benefit (GMDB) product. The product guarantees the following payment to the policyholder when he dies,

$$\text{Max}(S(T(x)),\, K)$$

where K is the guaranteed minimum amount. Since

$$\text{Max}(S(T(x)),\, K) = S(T(x)) + [K - S(T(x))]_+$$

the problem of valuing the guarantee becomes a problem of valuing a K-strike put option that is exercised at the time of death. Because some of the guarantees will not be exercised due to lapses, when an actuary prices a guarantee in a variable annuity, he or she needs to consider these lapses in order to pay less for the guarantee. To model the surrender and lapse behavior, we assume that the policyholder will surrender his contract at the first moment the account value hits a predetermined barrier. If we use $M_S(T(x))$ to denote the running maximum of the stock price until the time of death, we can buy a basket of barrier options corresponding to the following formula,

$$\sum_i p_i I(M_S(T(x)) < L_{i+1}) \left[K - S(T(x))\right]_+ \tag{1.1.2}$$

Here, we use $L_1 < L_2 < \cdots$ to denote the barriers, and p_i is the additional fraction of policyholders who will surrender when the maximum stock price is larger than L_i and smaller than L_{i+1}. From the above expression, we can see that the total cost of a basket of up-and-out put options is lower than the price of put options, since

$$\left[K - S(T(x))\right]_+ = \sum_i p_i \left[K - S(T(x))\right]_+ > \sum_i p_i I(M_S(T(x)) < L_{i+1}) \left[K - S(T(x))\right]_+$$

The closed-form expressions for each of the underlying options in (1.1.2) are available under the Black-Scholes model. Therefore, the valuation of a GMDB contract under lapse assumption and the computation of Greeks required for establishing a dynamic hedging strategy are both straightforward to perform.

1.2 Structure of the thesis

Chapter 2 serves as a brief introduction of some probability and stochastic processes concepts. Chapters 3-6 form the main part of this thesis.

In Chapter 3, we focus on the valuation problem of barrier options. Barrier options are useful for risk management of surrender and lapse behavior of policyholders. Applying the method of Esscher transforms, we calculate the price of outside barrier options, double barrier options with constant barriers, and double barrier options with time varying exponential barriers.

In Chapter 4, we consider the valuation problem of American options whose expiration date is exponentially distributed and independent of the underlying stock price process. The memory-less property of the exponential distribution implies that the exercise boundary is flat. We present two alternative probability methods for deriving the pricing formulas for this kind of American option. The examples of the put option, the exchange option, and the maximum option are provided.

In Chapter 5, the expiry date of the option is extended from the exponential distribution to the *Erlang* distribution. We calculate the American option price as the sum of the price of the European option and the early exercise premium. Since the optimal exercise boundary has a staircase form, the early exercise premium can be calculated through piece-wise integration. The pricing formula for the American option with an *Erlang* distributed expiry date takes the form of a triple sum. To determine the optimal exercise boundary, we recursively impose the "value matching" condition for the price of options at the optimal exercise boundary. By fixing the mean of the *Erlang* distribution and letting the shape parameter go to infinity, we can obtain the price of the American option with a fixed expiry date.

In Chapter 6, we numerically approximate the distribution of $T(x)$, the time-until-death random variable for a life aged x, by the combinations of exponential distributions and the mixtures of *Erlang* distributions. The problem is essentially an optimization problem. We propose a splitting method to estimate the parameters. Through splitting the whole optimization problem into two sub-problems, linear optimization and nonlinear optimization, the results are robust to the initial guess of parameters. We also apply the adjustment procedure provided in Lee and Lin (2010) to identify the shape parameters of *Erlang* distributions.

1.3 Notation

τ an independent exponential random variable with density function $\lambda e^{-\lambda t}$

$T_n,\, n \geq 1,$ an independent *Erlang* distributed random variable with density function $f_{T_n}(t) = e^{-\lambda t}\frac{\lambda^n t^{n-1}}{(n-1)!}$

$\{S(t),\ t \geq 0\}$ a single stock price process, which is modeled as $S(t) = S(0)e^{X(t)}$ and

$$X(t) = \mu t + \sigma B(t), \qquad t \geq 0 \tag{1.3.1}$$

where $B(t)$ is a standard Brownian motion (Wiener process), and μ and $\sigma > 0$ are constants.

$m(t)$ $\min\limits_{0\leq s\leq t} X(s)$, running minimum of $X(s)$, $s \geq 0$ until time t

$M(t)$ $\max\limits_{0\leq s\leq t} X(s)$, running maximum of $X(s)$, $s \geq 0$ until time t

$f^r_{X(\tau),\, m(\tau)}(x,\, y)$ discounted joint density function

$M_S(t)$ $\max\limits_{0\leq s\leq t} S(t)$ running maximum of the stock process

$m_S(t)$ $\min\limits_{0\leq s\leq t} S(t)$ running minimum of the stock process

$\{\boldsymbol{S}(t),\ t \geq 0\}$ a stock vector price process

$$\boldsymbol{S}(t) = (S_0(t),\, S_1(t), \cdots,\, S_{n-1}(t))^{'}$$

For each stock $S_i(t)$,

$$S_i(t) = S(0)e^{X_i(t)} \quad i = 0,\, 1,\, 2\cdots$$

$\{\boldsymbol{X}(t),\ t \geq 0\}$ an $n-$dimensional Brownian motion,

$$\boldsymbol{X}(t) = (X_0(t),\, X_1(t), \cdots,\, X_{n-1}(t))^{'};$$

starting at $\mathbf{0}$ with a drift vector $\boldsymbol{\mu}$

$$\boldsymbol{\mu} = (\mu_0,\, \mu_1, \cdots,\, \mu_{n-1})^{'}$$

and a diffusion matrix $\mathbf{\Sigma}$

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_0^2 & \cdots & \rho_{n-1}\sigma_0\sigma_{n-1} \\ \vdots & \vdots & \vdots \\ \rho_{n-1}\sigma_0\sigma_{n-1} & \cdots & \sigma_{n-1}^2 \end{bmatrix}_{n\times n}$$

where ρ_k is the correlation coefficient of $X_0(t)$ and $X_k(t)$.

first hitting time of one side for stock price $\quad T_U = \inf\left\{t \geq 0 \middle| S(t) = U\right\}$

first hitting time of two sides for stock price

$$T_{U,L} = \inf\left\{t \geq 0 \middle| S(t) = U \text{ or } S(t) = L\right\}$$

first hitting time of two exponential boundaries for stock price

$$T'_{U,L} = \inf\left\{t \geq 0 \middle| S(t) = U(t) \text{ or } S(t) = L(t)\right\}$$

first hitting time of one side for the ratio of two stocks

$$T^{ratio}_{\underset{\sim}{c}^*} = \inf\left\{t \geq 0 \middle| \frac{S_1(t)}{S_2(t)} = \underset{\sim}{c}^*\right\}$$

first hitting time of two sides for the ratio of two stocks

$$T^{ratio}_{\underset{\sim}{c}^*,\underset{\sim}{b}^*} = \inf\left\{t \geq 0 \middle| \frac{S_1(t)}{S_2(t)} = \underset{\sim}{c}^* \text{or } \frac{S_1(t)}{S_2(t)} = \underset{\sim}{b}^*\right\}$$

$\alpha < 0,\ \beta > 0$ $\quad$ two roots of the following quadratic equation

$$\frac{1}{2}\sigma^2\theta^2 + (r - \delta - \frac{1}{2}\sigma^2)\theta - (r + \lambda) = 0$$

$\alpha_1 < 0,\ \beta_1 > 0$ $\quad$ two roots of the following quadratic equation

$$\frac{1}{2}\sigma^2\theta^2 + (\delta_1 - \delta_2 - \frac{\sigma^2}{2})\theta - (\delta_1 + \lambda) = 0$$

where $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$.

$\alpha_2 < 0,\ \beta_2 > 0$ two roots of the following quadratic equation

$$\frac{1}{2}\sigma^2\theta^2 + (\delta_2 - \delta_1 - \frac{\sigma^2}{2})\theta - (\delta_2 + \lambda) = 0$$

where $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$.

$\alpha^* < 0,\ \beta^* > 0$ two roots of the following quadratic equation

$$\frac{1}{2}\sigma^2 x^2 + \mu x - \lambda = 0$$

CHAPTER 2

PRELIMINARIES

2.1 Esscher transforms

Esscher transforms is a time-honored technique in actuarial science. It was first introduced by F. Esscher in 1932. Gerber and Shiu (1994a) extends the concept of Esscher transforms from the case for a single random variable to that for a Levy process. Let $\{X(t)\}$ be a Levy process and h be a real number. The expectation of $g\left(X(t),\, 0 \leq t \leq T\right)$ with respect to the changed probability measure, indexed by h, is defined as

$$\mathrm{E}\left[g\left(X(t),\, 0 \leq t \leq T\right);\, h\right] = \frac{\mathrm{E}\left[g\left(X(t),\, 0 \leq t \leq T\right) e^{hX(T)}\right]}{\mathrm{E}\left[e^{hX(T)}\right]}$$

Esscher transforms have very elegant factorization formulas

$$\mathrm{E}\left[e^{kX(T)} g\left(X(t),\, 0 \leq t \leq T\right);\, h\right] = \mathrm{E}\left[e^{kX(T)};\, h\right] \times \mathrm{E}\left[g\left(X(t),\, 0 \leq t \leq T\right);\, h+k\right]$$

For an n-dimensional Levy process $\{\boldsymbol{X}(t)\} = \left\{\left(X_1(t),\, \cdots,\, X_n(t)\right)'\right\}$, and vector $h = \left(h_1,\, \cdots,\, h_n\right)'$, $k = \left(k_1,\, \cdots,\, k_n\right)'$. The expectation of $g\left(\boldsymbol{X}(t),\, 0 \leq t \leq T\right)$ with respect to the changed probability measure, indexed by $\boldsymbol{h}$, is defined as

$$\mathrm{E}\left[g\left(\boldsymbol{X}(t),\, 0 \leq t \leq T\right);\, \boldsymbol{h}\right] = \frac{\mathrm{E}\left[g\left(\boldsymbol{X}(t),\, 0 \leq t \leq T\right) e^{\langle \boldsymbol{h},\, \boldsymbol{X}(T)\rangle}\right]}{\mathrm{E}\left[e^{\langle \boldsymbol{h},\, \boldsymbol{X}(T)\rangle}\right]}$$

Correspondingly, the factorization formula can be extended to

$$\begin{aligned}\mathrm{E}\left[e^{\langle \boldsymbol{k},\, \boldsymbol{X}(T)\rangle} g\left(\boldsymbol{X}(t),\, 0 \leq t \leq T\right);\, \boldsymbol{h}\right] &= \mathrm{E}\left[e^{\langle \boldsymbol{k},\, \boldsymbol{X}(\boldsymbol{T})\rangle};\, \boldsymbol{h}\right] \\ &\quad \times \mathrm{E}\left[g\left(\boldsymbol{X}(t),\, 0 \leq t \leq T\right);\, \boldsymbol{h}+\boldsymbol{k}\right]\end{aligned}$$

Let $X(t) = \mu t + \sigma B(t)$, $t > 0$ be a Brownian motion, we have

$$\begin{aligned} \mathrm{E}\left[e^{zX(t)};\, h\right] &= \frac{\mathrm{E}\left[e^{(z+h)X(t)}\right]}{\mathrm{E}\left[e^{hX(t)}\right]} \\ &= \frac{M_{X(t)}(z+h)}{M_{X(t)}(h)} \\ &= \exp\left(z(\mu + h\sigma^2)t + \frac{z^2\sigma^2 t}{2}\right) \end{aligned}$$

The above means that under Esscher transform indexed by h, a Brownian motion with drift μ and volatility σ is still a Brownian motion with the same volatility σ, but with a changed drift $\mu + h\sigma^2$. If $h = -\frac{2\mu}{\sigma^2}$, the drift of $X(t)$ is changed to its negative. Therefore, $\{X(t),\, t > 0\}$ under Esscher transform indexed by $-\frac{2\mu}{\sigma^2}$ can be thought as a reflection of $\{X(t),\, t > 0\}$ under the original measure.

2.2 Reflection principle

Theorem 2.2.1. (*The reflection principle*). *Let $X(t)$ be a Brownian motion as defined in (1.3.1), if $\mu = 0$, we have*

$$\Pr\left[X(T) \le x \text{ and } M(T) > y\right] = \Pr\left[X(T) \le x - 2y\right], \qquad y \ge \max(x,\, 0). \tag{2.2.1}$$

If μ is not necessary 0, then

$$\Pr\left[X(T) \le x \text{ and } M(T) > y\right] = e^{Ry}\Pr\left[X(T) \le x - 2y\right], \qquad y \ge \max(x,\, 0). \tag{2.2.2}$$

where R equals to $\frac{2\mu}{\sigma^2}$.

Remark 2.2.1. When $\mu \neq 0$, $R = \frac{2\mu}{\sigma^2}$ is the non-zero number such that $\left\{e^{-RX(t)},\, t \ge 0\right\}$ is a martingale.

2.3 The Poisson process and the *Erlang* distribution

Consider a Poisson process $\{N(t)\}$ with rate λ, and denote the time of the first event by τ_1. Further, for $n > 1$, let τ_n denote the elapsed time between the $(n-1)$st and the nth event. We know τ_n, $n = 1, 2, \ldots$, are independent identically distributed exponential random variables with rate parameter λ. If we use T_n to denote the arrival time of the nth event, it is easily seen that

$$T_n = \sum_{i=1}^{n} \tau_i, \quad n \geq 1 \tag{2.3.1}$$

Therefore, T_n has a gamma distribution with parameters n and λ. That is, the probability density function of T_n is

$$f_{T_n}(t) = e^{-\lambda t} \frac{\lambda^n t^{n-1}}{(n-1)!}$$

The gamma distribution with a positive integer shape parameter is also called the *Erlang* distribution. Because of the equivalence of the following two events

$$T_n > t \quad \Longleftrightarrow \quad N(t) < n$$

the two events have the same probabilities,

$$P(T_n > t) = P(N(t) < n) \tag{2.3.2}$$

That is,

$$\int_t^{\infty} \lambda^n e^{-\lambda s} \frac{s^{n-1}}{(n-1)!} ds = \sum_{j=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!} \tag{2.3.3}$$

which can also be derived by integration by parts. Consequently,

$$P(T_n > t \geq T_{n-1}) = P(T_n > t) - P(T_{n-1} > t) = e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \tag{2.3.4}$$

2.4 Feynman–Kac theorem

Theorem 2.4.1. *Suppose $X(t)$ satisfies the stochastic differential equation*

$$dX(t) = \mu\left(X(t),\, t\right) dt + \sigma\left(X(t),\, t\right) dB(t)$$

Then the function

$$h(x,\, t) = E\left[e^{-\int_t^T r(X(u),\, u)\, du} g\left(X(T)\right) \mid X(t) = x\right]$$

solves the partial differential equation

$$h_t + \mu(x,\, t)h_x + \frac{1}{2}\sigma^2(x,\, t)h_{xx} = r(x,\, t)h(x,\, t) \tag{2.4.1}$$

with the boundary condition

$$h\left(x,\, T\right) = g\left(x\right).$$

CHAPTER 3

BARRIER OPTION

In this chapter, I will extend the reflection principle to a multivariate geometric Brownian motion using Esscher transforms. This method has been applied by Jun Yang to study the price of the barrier options and the lookback options. We first use it to calculate the price of the outside up-and-out barrier option. Through multiple reflections, the double barrier option could also be valued. At last, the valuation method of the double barrier option whose boundary is the time varying exponential function with different rates is provided.

3.1 Literature review

Prior works that have already been devoted to barrier options pricing include Merton (1973), Reiner and Rubinstein (1991), Heynen and Kat (1994a, 1994b), Kunitomo and Ikeda (1992), Geman and Yor (1996), Buchen and Konstandatos (2009). Merton (1973) first provided a closed-form formula for a down and out call option using the method of differential equations. Following Merton's paper, Reiner and Rubinstein (1991) used probabilistic methods to develop pricing formulas for eight types of standard barrier options.The barrier options hedging problem has been first proposed by Carr and Bowie (1994). They provide a static hedging method for studying path-dependent options. Later, in Carr, Ellis and Gupta (1998), they generalize this method to more general exotic options.

Both the method of differential equations and probabilistic methods have been greatly extended to some complex barrier options. For example, partial time barrier options, where the barrier is monitored only at the start or end of the life of the option were priced by Heynen and Kat (1994a), using differential equations. Another extention Heynen and Kat (1994b) made is to the case of two stocks (outside barrier option), where the payoff function depends on one stock and the barrier event depends on another stock. Later, Kwok, Wu and Yu (1998) consider options written on the maximum of several assets.

Working in a slightly different direction, using the probability density function, Kunitomo and Ikeda (1992) first derived the valuation formulas for double-knock-out call and put, which is an option with two distinct triggers that define the fluctuation of an underlying asset. Alternatively, Geman and Yor (1996) derived an expression for Laplace transformations of the double knock-out call and put and then numerically inverted these expression to obtain the required price. The innovation has continued; Buchen and Konstandatos (2009) considered an arbitrary double-knock-out barrier option with exponential time varying boundaries. The method they used was termed as the method of images for the Black-Scholes equation.

3.2 Generalized reflection principle

In this section, I will extend the reflection principle for Brownian motion to a multivariate geometric Brownian motion case. Theorem 3.2.1 can be seen as an application of the reflection principle for multivariate geometric Brownian motion. In this chapter, we use R_0 to denote a non-zero number such that $\left\{e^{-R_0 X_0(t)},\, t \geq 0\right\}$ is a martingale.

Proposition 3.2.1. *Let* $\left\{\begin{pmatrix} X_0(t) \\ X_1(t) \end{pmatrix};\, t \geq 0\right\}$ *be a two dimensional Brownian motion. For a given function* $g(\cdot)$*, we have*

$$\mathrm{E}\left[\left(e^{-R_0 X_0(t)}\right) g\left(-X_0(t),\, X_1(t) - 2\rho_1 \frac{\sigma_1}{\sigma_0} X_0(t)\right)\right] = \mathrm{E}\left[g\left(X_0(t),\, X_1(t)\right)\right] \quad (3.2.1)$$

Proof. By the factorization formula, we have

$$\begin{aligned}
&\mathrm{E}\left[\left(e^{-R_0 X_0(t)}\right) g\left(-X_0(t),\, X_1(t)-2\rho_1\frac{\sigma_1}{\sigma_0}X_0(t)\right)\right] \\
&= \mathrm{E}\left[e^{-R_0 X_0(t)}\right] \times \mathrm{E}\left[g\left(-X_0(t),\, X_1(t)-2\rho_1\frac{\sigma_1}{\sigma_0}X_0(t)\right);\begin{pmatrix} -R_0 \\ 0 \end{pmatrix}\right]
\end{aligned} \tag{3.2.2}$$

Since $\left\{e^{-R_0 X_0(t)},\, t \geq 0\right\}$ is a martingale and $X_0(0) = 0$, we have $\mathrm{E}\left[e^{-R_0 X_0(t)}\right] = 1$. Also, we have the following relationship

$$\begin{pmatrix} -X_0(t) \\ X_1(t) - 2\rho_1\frac{\sigma_1}{\sigma_0}X_0(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2\rho_1\frac{\sigma_1}{\sigma_0} & 1 \end{pmatrix}\begin{pmatrix} X_0(t) \\ X_1(t) \end{pmatrix}.$$

Under the Esscher measure indexed by the parameter vector $\begin{pmatrix} -R_0 \\ 0 \end{pmatrix}$,

$$\left\{\begin{pmatrix} -X_0(t) \\ X_1(t) - 2\rho_1\frac{\sigma_1}{\sigma_0}X_0(t) \end{pmatrix};\, t \geq 0\right\}$$

is a two dimensional Brownian motion with the drift vector

$$\begin{pmatrix} -1 & 0 \\ -2\rho_1\frac{\sigma_1}{\sigma_0} & 1 \end{pmatrix}\left[\begin{pmatrix} \mu_0 \\ \mu_1 \end{pmatrix} + \begin{pmatrix} \sigma_0^2 & \rho_1\sigma_0\sigma_1 \\ \rho_1\sigma_0\sigma_1 & \sigma_1^2 \end{pmatrix}\begin{pmatrix} -R_0 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} \mu_0 \\ \mu_1 \end{pmatrix}$$

and the diffusion matrix

$$\begin{pmatrix} -1 & 0 \\ -2\rho_1\frac{\sigma_1}{\sigma_0} & 1 \end{pmatrix}\begin{pmatrix} \sigma_0^2 & \rho_1\sigma_0\sigma_1 \\ \rho_1\sigma_0\sigma_1 & \sigma_1^2 \end{pmatrix}\begin{pmatrix} -1 & -2\rho_1\frac{\sigma_1}{\sigma_0} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sigma_0^2 & \rho_1\sigma_0\sigma_1 \\ \rho_1\sigma_0\sigma_1 & \sigma_1^2 \end{pmatrix}.$$

Since a normal distribution is only determined by its mean and variance, the distribution of process

$$\left\{ \begin{pmatrix} -X_0(t) \\ X_1(t) - 2\rho_1 \frac{\sigma_1}{\sigma_0} X_0(t) \end{pmatrix}; \, t \geq 0 \right\}$$

under the Esscher measure indexed by the parameter vector $\begin{pmatrix} -R_0 \\ 0 \end{pmatrix}$ is same as that of the process $\left\{ \begin{pmatrix} X_0(t) \\ X_1(t) \end{pmatrix}; \, t \geq 0 \right\}$ under the original measure. Therefore,

$$\mathrm{E}\left[g\left(-X_0(t), \, X_1(t) - 2\rho_1 \frac{\sigma_1}{\sigma_0} X_0(t) \right); \begin{pmatrix} -R_0 \\ 0 \end{pmatrix} \right] = \mathrm{E}\left[g\left(X_0(t), \, X_1(t) \right) \right].$$

Following (3.2.2), we obtain the desired result. □

Using Proposition 3.2.1, I will give the poof for Theorem 3.2.1 with two stocks case. The proof can be easily generated to n stocks.

Theorem 3.2.1. *For $t \in [0, T]$ and a given function $g(\cdot)$, define*

$$f\left(S_0(t), \, S_1(t), \cdots, \, S_{n-1}(t), \, t\right) = \mathrm{E}_t\left[g\left(S_0(T), \, S_1(T), \cdots, \, S_{n-1}(T) \right) \right]$$

Then, for each positive constant B,

$$\mathrm{E}_t\left[\left(S_0(T)^{-R_0} \right) g\left(\frac{B}{S_0(T)}, \, S_1(T) S_0(T)^{-2\rho_1 \frac{\sigma_1}{\sigma_0}}, \cdots, \, S_{n-1}(T) S_0(T)^{-2\rho_{n-1} \frac{\sigma_n}{\sigma_0}} \right) \right]$$

$$= \left(S_0(t)^{-R_0} \right) f\left(\frac{B}{S_0(t)}, \, S_1(t) S_0(t)^{-2\rho_1 \frac{\sigma_1}{\sigma_0}}, \cdots, \, S_{n-1}(t) S_0(t)^{-2\rho_{n-1} \frac{\sigma_n}{\sigma_0}}, \, t \right)$$

Proposition 3.2.2 is the two-stock case of Theorem 3.2.1 and Proposition 3.2.3 is the special case with one stock.

Proposition 3.2.2. *For $t \in [0,\, T]$ and a given function $g\,(\cdot, \cdot)$, define*

$$f\,(S_0(t),\, S_1(t),\, t) = \mathrm{E}_t\,[g\,(S_0(T),\, S_1(T))]$$

Then, for each positive constant B,

$$\begin{aligned} &\mathrm{E}_t\left[\left(S_0(T)^{-R_0}\right) g\left(\frac{B}{S_0(T)},\, S_1(T)S_0(T)^{-2\rho_1\frac{\sigma_1}{\sigma_0}}\right)\right] \\ &= \left(S_0(t)^{-R_0}\right) f\left(\frac{B}{S_0(t)},\, S_1(t)S_0(t)^{-2\rho_1\frac{\sigma_1}{\sigma_0}},\, t\right) \end{aligned} \tag{3.2.3}$$

Proof. Since $S_0(T)$, $S_1(T)$ can be written as $S_0(t)e^{X_0(T)-X_0(t)}$ and $S_1(t)e^{X_1(T)-X_1(t)}$, the expectation of

$$\left(S_0(T)^{-R_0}\right) g\left(\frac{B}{S_0(T)},\, S_1(T)S_0(T)^{-2\rho_1\frac{\sigma_1}{\sigma_0}}\right)$$

can be written as

$$\begin{aligned} \mathrm{E}_t\Bigg[&S_0(t)^{-R_0}\left(e^{X_0(T)-X_0(t)}\right)^{-R_0} g\Bigg(\frac{B}{S_0(t)}e^{-(X_0(T)-X_0(t))}, \\ &S_1(t)S_0(t)^{-2\rho_1\frac{\sigma_1}{\sigma_0}} \exp\left[(X_1(T)-X_1(t)) - 2\rho_1\frac{\sigma_1}{\sigma_0}\,(X_0(T)-X_0(t))\right]\Bigg)\Bigg] \end{aligned}$$

Following proposition 3.2.1, the above equals

$$\begin{aligned} &\mathrm{E}_t\left[S_0(t)^{-R_0} g\left(\frac{B}{S_0(t)}e^{(X_0(T)-X_0(t))},\, S_1(t)S(t)^{-2\rho_1\frac{\sigma_1}{\sigma_0}}e^{X_1(T)-X_1(t)}\right)\right] \\ &= \left(S_0(t)^{-R_0}\right) f\left(\frac{B}{S_0(t)},\, S_1(t)S_0(t)^{-2\rho_1\frac{\sigma_1}{\sigma_0}},\, t\right). \end{aligned}$$

□

Proposition 3.2.3. *For $t \in [0,\, T]$ and a given (payoff) function $g\,(.)$, if we define $f\,(S(t), t) = \mathrm{E}_t\,[g(S(T))]$. Then, for each positive constant B,*

$$\mathrm{E}_t\left[\left(S(T)^{-R}\right) g\left(\frac{B}{S(T)}\right)\right] = \left(S(t)^{-R}\right) f\left(\frac{B}{S(t)},\, t\right)$$

Remark 3.2.1. The result above can also be derived through the Feynman-Kac Theorem 2.4.1. If we define

$$h(s,\, t) = s^{-R} f\left(\frac{B}{s},\, t\right),$$

we can show that it satisfies the following partial differential equation

$$h_t + (\mu + \frac{1}{2}\sigma^2) s h_s + \frac{1}{2}\sigma^2 s^2 h_{ss} = 0. \tag{3.2.4}$$

The derivatives of $h(s,\, t)$ are

$$h_t = s^{-R} f_t\left(\frac{B}{s},\, t\right) \tag{3.2.5}$$

$$h_s = -R s^{-R-1} f\left(\frac{B}{s},\, t\right) - B s^{-R-2} f_s\left(\frac{B}{s},\, t\right) \tag{3.2.6}$$

$$\begin{aligned} h_{ss} = {} & R(R+1) s^{-R-2} f\left(\frac{B}{s},\, t\right) + BR s^{-R-3} f_s\left(\frac{B}{s},\, t\right) \\ & + B(R+2) s^{-R-3} f_s\left(\frac{B}{s},\, t\right) + B^2 s^{-R-4} f_{ss}\left(\frac{B}{s},\, t\right) \end{aligned} \tag{3.2.7}$$

Substituting the above derivatives into the left-hand side of (3.2.4), we have

$$\begin{aligned} & s^{-R} f_t\left(\frac{B}{s},\, t\right) - (\mu + \frac{1}{2}\sigma^2) R s^{-R} f\left(\frac{B}{s},\, t\right) - (\mu + \frac{1}{2}\sigma^2) B s^{-R-1} f_s\left(\frac{B}{s},\, t\right) \\ & + \frac{1}{2}\sigma^2 R(R+1) s^{-R} f\left(\frac{B}{s},\, t\right) + \frac{1}{2}\sigma^2 B(2R+2) s^{-R-1} f_s\left(\frac{B}{s},\, t\right) \\ & + \frac{1}{2}\sigma^2 B^2 s^{-R-2} f_{ss}\left(\frac{B}{s},\, t\right) \end{aligned}$$

The above can be simplified as

$$s^{-R}\left(f_t\left(\frac{B}{s},\, t\right) + (\mu + \frac{1}{2}\sigma^2)\frac{B}{s} f_s\left(\frac{B}{s},\, t\right) + \frac{1}{2}\sigma^2\left(\frac{B}{s}\right)^2 f_{ss}\left(\frac{B}{s},\, t\right)\right) \tag{3.2.8}$$

Since $f\left(S(t),t\right)=\mathrm{E}_t\left[g(S(T))\right]$ is a martingale, $f(s,\,t)$ satisfies

$$f_t+(\mu+\frac{1}{2}\sigma^2)sf_s+\frac{1}{2}\sigma^2s^2f_{ss}=0.$$

Therefore, (3.2.8) equals zero. Also $h(s,\,t)$ satisfies the terminal condition

$$h(s,\,T)=s^{-R}f\left(\frac{B}{s},\,T\right)=s^{-R}g\left(\frac{B}{s}\right)$$

Therefore, according to the Feynman-Kac Theorem, we have

$$h(S(t),\,t)=\left(S(t)^{-R}\right)f\left(\frac{B}{S(t)},\,t\right)=\mathrm{E}_t\left[\left(S(T)^{-R}\right)g\left(\frac{B}{S(T)}\right)\right]$$

3.3 Main results

In this section, we define $M_S(T)=\max_{0\le t\le T}S(t)$ as the running maximum of the stock price process and $m_S(T)=\min_{0\le t\le T}S(t)$ as the running minimum of the stock price process. Correspondingly, $M(T)$ without subscript S is defined as $\max_{0\le t\le T}X(t)$ and $m(T)$ without subscript S is defined as $\min_{0\le t\le T}X(t)$. Claim 3.3.1 considers the pricing problem of barrier options whose payoff depends on the value of two stocks, the barrier stock being one of them. This can be thought as a slight generalization of the outside barrier option, which is studied in Heynen and Kat (1994b).

Claim 3.3.1. If the initial stock price $S_0(0)$ is lower than level U, the payoff function

$$\pi\left(S_0(T),\,S_1(T)\right)\times I(M_{S_0}(T)<U) \tag{3.3.1}$$

has the same expectation as that of

$$\begin{aligned}&\pi\left(S_0(T),\,S_1(T)\right)\times I(S_0(T)<U)\\&-\left(\frac{S_0(T)}{U}\right)^{-R_0}\pi\left(\frac{U^2}{S_0(T)},\,S_1(T)\left(\frac{S_0(T)}{U}\right)^{-2\rho_{10}\frac{\sigma_1}{\sigma_0}}\right)\times I(S_0(T)>U)\end{aligned} \tag{3.3.2}$$

Proof. There are two situations over the time interval $[0, T]$.

a, The barrier U is never hit before time T, then the payoff from (3.3.1) is $\pi(S_0(T),\, S_1(T))$, which is the same as the payoff from (3.3.2);

b, The barrier U is hit before time T. At the hitting time, the up-and-out option becomes void, and hence there is no longer any payoff. We need to show that the expectation of (3.3.2) taken at the hitting time is also zero.

Consider $g(s_0,\, s_1) = \pi(s_0,\, s_1) \times I(s_0 < U)$, and T_U is the first passage time (hitting time) when $S_0(t)$ hits the barrier U, which is defined as

$$T_U = \inf\left\{t \geq 0 \middle| S_0(t) = U\right\} \tag{3.3.3}$$

According to the definition in proposition 3.2.2, the expectation of the first term of (3.3.2) can be written as

$$f\left(U,\, S_1(T_U),\, T_U\right) = \mathrm{E}_{T_U}[g(S_0(T),\, S_1(T))]$$

and the second term of (3.3.2) can be written as

$$\left(\frac{S_0(T)}{U}\right)^{-R_0} \pi\left(\frac{U^2}{S_0(T)},\, S_1(T)\left(\frac{S_0(T)}{U}\right)^{-2\rho_{10}\frac{\sigma_1}{\sigma_0}}\right) \times I\left(\frac{U^2}{S_0(T)} < U\right)$$

According to the definition of $g\left(\cdot, \cdot\right)$, the expectation of the above can be written as

$$\mathrm{E}_{T_U}\left[\left(\frac{S_0(T)}{U}\right)^{-R_0} g\left(\frac{U^2}{S_0(T)},\, S_1(T)\left(\frac{S_0(T)}{U}\right)^{-2\rho_{10}\frac{\sigma_1}{\sigma_0}}\right)\right].$$

Following proposition 3.2.2, given $B = U^2$, the above equals to

$$\left(\frac{S_0(T_U)}{U}\right)^{-R_0} f\left(\frac{U^2}{S_0(T_U)},\, S_1(T_U)\left(\frac{S_0(T_U)}{U}\right)^{-2\rho_{10}\frac{\sigma_1}{\sigma_0}},\, T_U\right).$$

Since at time T_U, $S_0(T_U)$ equals U, we have

$$\left(\frac{S_0(T_U)}{L}\right)^{-R_0} f\left(\frac{U^2}{S_0(T_U)},\, S_1(T_U)\left(\frac{S_0(T_U)}{U}\right)^{-2\rho_{10}\frac{\sigma_1}{\sigma_0}},\, T_U\right) = f\left(U,\, S_1(T_U),\, T_U\right).$$

The expectation of the second term in (3.3.2) equals to $f\left(U,\, S_1(T_U),\, T_U\right)$, which is same as the expectation of the first term in (3.3.2). Therefore, the expectation of (3.3.2) taken at the first passage time T_U is zero. □

Remark 3.3.1. The intuition is that the up-and-out payoff can be replicated by a "buy/sell-and-hold" strategy, i.e., static hedging instead of dynamic hedging. At time 0, an investor would long a security corresponding to the first term in (3.3.2), and short a security corresponding to the second term in (3.3.2). Once the stock price hits the barrier, the investor shorts the security corresponding to the first term in (3.3.2), and uses the revenue to buy back security corresponding to the second term in (3.3.2). Since the result that the expectation of (3.3.2) taken at the time T_U is zero will be frequently used, we refer to it as Fact 3.3.1.

Fact 3.3.1. *If $T_U < T$, taken at the time T_U, the expectation of*

$$\begin{aligned}&\pi\left(S_0(T),\, S_1(T)\right) \times I(S_0(T) < U)\\ &- \left(\frac{S_0(T)}{U}\right)^{-R_0} \pi\left(\frac{U^2}{S_0(T)},\, S_1(T)\left(\frac{S_0(T)}{U}\right)^{-2\rho_{10}\frac{\sigma_1}{\sigma_0}}\right) \times I(S_0(T) > U)\end{aligned}$$

is zero.

Correspondingly, for one stock case, we have the following fact. We will use it to prove Claim 3.3.2, which is the price of the double barrier option with arbitrary payoffs.

Fact 3.3.2. *If the stock price $S(t)$ hits the barrier U before expire date T, taken at the hitting time, the expectation of*

$$\pi\left(S(T)\right) \times I(S(T) < U) - \left(\frac{S(T)}{U}\right)^{-R} \pi\left(\frac{U^2}{S(T)}\right) \times I(S(T) > U) \qquad (3.3.4)$$

is zero.

Example 3.3.1. (Up-and-Out exchange option) We calculate the price under risk-neutral measure and assume two stocks do not pay dividends. If $S_2(0) < U$, the payoff function

$$(S_1(T) - S_2(T))_+ \times I(M_{S_2}(T) < U) \tag{3.3.5}$$

has the same expectation as that of

$$\begin{aligned}&(S_1(T) - S_2(T))_+ \times I(S_2(T) < U)\\ &- \left(\frac{S_2(T)}{U}\right)^{-R_2} \left(S_1(T) \left(\frac{S_2(T)}{U}\right)^{-2\rho_{12}\frac{\sigma_1}{\sigma_2}} - \frac{U^2}{S_2(T)}\right)_+ \times I(\frac{U^2}{S_2(T)} < U)\end{aligned} \tag{3.3.6}$$

The expectation of the discounted value of

$$(S_1(T) - S_2(T))_+ \times I(S_2(T) < U)$$

can be calculated straightforwardly by Esscher Transform. It equals

$$\begin{aligned}&S_1(0)NN\left(d_1(\ln\frac{S_2(0)}{U}),\ d_2(\ln\frac{S_1(0)}{S_2(0)}),\ \rho\right)\\ &- S_2(0)NN\left(d_3(\ln\frac{S_2(0)}{U}),\ d_4(\ln\frac{S_1(0)}{S_2(0)}),\ \rho\right)\end{aligned} \tag{3.3.7}$$

where

$$
\begin{aligned}
d_1(x) &= -\frac{\ln x + \left(r - \frac{1}{2}\sigma_2^2 + \rho_{12}\sigma_1\sigma_2\right)T}{\sigma_2\sqrt{T}} \\
d_2(x) &= \frac{\ln x + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\
d_3(x) &= -\frac{\ln x + \left(r + \frac{1}{2}\sigma_2^2\right)T}{\sigma_2\sqrt{T}} \\
d_4(x) &= d_2(x) - \sigma^2\sqrt{T}
\end{aligned}
$$

$$
\rho = \frac{\sigma_2 - \rho_{12}\sigma_1}{\sigma}, \text{ and } \sigma^2 = \sigma_1^2 + 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2
$$

The expectation of the discounted value of the second line in (3.3.6) can be calculated using proposition 3.2.2 and the result in (3.3.7), it equals to

$$
\begin{aligned}
&S_1(0)\left(\frac{S_2(0)}{U}\right)^{-R_2-2\rho_{12}\frac{\sigma_1}{\sigma_2}} NN\left(d_1(\ln\frac{U}{S_2(0)}), d_2\left(\ln\left(\frac{S_1(0)}{U}\left(\frac{S_2(0)}{U}\right)^{1-2\rho_{12}\frac{\sigma_1}{\sigma_2}}\right)\right), \rho\right) \\
&-\left(\frac{S_2(0)}{U}\right)^{-R_2}\frac{U^2}{S_2(0)} NN\left(d_3(\ln\frac{U}{S_2(0)}), d_4\left(\ln\left(\frac{S_1(0)}{U}\left(\frac{S_2(0)}{U}\right)^{1-2\rho_{12}\frac{\sigma_1}{\sigma_2}}\right)\right), \rho\right)
\end{aligned}
\tag{3.3.8}
$$

Therefore the price of the up-and-out exchange option equals

$$
\begin{aligned}
&S_1(0)NN\left(d_1(\ln\frac{S_2(0)}{U}),\ d_2(\ln\frac{S_1(0)}{S_2(0)}),\ \rho\right) \\
&-S_2(0)NN\left(d_3(\ln\frac{S_2(0)}{U}),\ d_4(\ln\frac{S_1(0)}{S_2(0)}),\ \rho\right) \\
&-S_1(0)\left(\frac{S_2(0)}{U}\right)^{-R_2-2\rho_{12}\frac{\sigma_1}{\sigma_2}} NN\left(d_1(\ln\frac{U}{S_2(0)}), d_2\left(\ln\left(\frac{S_1(0)}{U}\left(\frac{S_2(0)}{U}\right)^{1-2\rho_{12}\frac{\sigma_1}{\sigma_2}}\right)\right), \rho\right) \\
&+\left(\frac{S_2(0)}{U}\right)^{-R_2}\frac{U^2}{S_2(0)} NN\left(d_3(\ln\frac{U}{S_2(0)}), d_4\left(\ln\left(\frac{S_1(0)}{U}\left(\frac{S_2(0)}{U}\right)^{1-2\rho_{12}\frac{\sigma_1}{\sigma_2}}\right)\right), \rho\right)
\end{aligned}
$$

Claim 3.3.2. If the initial stock price $S(0)$ is between level U and level L, the double knock-out option with constant barrier whose payoff function is

$$\pi\left(S(T)\right) \times I(M_S(T) < U)I(m_S(T) > L) \tag{3.3.9}$$

has the same expectation as that of

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}\left(\frac{L}{U}\right)^{-Rk}\pi\left(\frac{U^{2k}S(T)}{L^{2k}}\right) \times I(L < \frac{U^{2k}S(T)}{L^{2k}} < U) \\
&-\sum_{k=-\infty}^{\infty}\left(\frac{L^{k-1}S(T)}{U^{k}}\right)^{-R}\pi\left(\frac{U^{2k}}{L^{2(k-1)}S(T)}\right) \times I(L < \frac{U^{2k}}{L^{2(k-1)}S(T)} < U)
\end{aligned} \tag{3.3.10}$$

Proof. Same as the proof of Claim 3.3.1, two situations are considered over the time interval $[0, T]$.

a. The barriers U and L are never hit before time T, then the payoff from (3.3.9) is $\pi(S(T))$, which is the same as the payoff from (3.3.10).

b. The barriers U or L is hit before time T. At the hitting time, the double barrier option becomes void, and hence there is no longer any payoff. We need to show that the expectation of (3.3.10) taken at the time when the stock price hits either U or L is zero. Here the first passage (hitting) time is defined as

$$T_{U,L} = \inf\left\{t \geq 0 \middle| S(t) = U \text{ or } S(t) = L\right\}$$

I will show the case at $T_{U,L}$, the stock price equals U. The case of $S(T_{U,L}) = L$ can be proved similarly . According to Fact 3.3.2, if we define

$$\pi_k(s) = \left(\frac{L^k}{U^k}\right)^{-R}\pi\left(\frac{U^{2k}s}{L^{2k}}\right) \times I(L < \frac{U^{2k}s}{L^{2k}})$$

taken at the time when the stock price first hit level U, the expectation of

$$\pi_k\left(S(T)\right) \times I(\frac{U^{2k}S(T)}{L^{2k}} < U)$$

equals to the expectation of

$$\left(\frac{S(T)}{U}\right)^{-R} \pi_k \left(\frac{U^2}{S(T)}\right) \times I(\frac{U^{2k}U^2}{L^{2k}S(T)} < U)$$
$$= \left(\frac{S(T)L^k}{U^{k+1}}\right)^{-R} \pi \left(\frac{U^{2(k+1)}}{L^{2k}S(T)}\right) \times I(L < \frac{U^{2(k+1)}}{L^{2k}S(T)} < U)$$

Therefore, the expectation of (3.3.10) taken at the time when the stock price hits level U is zero. Now we show the expectation of the first sum in (3.3.10) converges. The convergence of the second sum in (3.3.10) can be shown similarly. Here, we assume $\pi(x)$ is bounded by M when $L \leq x \leq U$. Since $\left(\frac{L}{U}\right)^{-R} > 1$, the convergence of the expectation of the sum from $-\infty$ to 0 can be easily shown. We shall use the ratio test to prove the convergence of the expectation of the sum from 1 to ∞. Since when k goes to infinity,

$$\frac{\mathrm{E}\left[\left(\frac{L}{U}\right)^{-R(k+1)} I(L < \frac{U^{2(k+1)}S(T)}{L^{2(k+1)}} < U)\right]}{\mathrm{E}\left[\left(\frac{L}{U}\right)^{-Rk} I(L < \frac{U^{2k}S(T)}{L^{2k}} < U)\right]}$$
$$= \left(\frac{L}{U}\right)^{-R} \frac{\Pr\left(\ln(\frac{L^{2k+3}}{U^{2k+2}S(0)}) < X(T) < \ln(\frac{L^{2k+2}}{U^{2k+1}S(0)})\right)}{\Pr\left(\ln(\frac{L^{2k+1}}{U^{2k}S(0)}) < X(T) < \ln(\frac{L^{2k}}{U^{2k-1}S(0)})\right)}$$

Both the denominator and the numerator of the above go to zero. To keep the calculation simple, we assume $X(T) = B(T)$ ($\mu = 0$ and $\sigma = 1$). Applying L'Hospital's rule, the above is proportional to

$$\frac{\phi\left(\ln(\frac{L^{2k+2}}{U^{2k+1}S(0)})\right) - \phi\left(\ln(\frac{L^{2k+3}}{U^{2k+2}S(0)})\right)}{\phi\left(\ln(\frac{L^{2k}}{U^{2k-1}S(0)})\right) - \phi\left(\ln(\frac{L^{2k+1}}{U^{2k}S(0)})\right)}$$

where $\phi(x)$ is the density function of the standard normal distribution. The result that the above converges to zero as k goes to infinity can be obtained by algebra. Therefore, the expectation of the first sum in (3.3.10) converges. □

Combining the Fact 3.3.1 and the proving of Claim 3.3.2, we are able to calculate the value of the double outside knock-out option with flat bound. We give the result without proof.

Claim 3.3.3. If the initial stock price $S_0(0)$ is between level U and level L, double outside knock-out option with constant barrier whose payoff function is

$$\pi\left(S_1(T),\, S_0(T)\right) \times I(M_{S_0}(T) < U) I(m_{S_0}(T) > L) \tag{3.3.11}$$

has the same expectation as that of

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \left(\frac{L}{U}\right)^{-R_0 k} \pi\left(S_1(T)\left(\frac{L^k}{U^k}\right)^{-2\rho_{10}\frac{\sigma_1}{\sigma_0}}, \frac{U^{2k} S_0(T)}{L^{2k}}\right) \times I(L < \frac{U^{2k} S_0(T)}{L^{2k}} < U) \\
&- \sum_{k=-\infty}^{\infty} \left(\frac{L^{k-1} S_0(T)}{U^k}\right)^{-R_0} \pi\left(S_1(T)\left(\frac{S_0(T) L^{k-1}}{U^k}\right)^{-2\rho_{10}\frac{\sigma_1}{\sigma_0}}, \frac{U^{2k}}{L^{2(k-1)} S_0(T)}\right) \\
&\qquad \times I(L < \frac{U^{2k}}{L^{2(k-1)} S_0(T)} < U)
\end{aligned} \tag{3.3.12}$$

Now we generalize the boundaries of the double barrier option to be timed-dependent. We assume the upper and lower boundaries have the form $U(t) = Ue^{\beta t}$ and $L(\tau) = Le^{\alpha t}$, with $U > L > 0$ and $\beta > \alpha > 0$. We define the first passage(hitting) time as

$$T'_{U,L} = \inf\left\{t \geq 0 \,\middle|\, S(t) = U(t) \,\text{or}\, S(t) = L(t)\right\} \tag{3.3.13}$$

Since at time $T'_{U,L}$, the stock price can be either $U(T'_{U,L})$ or $L(T'_{U,L})$, to distinguish between two cases, we refer T'_U as the first hitting time $S(T'_U) = U(T'_U)$ and for all $t < T'_U$, $S(t) \neq L(t)$, and refer T'_L as the first hitting time $S(T'_L) = L(T'_L)$ and for all $t < T'_L$, $S(t) \neq U(t)$. Therefore the first passage $T'_{U,L}$ either equals to T'_U or T'_L. Similar to Fact *3.3.2*, we have the following result.

Fact 3.3.3. *If $T'_U < T$, taken at the time T'_U, the expectation of*

$$\pi\left(S(T)\right) \times I(S(T) < U(T)) \tag{3.3.14}$$

is same as the expectation of

$$\left(\frac{S(T)}{U(T)}\right)^{-R'} \pi\left(\frac{U(T)^2}{S(T)}\right) \times I(\frac{U(T)^2}{S(T)} < U(T)) \tag{3.3.15}$$

where $R' = \frac{2(\mu-\beta)}{\sigma^2}$.

Proof. If we define $g(s) = \pi(s) \times I(s < U(T))$, according to the definition in proposition 3.2.3

$$f\left(S(T'_U), T'_U\right) = \mathrm{E}_{T'_U}\left(g(S(T)\right) \tag{3.3.16}$$

By the factorization formula,

$$\begin{aligned}&\mathrm{E}_{T'_U}\left(\left(\frac{S(T)}{U(T)}\right)^{-R'} \pi\left(\frac{U(T)^2}{S(T)}\right) \times I(\frac{U(T)^2}{S(T)} < U(T))\right)\\ &= \mathrm{E}_{T'_U}\left(\left(\frac{S(T)}{U(T)}\right)^{-R'}\right) \times \mathrm{E}_{T'_U}\left(\pi\left(\frac{U(T)^2}{S(T)}\right) \times I(\frac{U(T)^2}{S(T)} < U(T)), -R'\right)\end{aligned}$$

Because of the martingale property,

$$\mathrm{E}_{T'_U}\left(\left(\frac{S(T)}{U(T)}\right)^{-R'}\right) = 1$$

and

$$\begin{aligned}&\mathrm{E}_{T'_U}\left(\pi\left(\frac{U(T)^2}{S(T)}\right) \times I(\frac{U(T)^2}{S(T)} < U(T)), -R'\right)\\ &= \mathrm{E}_{T'_U}\left(g\left(\frac{U(T)^2}{S(T)}\right), -R'\right)\\ &= \mathrm{E}_{T'_U}\left(g\left(\frac{U(T'_U)^2 e^{2\beta(T-T'_U)}}{S(T'_U)} e^{-[X(T)-X(T'_U]}\right), -R'\right).\end{aligned}$$

Under the changed probability measure indexed by $-R'$, the drift of the linear

Brownian motion $\{X(t)\}$ is changed from μ to $\mu - \frac{2(\mu-\beta)}{\sigma^2}\sigma^2 = -\mu + 2\beta$. Hence

$$\begin{aligned}
&\mathrm{E}_{T_U'}\left(g\left(\frac{U(T_U')^2 e^{2\beta(T-T_U')}}{S(T_U')}e^{-[X(T)-X(T_U')]}\right), -R'\right) \\
&= \mathrm{E}_{T_U'}\left(g\left(\frac{U(T_U')^2 e^{2\beta(T-T_U')}}{S(T_U')}e^{[X(T)-X(T_U')]-2\beta(T-T_U')}\right)\right) \\
&= f\left(\frac{U(T_U')^2}{S(T_U')}, T_U'\right)
\end{aligned}$$

Since $S(T_U') = U(T_U')$, we get the desired result. □

Now we can calculate the price of a double barrier option with exponential time varying upper and lower barrier levels.

Claim 3.3.4. If the initial stock price $S(0)$ is between level U and level L, the double knock-out option with exponential time varying barrier whose payoff function is

$$\pi(S) \times I\left(Le^{\alpha t} < S(t) < Ue^{\beta t}, \text{ for any } t \in (0,T]\right) \tag{3.3.17}$$

has the same expectation as that of

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}\left(\frac{L^{k(k+1)}}{U^{k^2}S^k}\right)^{-R'}\left(\frac{U^{k(k-1)}S^k}{L^{k^2}}\right)^{-R''}\pi\left(\frac{U^{2k}S}{L^{2k}}\right) \times I(L < \frac{U^{2k}S}{L^{2k}} < U) \\
&- \sum_{k=-\infty}^{\infty}\left(\frac{L^{k(k+1)}S^{k+1}}{U^{(k+1)^2}}\right)^{-R'}\left(\frac{U^{k(k+1)}}{L^{k^2}S^k}\right)^{-R''}\pi\left(\frac{U^{2(k+1)}}{L^{2k}S}\right) \times I(L < \frac{U^{2(k+1)}}{L^{2k}S} < U)
\end{aligned} \tag{3.3.18}$$

where $R' = \frac{2(\mu-\beta)}{\sigma^2}$ and $R'' = \frac{2(\mu-\alpha)}{\sigma^2}$. To simplify the notation, we use S, U and L to denote $S(T)$, $U(T)$ and $L(T)$.

Proof. Similar to the previous proofs, there are two cases over the time interval $[0,T]$. If the barrier $U(t)$ and $L(t)$ are never hit before time T, the payoff function of (3.3.17) is $\pi(S)$, which is same as the payoff function in (3.3.18). Conversely, if either barrier $U(t)$ and $L(t)$ is hit before time T, the payoff function of (3.3.17) is 0. We need to show at the hitting time, the expectation of (3.3.18) also equals to zero. As the proof in Claim 3.3.3, we will show the case when $S(T'_{U,L}) = U(T'_{U,L})$, namely, at time T'_U, the expectation of (3.3.18) equals zero. The case of $S(T'_{U,L}) = L(T'_{U,L})$, namely, at time T'_L can be derived using the same method. According to Fact 3.3.3, if we define

$$\pi_k(s) = \left(\frac{L^{k(k+1)}}{U^{k^2}s^k}\right)^{-R'} \left(\frac{U^{k(k-1)}s^k}{L^{k^2}}\right)^{-R''} \pi\left(\frac{U^{2k}s}{L^{2k}}\right) \times I(L < \frac{U^{2k}s}{L^{2k}}),$$

taken at the time T'_U, the expectation of

$$\pi_k(S) \times I(\frac{U^{2k}s}{L^{2k}} < U)$$

equals to the expectation of

$$\left(\frac{S}{U}\right)^{-R'} \pi_k\left(\frac{U^2}{S}\right) \times I(\frac{U^{2k}U^2}{L^{2k}S} < U)$$

the above equals to

$$\left(\frac{L^{k(k+1)}S^{k+1}}{U^{(k+1)^2}}\right)^{-R'} \left(\frac{U^{k(k+1)}}{L^{k^2}S^k}\right)^{-R''} \pi\left(\frac{U^{2(k+1)}}{L^{2k}S}\right) \times I(L < \frac{U^{2(k+1)}}{L^{2k}S} < U)$$

which is the term in the second line of (3.3.18). The convergence of the expectation of each line in (3.3.18) can be proved similarly to the proof in Claim 3.3.2. Therefore, the expectation of (3.3.18) taken at time T'_U is zero. □

Example 3.3.2. The time-0 price of the call option, which is nullified before its expiry date whenever the underlying asset price reaches the upper boundary $Be^{\beta t}$ or the lower boundary $Ae^{\alpha t}$ for any $t \in (0, T]$. The corresponding payoff function is

$$(S(T) - K)_+ \times I\left(Le^{\alpha t} < S(t) < Ue^{\beta t}, \text{ for any } t \in (0, T]\right) \tag{3.3.19}$$

Here we assume $L < K < U$ and $0 < \alpha < \beta$. According to Claim 3.3.4, it has the same expectation as that of

$$\sum_{k=-\infty}^{\infty}\left(\frac{\left(Le^{\alpha T}\right)^{k(k+1)}}{(Ue^{\beta T})^{k^2}\,S(T)^k}\right)^{-R'}\left(\frac{\left(Ue^{\beta T}\right)^{k(k-1)}\,S(T)^k}{(Le^{\alpha T})^{k^2}}\right)^{-R''}\left(\frac{\left(Ue^{\beta T}\right)^{2k}\,S(T)^k}{(Le^{\alpha T})^{2k}}-K\right)_+$$
$$\times I(Le^{\alpha T}<\frac{\left(Ue^{\beta T}\right)^{2k}\,S(T)^k}{(Le^{\alpha T})^{2k}}<Ue^{\beta T}) \tag{3.3.20}$$
$$-\sum_{k=-\infty}^{\infty}\left(\frac{\left(Le^{\alpha T}\right)^{k(k+1)}\,S(T)^{k+1}}{(Ue^{\beta T})^{(k+1)^2}}\right)^{-R'}\left(\frac{\left(Ue^{\beta T}\right)^{k(k+1)}}{(Le^{\alpha T})^{k^2}\,S(T)^k}\right)^{-R''}\left(\frac{\left(Ue^{\beta T}\right)^{2(k+1)}}{(Le^{\alpha T})^{2k}\,S(T)}-K\right)_+$$
$$\times I(Le^{\alpha T}<\frac{\left(Ue^{\beta T}\right)^{2(k+1)}}{(Le^{\alpha T})^{2k}\,S(T)}<Ue^{\beta T}) \tag{3.3.21}$$

Here we calculate the price under risk-neutral measure and assume the stock does not pay dividends, therefore $\mu = r - \frac{1}{2}\sigma^2$. Since $D < K$, (3.3.20) equals

$$\sum_{k=-\infty}^{\infty}\left(\frac{\left(Le^{\alpha T}\right)^{(k+1)}}{(Ue^{\beta T})^{k}\,S(T)}\right)^{-kR'}\left(\frac{\left(Ue^{\beta T}\right)^{(k-1)}\,S(T)}{(Le^{\alpha T})^{k}}\right)^{-kR''}\left(\frac{\left(Ue^{\beta T}\right)^{2k}\,S(T)^k}{(Le^{\alpha T})^{2k}}-K\right)$$
$$\times I(K<\frac{\left(Ue^{\beta T}\right)^{2k}\,S(T)^k}{(Le^{\alpha T})^{2k}}<Ue^{\beta T})$$

The expectation of the above can be calculated straightforwardly by Esscher Transform. It equals

$$S(0)e^{rT}\sum_{k=-\infty}^{\infty}\left(\frac{U^k}{L^k}\right)^{kR'-(k-1)R''+2}\left(\frac{L}{S(0)}\right)^{kR''-kR'}$$
$$\times\left(N\left[d_1(\ln\frac{U^{2k}S(0)}{L^{2k}K})\right]-N\left[d_1(\ln\frac{U^{2k}S(0)}{L^{2k}Ue^{\beta T}})\right]\right)$$
$$-K\sum_{k=-\infty}^{\infty}\left(\frac{U^k}{L^k}\right)^{kR'-(k-1)R''}\left(\frac{L}{S(0)}\right)^{kR''-kR'}$$
$$\times\left(N\left[d_2(\ln\frac{U^{2k}S(0)}{L^{2k}K})\right]-N\left[d_2(\ln\frac{U^{2k}S(0)}{L^{2k}Ue^{\beta T}})\right]\right)$$

with

$$d_1(x)=\frac{\ln x+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2(x) = d_1(x) - \sigma\sqrt{T}$$

Through algebra calculation, (3.3.21) equals

$$\sum_{k=-\infty}^{\infty} \left(\frac{\left(Le^{\alpha T}\right)^{k(k-1)}}{(Ue^{\beta T})^{(k-1)^2} S(T)^{k-1}} \right)^{-R'} \left(\frac{\left(Ue^{\beta T}\right)^{k(k-1)} S(T)^k}{(Le^{\alpha T})^{k^2}} \right)^{-R''}$$
$$\times \left(\frac{\left(Le^{\alpha T}\right)^{2k}}{(Ue^{\beta T})^{2(k-1)} S(T)} - K \right) \times I(K < \frac{\left(Le^{\alpha T}\right)^{2k}}{(Ue^{\beta T})^{2(k-1)} S(T)} < Ue^{\beta T})$$

The expectation of the above equals

$$S(0)e^{rT} \sum_{k=-\infty}^{\infty} \left(\frac{L^k}{U^{k-1}S(0)} \right)^{R''k - R'(k-1)+2}$$
$$\times \left(N\left[d_1(\ln \frac{L^{2k}S(0)}{U^{2k-2}SK}) \right] - N\left[d_1(\ln \frac{L^{2k}}{U^{2k-2}S(0)Ue^{\beta T}}) \right] \right)$$
$$-K \sum_{k=-\infty}^{\infty} \left(\frac{L^k}{U^{k-1}S(0)} \right)^{R''k - R'(k-1)}$$
$$\times \left(N\left[d_2(\ln \frac{L^{2k}S(0)}{U^{2k-2}SK}) \right] - N\left[d_2(\ln \frac{L^{2k}}{U^{2k-2}S(0)Ue^{\beta T}}) \right] \right)$$

Therefore, the time-0 price of the double exponential barrier call option equals

$$
\begin{aligned}
S(0) \sum_{k=-\infty}^{\infty} \Bigg\{ & \left(N\left[d_1(\ln \frac{U^{2k}S(0)}{L^{2k}K}) \right] - N\left[d_1(\ln \frac{U^{2k}S(0)}{L^{2k}Ue^{\beta T}}) \right] \right) \\
& \times \left(\frac{U^k}{L^k} \right)^{kR^{'}-(k-1)R^{''}+2} \left(\frac{L}{S(0)} \right)^{kR^{''}-kR^{'}} \\
& - \left(N\left[d_1(\ln \frac{L^{2k}S(0)}{U^{2k-2}SK}) \right] - N\left[d_1(\ln \frac{L^{2k}}{U^{2k-2}S(0)Ue^{\beta T}}) \right] \right) \\
& \times \left(\frac{L^k}{U^{k-1}S(0)} \right)^{R^{''}k-R^{'}(k-1)+2} \Bigg\} \\
- Ke^{-rT} \sum_{k=-\infty}^{\infty} \Bigg\{ & \left(N\left[d_2(\ln \frac{U^{2k}S(0)}{L^{2k}K}) \right] - N\left[d_2(\ln \frac{U^{2k}S(0)}{L^{2k}Ue^{\beta T}}) \right] \right) \\
& \times \left(\frac{U^k}{L^k} \right)^{kR^{'}-(k-1)R^{''}} \left(\frac{L}{S(0)} \right)^{kR^{''}-kR^{'}} \\
& - \left(N\left[d_2(\ln \frac{L^{2k}S(0)}{U^{2k-2}SK}) \right] - N\left[d_2(\ln \frac{L^{2k}}{U^{2k-2}S(0)Ue^{\beta T}}) \right] \right) \\
& \times \left(\frac{L^k}{U^{k-1}S(0)} \right)^{R^{''}k-R^{'}(k-1)} \Bigg\}
\end{aligned}
$$

The result above corresponds to the result in Ikeda and Kuintomo (1992). The essential difference ($d_2 \to d_1, R^{'} \to R^{'}+2, R^{''} \to R^{''}+2$) between the two sum in the above expression is due to the change of the measure.

CHAPTER 4

AMERICAN OPTION WITH EXPONENTIAL EXPIRY DATE

4.1 Introduction

American options can be exercised at any time prior to the expiry date, whereas European options can only be exercised at the expiry date. Since the additional right should not be worthless, we expect American options to be worth more than their European counterparts. For a standard American call option without dividends, there is no advantage to exercise it prematurely. Therefore, it can be valued in the same way as a European call option. However, the majority of American options are subject to early exercise. An exercise boundary is a time path of critical stock prices at which early exercise occurs. The optimal exercise boundary of an American option is not known in advance but has to be determined as part of the solution to the valuation problem. For general American options, due to the absence of simple expressions for the optimal exercise boundary, we have difficulty deriving closed form formulas.

In this chapter we consider the valuation problem of American options whose expiry dates are exponentially distributed and independent of the underlying stock price process. The owner of this kind of American options can exercise at any time up to and including some random expiry date. Since the memory-less property of the exponential distribution implies the exercise boundary is flat, the analytic pricing formulas for American options with exponentially distributed expiry dates are obtainable. Carr (1998) used the Black-Scholes PDE method to obtain the pricing formula of an American put option with an exponentially distributed expiry date. Motivated by Gerber and Shiu (1994b) and Gerber, Shiu and Yang (2012), I shall provide two alternative probability methods for deriving the pricing formulas for

American options with exponentially distributed expiry dates. The first method calculates the price of American options as the sum of rebate options and barrier options. The second method decomposes the value of American options into the sum of the value of European options and the early exercise premium. The structure of this Chapter is as follows. In section 4.2, I use the American put option as an example to illustrate how these two probability methods work. In section 4.3, two methods are generalized to evaluate an exchange option. In section 4.4, the pricing problem of American maximum options is considered.

4.2 Put option

4.2.1 Two methodologies

We assume under the risk neutral probability measure, the time t stock price is also modeled as

$$S(t) = S(0)e^{X(t)}, \qquad t \geq 0$$

and

$$X(t) = (r - \delta - \frac{1}{2}\sigma^2)t + \sigma B(t), \qquad t \geq 0$$

where $\{B(t)\}$ is a standard Brownian motion, and r, δ, σ are the constant risk-free interest rate, the continuous dividend rate of the stock and the volatility of the stock. We also define τ as an exponential random variable independent of $\{B(t)\}$ with the density function $f_\tau(t) = \lambda e^{-\lambda t}$. Since the American option holder can exercise the option at any time during the life of the option, in order to determine the price of an American put option with a strike price K, we need to choose $\underset{\sim}{S}$ to maximize the following representation

$$\sup_{\underset{\sim}{S}} \mathrm{E}\left[e^{-r(T_{\underset{\sim}{S}} \wedge \tau)}[K - S(T_{\underset{\sim}{S}} \wedge \tau)]_+\right], \quad S(0) > \underset{\sim}{S} \tag{4.2.1}$$

where $T_{\underset{\sim}{S}}$ is defined as the first passage time with respect to the exercise boundary

$\underset{\sim}{S}$,

$$T_{\underset{\sim}{S}} = \inf\left\{t \geq 0 \middle| S(t) = \underset{\sim}{S}\right\}$$

For the fixed time of expiry date, the optimal exercise boundary is time-dependent. However, if the expiry date is exponentially distributed, the memoryless property of the exponential distribution implies that as calendar time elapses, the option gets no closer to its expiry date, and thus the passage of time has no effect on its optimal exercise boundary. Therefore, the exercise boundary becomes flat. If we use $\underline{S}$ to denote the optimal exercise boundary, we can rewrite (4.2.1) as an iterated expectation:

$$\begin{aligned}
&\mathrm{E}\left\{\mathrm{E}\left[e^{-r(T_{\underline{S}}\wedge\tau)}[K - S(T_{\underline{S}} \wedge \tau)]_+ \middle| \tau\right]\right\} \\
&= \int_0^\infty \mathrm{E}\left[e^{-r(T_{\underline{S}}\wedge t)}[K - S(T_{\underline{S}} \wedge t)]_+ \middle| \tau = t\right] f_\tau(t)dt
\end{aligned}$$

Because of the independence of the exponential random variable τ and the stock price process, the above can be simplified as:

$$\begin{aligned}
&\int_0^\infty \mathrm{E}\left[e^{-r(T_{\underline{S}}\wedge t)}[K - S(T_{\underline{S}} \wedge t)]_+\right] \lambda e^{-\lambda t} dt \\
&= \int_0^\infty \mathrm{E}\left[e^{-rT_{\underline{S}}}[K - \underline{S}]I(T_{\underline{S}} < t)\right] \lambda e^{-\lambda t} dt \\
&+ \int_0^\infty \mathrm{E}\left[e^{-rt}[K - (S(t)]_+ I(T_{\underline{S}} > t)\right] \lambda e^{-\lambda t} dt \\
&= \mathrm{E}\left[e^{-rT_{\underline{S}}}[K - \underline{S}]I(T_{\underline{S}} < \tau)\right] + \mathrm{E}\left[e^{-r\tau}[K - (S(\tau)]_+ I(T_{\underline{S}} > \tau)\right] \qquad (4.2.2)
\end{aligned}$$

From the above, we can see that the value of an American put with an exponentially distributed expiration date is the Laplace-Carson transform of a fixed expiration date option which is the sum of a down-and-out put with barrier $\underline{S}$ and rebate $K - \underline{S}$, maximized over a barrier. For an American option with a fixed expiration date, there is normally no closed form formula. However, for some of these options there are closed formulas for their Laplace-Carson transforms. Hence there is at least one way to price them numerically by inverting the transform.

The second approach for determining the price of an American put option with an exponentially distributed expiry date is motivated by another representation of the price of American options [a rigorous proof is given in Krylov (1980)]

$$\mathrm{E}[e^{-r\tau}\left[K - S(\tau)\right]_{+}] + \int_{0}^{\tau} e^{-rt}\mathrm{E}\left[\left[Kr - \delta S(t)\right] I(S(t) < \underline{S})\right] dt \tag{4.2.3}$$

The first term of (4.2.3) represents the value of the usual European put option while the second term of (4.2.3) represents the early exercise premium. The early exercise premium can be understood as the compensation paid to the holder when the early exercise right is forfeited. It can be expressed in terms of the exercise boundary in the form of an integral. The optimal exercise boundary $\underline{S}$ can be determined by applying the value matching condition at the optimal exercise boundary $\underline{S}$. This means at the optimal exercise boundary $\underline{S}$, the value of the American put option equals to its exercise value $K - \underline{S}$.

4.2.2 Rebate option

We start to evaluate the first part of (4.2.2)

$$\mathrm{E}\left[e^{-rT_{\underline{S}}}[K - \underline{S}]I(T_{\underline{S}} < \tau)\right]$$

which is an immediate rebate option with the payoff $K - \underline{S}$. We will use the martingale approach to derive the formula. This approach is first developed in Gerber and Shiu (1994b). We consider the stochastic process $\left\{e^{-rt}S(t)^{\theta}I(t < \tau),\, t > 0\right\}$. Because of the independence of the exponential random variable τ and the stock price process, the martingale condition is equivalent to choosing θ such that

$$\frac{1}{2}\sigma^2\theta^2 + (r - \delta - \frac{1}{2}\sigma^2)\theta - (r + \lambda) = 0 \tag{4.2.4}$$

Let $\alpha < 0$, $\beta > 0$ be the two roots of the above quadratic equation. Since we evaluate the price of the an put option, the initial stock price $S(0) > \underline{S}$. Following martingale with power α

$$\left\{e^{-rt} S(t)^{\alpha} I\left(t<\tau\right), \, 0<t<T_{\underline{S}}\right\} \tag{4.2.5}$$

is bounded. Applying the optional sampling theorem, we have

$$\mathrm{E}\left[e^{-rT_{\underline{S}}} S(T_{\underline{S}})^{\alpha} I(T_{\underline{S}}<\tau)\right]=S(0)^{\alpha} \tag{4.2.6}$$

Since $S(T_{\underline{S}})=\underline{S}$, the time-0 value of immediate rebate $K-\underline{S}$ is,

$$(K-\underline{S})\,\mathrm{E}[e^{-rT_{\underline{S}}} I(T_{\underline{S}}<\tau)]=(K-\underline{S})\left(\frac{S(0)}{\underline{S}}\right)^{\alpha}$$

Remark 4.2.1. For the case $S(0)<\underline{S}$, the following process (the power of $S(t)$ is β)

$$\left\{e^{-rt} S(t)^{\beta} I\left(t<\tau\right), \, 0<t<T_{\underline{S}}\right\} \tag{4.2.7}$$

is a bounded martingale. Applying the optional sampling theorem and the condition $S(T_{\underline{S}})=\underline{S}$, we have

$$\mathrm{E}[e^{-rT_{\underline{S}}} I(T_{\underline{S}}<\tau)]=\left(\frac{S(0)}{\underline{S}}\right)^{\beta}$$

4.2.3 Barrier option

The second part of (4.2.2)

$$\mathrm{E}\left[e^{-r\tau}[K-(S(\tau)]_{+} I(T_{\underline{S}}>\tau)\right]$$

is the time-0 value of a down-and-out put option with exponential expiration date. The valuation problem of this option has been studied in Gerber, Shiu and Yang

(2012). Here, we give a quick review. Since the following two events are equivalent,

$$T_{\underline{S}} > \tau \iff \min_{0\le t\le \tau} S(t) > \underline{S}$$

The payoff of a down-and-out put option equals

$$[K - S(\tau)]_{+} I(\min_{0\le t\le\tau} S(t) > \underline{S}) \tag{4.2.8}$$

Using the notation in 1.3, the price of a down-and-out put option equals to the expected discounted value of (4.2.8), which is

$$\mathrm{E}\left[e^{-r\tau}[K - S(0)e^{X(\tau)}]_{+} I(S(0)e^{m(\tau)} > \underline{S})\right]. \tag{4.2.9}$$

To evaluate the above, the key is to calculate so called discounted density function, which is defined as

$$f^{r}_{X(\tau),\,m(\tau)}(x,\,y) = \int_0^\infty e^{-rt} f_{X(t),\,m(t)}(x,\,y) f_\tau(t)\mathrm{d}t, \qquad y \le \min(x,\,0)$$

where $f_{X(t),\,m(t)}(x,\,y)$ is the joint probability density function of $X(t)$ and $m(t)$. We give the detailed calculation of the above in the Appendix. Here, we give the result directly

$$f^{r}_{X(\tau),\,m(\tau)}(x,\,y) = \frac{\lambda}{\frac{1}{2}\sigma^2} e^{-\beta x + (\beta-\alpha)y}, \qquad y \le \min(x,\,0). \tag{4.2.10}$$

where β and α is defined in the previous section.

Remark 4.2.2. Integrating (4.2.10) over y, we obtain the discounted density function of $X(\tau)$.

$$f^{r}_{X(\tau)}(x) = \begin{cases} \kappa e^{-\alpha x}, & x \le 0 \\ \kappa e^{-\beta x}, & x > 0 \end{cases} \tag{4.2.11}$$

where $\kappa = \frac{\lambda}{\frac{1}{2}\sigma^2(\beta-\alpha)}$.

Once we know the discounted density function, (4.2.9) equals

$$\frac{\lambda}{\frac{1}{2}\sigma^2}\int_{\ln\left(\frac{\underline{S}}{S(0)}\right)}^{0\wedge\ln\left(\frac{K}{S(0)}\right)}\left[\int_y^{\ln\left(\frac{K}{S(0)}\right)}\left[K-S(0)e^x\right]_+e^{-\beta x}dx\right]e^{(\beta-\alpha)y}\mathrm{d}y$$

Depending on the different values of the initial stock price, the time-0 value of down-and-out put options with an exponentially distributed expiry date have different representation. If $S(0) > K$, it equals

$$\begin{aligned}
&\mathrm{E}[e^{-r\tau}\left[K-S(\tau)\right]_+ I(T_{\underline{S}}>\tau)] \\
&=\left(\frac{S(0)}{K}\right)^{\alpha}\left(\frac{\kappa K}{-\alpha(1-\alpha)}\right) \\
&\quad+\left(-\frac{\lambda}{\frac{1}{2}\sigma^2}\frac{1}{-\alpha\beta}K+\frac{\lambda}{\frac{1}{2}\sigma^2}\frac{\underline{S}}{(1-\alpha)(\beta-1)}-\frac{\kappa K^{(1-\beta)}\underline{S}^{\beta}}{\beta(\beta-1)}\right)\left(\frac{S(0)}{\underline{S}}\right)^{\alpha}
\end{aligned}\tag{4.2.12}$$

If $\underline{S} < S(0) \le K$, it equals

$$\begin{aligned}
&\mathrm{E}[e^{-r\tau}\left[K-S(\tau)\right]_+ I(T_{\underline{S}}>\tau)] \\
&=K\frac{\lambda}{\lambda+r}-S(0)\frac{\lambda}{\lambda+\delta}+\left(\frac{S(0)}{K}\right)^{\beta}\left(\frac{\kappa K}{\beta(\beta-1)}\right) \\
&\quad+\left(-\frac{\lambda}{\frac{1}{2}\sigma^2}\frac{1}{-\alpha\beta}K+\frac{\lambda}{\frac{1}{2}\sigma^2}\frac{\underline{S}}{(1-\alpha)(\beta-1)}-\frac{\kappa K^{(1-\beta)}\underline{S}^{\beta}}{\beta(\beta-1)}\right)\left(\frac{S(0)}{\underline{S}}\right)^{\alpha}
\end{aligned}\tag{4.2.13}$$

Remark 4.2.3. Following Gerber, Shiu and Yang (2012), the first part of (4.2.12), $\left(\frac{S(0)}{K}\right)^{\alpha}\left(\frac{\kappa K}{-\alpha(1-\alpha)}\right)$, is the price of an out-of-the-money European put with exponentially distributed expiry date. Similarly, the price of an out-of-the-money European call with exponentially distributed expiry date equals $\left(\frac{S(0)}{K}\right)^{\beta}\left(\frac{\kappa K}{\beta(\beta-1)}\right)$. We use the put-call parity to calculate the price of an in-the-money European put with exponentially distributed expiry date, which is the second line of the equation (4.2.13).

$$K\frac{\lambda}{\lambda+r}-S(0)\frac{\lambda}{\lambda+\delta}+\left(\frac{S(0)}{K}\right)^{\beta}\left(\frac{\kappa K}{\beta(\beta-1)}\right)$$

Remark 4.2.4. Because of the identity

$$\text{Knock-out Option} = \text{Ordinary Option} - \text{Knock-in option}$$

The last line of (4.2.12) and (4.2.13) is the price of the knock-in put option.

Remark 4.2.5. Through (4.2.6), we have

$$\mathrm{E}[e^{-rT_{\underline{S}}} I(T_{\underline{S}} < \tau)] = \left(\frac{S(0)}{\underline{S}}\right)^{\alpha}$$

The above implies the price of one dollar payable at the first hitting barrier time equals $\left(\frac{S(0)}{\underline{S}}\right)^{\alpha}$. Because of the following relationships

$$\alpha + \beta = -\frac{(r - \delta - \frac{1}{2}\sigma^2)}{\frac{1}{2}\sigma^2} \qquad \text{and} \qquad \alpha\beta = \frac{-(r+\lambda)}{\frac{1}{2}\sigma^2}$$

conditional on the initial stock price equals $\underline{S}$, the price of the put option is

$$\begin{aligned} &\mathrm{E}[e^{-r\tau} \left[K - S(\tau)\right]_+ |S(0) = \underline{S}] \\ &= \frac{\lambda}{\frac{1}{2}\sigma^2} \frac{1}{-\alpha\beta} K - \frac{\lambda}{\frac{1}{2}\sigma^2} \frac{\underline{S}}{(1-\alpha)(\beta-1)} + \frac{\kappa K^{(1-\beta)} \underline{S}^{\beta}}{\beta(\beta-1)} \end{aligned}$$

Remark 4.2.6. Following Remarks 4.2.4 and 4.2.5, we have the following relationship

$$\begin{aligned} &\mathrm{E}[e^{-r\tau} \left[K - S(\tau)\right]_+ I(T_{\underline{S}} < \tau)] \\ &= \mathrm{E}_{T_{\underline{S}}}[e^{-r\left(\tau - T_{\underline{S}}\right)} \left[K - S(\tau)\right]_+ |S(T_{\underline{S}}) = \underline{S}] \times \mathrm{E}[e^{-rT_{\underline{S}}} I(T_{\underline{S}} < \tau)] \\ &= \mathrm{E}_0[e^{-r\tau} \left[K - S(\tau)\right]_+ |S(0) = \underline{S}] \times \mathrm{E}[e^{-rT_{\underline{S}}} I(T_{\underline{S}} < \tau)] \end{aligned} \tag{4.2.14}$$

Intuitively, to calculate the price of the down-and-in option with an exponentially distributed expiry date, we first discount the payoff function from the expiry date to the first hitting barrier time, then discount it to the initial time. The second equality of (4.2.14) is due to the memoryless property of the exponential distribution and the strong Markov property of the stock process.

4.2.4 First methodology

According to (4.2.2), the value of an American put with an exponentially distributed expiry date equals to the sum of the value of an immediate rebate $K - \underline{S}$ and a down-and-out put option. We have calculated each of them in the previous sections. The sum of them equals, if $K < S(0)$

$$\left(\frac{S(0)}{K}\right)^{\alpha}\left(\frac{\kappa K}{-\alpha(1-\alpha)}\right) + \left(-\frac{\lambda}{\frac{1}{2}\sigma^2}\frac{1}{-\alpha\beta}K + \frac{\lambda}{\frac{1}{2}\sigma^2}\frac{\underline{S}}{(1-\alpha)(\beta-1)} - \frac{\kappa K^{(1-\beta)}\underline{S}^{\beta}}{\beta(\beta-1)} + K - \underline{S}\right)\left(\frac{S(0)}{\underline{S}}\right)^{\alpha} \tag{4.2.15}$$

if $\underline{S} < S(0) \leq K$, it equals

$$K\frac{\lambda}{\lambda+r} - S(0)\frac{\lambda}{\lambda+\delta} + \left(\frac{S(0)}{K}\right)^{\beta}\left(\frac{\kappa K}{\beta(\beta-1)}\right) + \left(-\frac{\lambda}{\frac{1}{2}\sigma^2}\frac{1}{-\alpha\beta}K + \frac{\lambda}{\frac{1}{2}\sigma^2}\frac{\underline{S}}{(1-\alpha)(\beta-1)} - \frac{\kappa K^{(1-\beta)}\underline{S}^{\beta}}{\beta(\beta-1)} + K - \underline{S}\right)\left(\frac{S(0)}{\underline{S}}\right)^{\alpha} \tag{4.2.16}$$

If $S(0) \leq \underline{S}$, we should optimally exercise the option and the value of the American option equals to its exercise value $K - S(0)$. To determine the optimal exercise boundary $\underline{S}$, we would choose $\underline{S}$ to maximize (4.2.15) or (4.2.16). That is to choose $\underline{S}$ to maximize

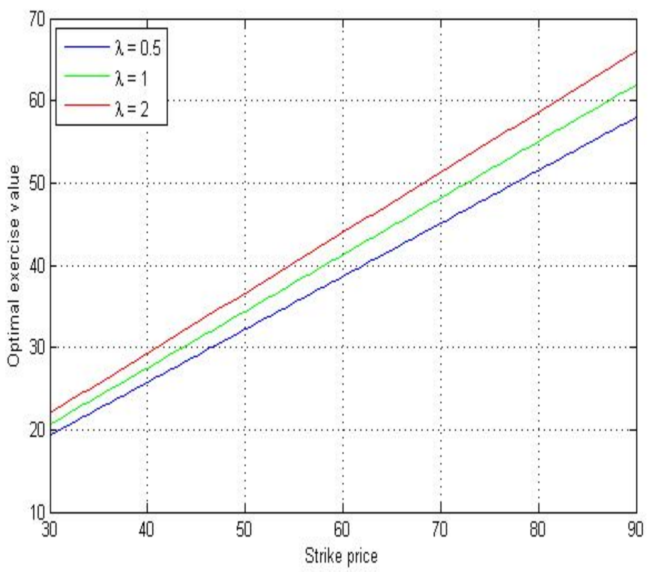
$$\left(-\frac{\lambda}{\frac{1}{2}\sigma^2}\frac{1}{-\alpha\beta}K + \frac{\lambda}{\frac{1}{2}\sigma^2}\frac{\underline{S}}{(1-\alpha)(\beta-1)} - \frac{\kappa K^{(1-\beta)}\underline{S}^{\beta}}{\beta(\beta-1)} + K - \underline{S}\right)\left(\frac{S(0)}{\underline{S}}\right)^{\alpha} \tag{4.2.17}$$

If $\delta = 0$, the optimal exercise boundary $\underline{S}$ has an explicit expression

$$\underline{S} = K \left(\frac{r(\beta - 1)}{\lambda} \right)^{\frac{1}{\beta}}$$

Otherwise, we need numerically calculate it. If $S(0) > \underline{S}$, the time-0 value of an American put option with an exponentially distributed expiration date is obtained by substituting the value of $\underline{S}$ into (4.2.15) or (4.2.16). We set the parameters $\delta = 0.01$, $\sigma = 0.3$, $r = 0.05$. Figure 4.1 shows the relationship between the optimal exercise boundary and the strike price. The ratio of the optimal exercise boundary and the strike price is a fixed positive number. Figure 4.2 shows the relationship between the optimal exercise boundary and the parameter of the exponential distribution λ. Figure 4.3 and 4.4 graph the value of the American put option with an exponential expiry date against the stock price.

Figure 4.1: Optimal exercise boundary vs Strike price



Remark 4.2.7. If $\lambda \to 0$, the optimal critical stock price will converge to $K\frac{\alpha}{\alpha-1}$. It equals to $K\frac{r}{r+\frac{1}{2}\sigma^2}$, when the dividend rate equals zero.

Figure 4.2: Optimal exercise boundary vs λ

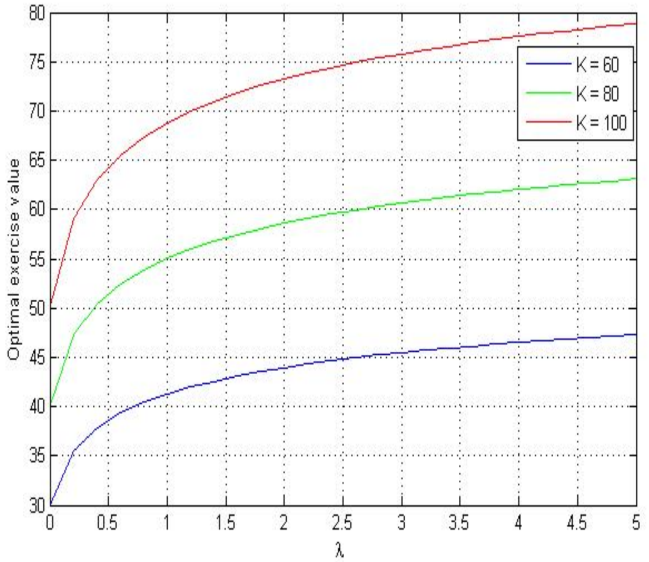
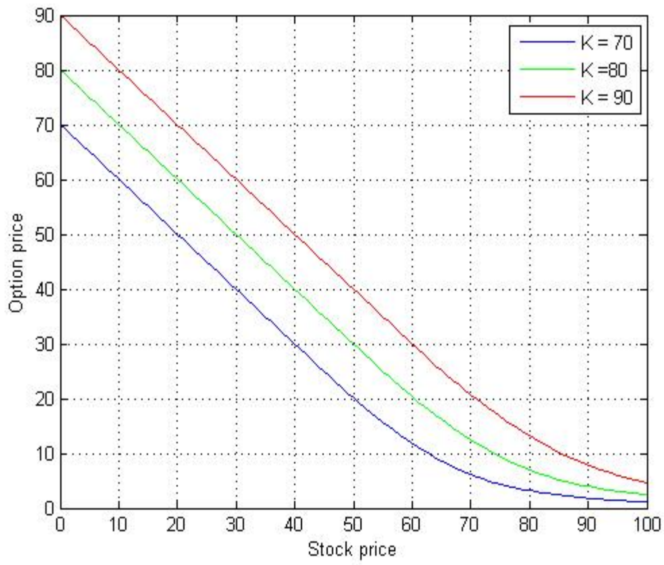


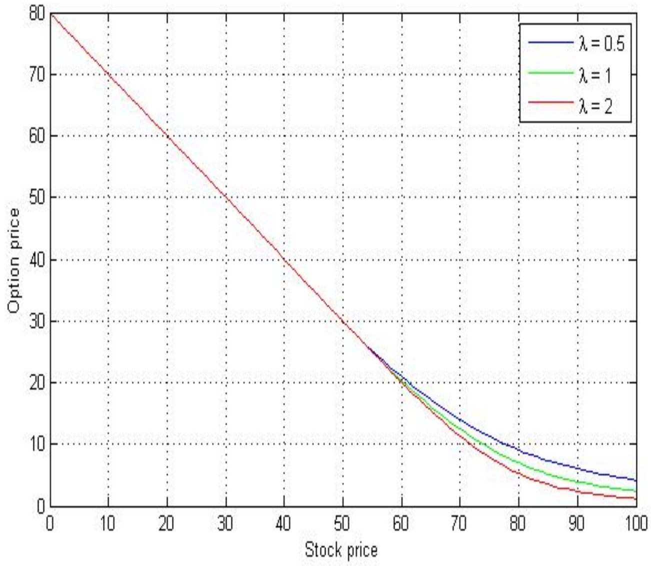
Figure 4.3: American put option vs stock price with $\lambda = 1$



4.2.5 Second methodology

The second approach for determining the price of an American put option is to decompose it into the sum of the value of the European put option and the early exercise premium. For American put options with an exponentially distributed

Figure 4.4: American put option vs stock price with $K = 80$



expiry date, the early exercise premium has the following integral representation,

$$KrE\left[\int_0^\tau e^{-rt}I(S(t) < \underline{S})dt\right] - \delta E\left[\int_0^\tau e^{-rt}S(t)I(S(t) < \underline{S})dt\right] \quad (4.2.18)$$

An intuitive argument for the above representation is: Before the expiration date, when the stock price is smaller than an exercise boundary, the option holder should exercise the option. The option holder will receive $[Kr - S(t)\delta]dt$ at those times t when the option has been exercised optimally. This is because the option holder would have earned interest $Krdt$ from the strike price received and lost dividend $S(t)\delta dt$ from the short position of the stock if he were to choose to exercise his put option. In general the exercise boundary should depend on time t, but because of the memory-less property of the exponential distribution, the exercise boundary is flat here. (4.2.18) equals

$$
\begin{aligned}
&Kr\mathrm{E}\left[\int_0^\infty I(t<\tau)e^{-rt}I(S(t)<\underline{S})\mathrm{d}t\right]-\delta\mathrm{E}\left[\int_0^\infty I(t<\tau)e^{-rt}S(t)I(S(t)<\underline{S})\mathrm{d}t\right]\\
&=Kr\mathrm{E}\left[\int_0^\infty e^{-\lambda t}e^{-rt}I(S(t)<\underline{S})\mathrm{d}t\right]-\delta\mathrm{E}\left[\int_0^\infty e^{-\lambda t}e^{-rt}S(t)I(S(t)<\underline{S})\mathrm{d}t\right]\\
&=\frac{Kr}{\lambda}\mathrm{E}\left[e^{-r\tau}I(S(\tau)<\underline{S})\right]-\frac{\delta}{\lambda}\mathrm{E}\left[e^{-r\tau}S(\tau)I(S(\tau)<\underline{S})\right]\\
&=\frac{Kr}{\lambda}\left(\frac{S(0)}{\underline{S}}\right)^{\alpha}\frac{\kappa}{-\alpha}-\frac{S(0)\delta}{\lambda}\left(\frac{S(0)}{\underline{S}}\right)^{\alpha-1}\frac{\kappa}{1-\alpha}
\end{aligned}
$$

The first equality is due to the independence of the expiry time and the stock price process. The third equality is obtained by writing the expected value as an integral with respect to the discounted density function of $X(\tau)$ (4.2.11). Following Gerber, Shiu and Yang (2012), if $S(0)>K$, the price of an out-of-the-money European put with an exponentially distributed expiration date equals

$$\mathrm{E}[e^{-r\tau}\left[K-S(\tau)\right]_+]=\left(\frac{S(0)}{K}\right)^{\alpha}\left(\frac{\kappa K}{-\alpha(1-\alpha)}\right)$$

If $S(0)<K$, we use put-call parity to generate the price of in-the-money case

$$\mathrm{E}[e^{-r\tau}\left[K-S(\tau)\right]_+]=K\frac{\lambda}{\lambda+r}-S(0)\frac{\lambda}{\lambda+\delta}+\left(\frac{S(0)}{K}\right)^{\beta}\left(\frac{\kappa K}{\beta(\beta-1)}\right)$$

Therefore, the value of an American put option with an exponentially distributed expiry date equals

$$\begin{cases} \left(\frac{S(0)}{K}\right)^{\alpha}\left(\frac{\kappa K}{-\alpha(1-\alpha)}\right) \\ +\left(\frac{S(0)}{\underline{S}}\right)^{\alpha} K\frac{r}{\lambda}\frac{\kappa}{-\alpha} - \frac{S(0)\delta}{\lambda}\left(\frac{S(0)}{\underline{S}}\right)^{\alpha-1}\frac{\kappa}{1-\alpha} & \text{if } K < S(0) \\ \\ K\frac{\lambda}{\lambda+r} - S(0)\frac{\lambda}{\lambda+\delta} + \left(\frac{S(0)}{K}\right)^{\beta}\left(\frac{\kappa K}{\beta(\beta-1)}\right) \\ +\left(\frac{S(0)}{\underline{S}}\right)^{\alpha} K\frac{r}{\lambda}\frac{\kappa}{-\alpha} - \frac{S(0)\delta}{\lambda}\left(\frac{S(0)}{\underline{S}}\right)^{\alpha-1}\frac{\kappa}{1-\alpha}, & \text{if } \underline{S} < S(0) \le K \\ \\ K - S(0) & \text{if } S(0) \le \underline{S} \end{cases} \tag{4.2.19}$$

To determine $\underline{S}$, we need to impose the continuity condition in (4.2.19) at the optimal exercise boundary $\underline{S}$, namely, the option value should be equal to the exercise value at the optimal exercise boundary $\underline{S}$. A financial interpretation of the necessity of the continuity of the price is provided by Carr, P., Jarrow, R., and Myneni, R. (1992). After purchasing the American put option, the investor would instantaneously exercise the option whenever the stock price falls to the optimal exercise price and purchase back the option whenever the stock price rises to the optimal exercise price. We require the high-contact condition in order to ensure that these transactions are self-financing. Therefore we have the following equation

$$K\frac{\lambda}{\lambda+r} - \underline{S}\frac{\lambda}{\lambda+\delta} + \left(\frac{\underline{S}}{K}\right)^{\beta}\left(\frac{\kappa K}{\beta(\beta-1)}\right) + K\frac{r}{\lambda}\frac{\kappa}{-\alpha} - \underline{S}\frac{\delta}{\lambda}\frac{\kappa}{1-\alpha} = K - \underline{S}$$

If the dividend rate equals to zero, the optimal exercise boundary $\underline{S}$ has an explicit expression which is the same as the result of the first method.

4.3 Exchange option

In this section, I generalize the above pricing methods to the case of the American exchange option with an exponentially distributed expiration date. In the remaining section, we assume the time t stock $i,\ i = 1, 2$ price is also modeled as

$$S_i(t) = S_i(0)e^{X_i(t)}, \qquad t \geq 0$$

and

$$X_i(t) = (r - \delta_i - \frac{1}{2}\sigma_i^2)t + \sigma_i B_i(t), \qquad t \geq 0$$

where r, δ_i, σ_i are the constant risk-free interest rate, the continuous dividend rate of the stock i and the volatility of the stock i, respectively. Here, $\{B_1(t),\ B_2(t)\}$ is a two dimensional Wiener process with its correlation coefficient $\mathrm{corr}(B_1(t),\ B_2(t)) = \rho$. For $\frac{S_1(0)}{S_2(0)} < \underset{\sim}{c}^*$, the value of an American exchange option with exponentially distributed expiry date can be represented as

$$\sup_{\underset{\sim}{c}^*} \mathrm{E}\left[e^{-r(T^{ratio}_{\underset{\sim}{c}^*} \wedge \tau)}[S_1(T^{ratio}_{\underset{\sim}{c}^*} \wedge \tau) - S_2(T^{ratio}_{\underset{\sim}{c}^*} \wedge \tau)]_+\right], \tag{4.3.1}$$

where $T^{ratio}_{\underset{\sim}{c}^*}$ is the first passage (hitting) time, which is defined as

$$\begin{aligned} T^{ratio}_{\underset{\sim}{c}^*} &= \inf\left\{t \geq 0 \middle| \frac{S_1(t)}{S_2(t)} = \underset{\sim}{c}^*\right\} \\ &= \inf\left\{t \geq 0 \middle| S_1(0)e^{X_1(t) - X_2(t)} \geq \underset{\sim}{c}^* S_2(0)\right\} \end{aligned} \tag{4.3.2}$$

The definition of $T^{ratio}_{\underset{\sim}{c}^*}$ indicates that the decision to exercise the option depends on the ratio of two stock prices. If we use c^* to denote the unknown optimal ratio, depending on whether the first passage time is earlier than the expiry time or not, (4.3.1) is decomposed as the following

$$\mathrm{E}\left[e^{-rT_{c^*}^{ratio}}[S_1(T_{c^*}^{ratio}) - S_2(T_{c^*}^{ratio})]_+ I(T_{c^*}^{ratio} < \tau)\right]$$
$$+\,\mathrm{E}\left[e^{-r\tau}[S_1(\tau) - S_2(\tau)]_+ I(T_{c^*}^{ratio} \geq \tau)\right] \tag{4.3.3}$$

We regard the first part of the above as a rebate option and the second part as a barrier option.

4.3.1 Perpetual exchange option

To evaluate the first part of (4.3.3), we used the martingale approach which has been used in Gerber and Shiu (1996a) to evaluate the price of a perpetual exchange option. The first part of (4.3.3) is

$$\mathrm{E}\left[e^{-rT_{c^*}^{ratio}}[S_1(T_{c^*}^{ratio}) - S_2(T_{c^*}^{ratio})]_+ I(T_{c^*}^{ratio} < \tau)\right] \tag{4.3.4}$$

where $T_{c^*}^{ratio}$ is defined according to (4.3.2). To evaluate (4.3.4), we consider the stochastic process $\left\{e^{-rt}S_1(t)^{\theta}S_2(t)^{1-\theta}I(t < \tau),\, t > 0\right\}$. The martingale condition is equivalent to choosing θ such that

$$\frac{\sigma^2}{2}\theta^2 + (\delta_2 - \delta_1 - \frac{\sigma^2}{2})\theta - (\delta_2 + \lambda) = 0 \tag{4.3.5}$$

where $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$. Let $\beta_2 > 0$ and $\alpha_2 < 0$ be the two roots of the equation (4.3.5). Since the initial ratio of the two stocks is smaller than c^*, we consider the martingale

$$\left\{e^{-rt}S_1(t)^{\beta_2}S_2(t)^{1-\beta_2}I(t < \tau),\, t > 0\right\}$$

Stopping the martingale at the finite stopping time $t \wedge T_{c^*}^{ratio}$ and applying the optional sampling theorem yields

$$\begin{aligned}
\left(\frac{S_1(0)}{S_2(0)}\right)^{\beta_2} S_2(0) &= \mathrm{E}\left[e^{-rT_{c^*}^{r}} S_1(T_{c^*}^{ratio})^{\beta_2} S_2(T_{c^*}^{ratio})^{1-\beta_2} I(T_{c^*}^{ratio} < \tau) I(T_{c^*}^{ratio} \le t)\right] \\
&+ \mathrm{E}\left[e^{-rt} S_1(t)^{\beta_2} S_2(t)^{1-\beta_2} I(t < \tau) I(t < T_{c^*}^{ratio})\right] \\
&= c^{*\beta_2} \mathrm{E}\left[e^{-rT_{c^*}^{r}} S_2(T_{c^*}^{ratio}) I(T_{c^*}^{ratio} < \tau) I(T_{c^*}^{ratio} \le t)\right] \\
&+ e^{-rt} \mathrm{E}\left[S_1(t)^{\beta_2} S_2(t)^{1-\beta_2} I(t < \tau) I(t < T_{c^*}^{ratio})\right]
\end{aligned}$$

If $T_{c^*}^{r} > t$, then $S_1(t) < c^* S_2(t)$. Since $\beta_2 > 0$, we have

$$\left(\frac{S_1(t)}{S_2(t)}\right)^{\beta_2} I(t < T_{c^*}^{ratio}) < c^{*\beta_2}$$

Hence

$$e^{-rt} \mathrm{E}\left[S_1(t)^{\beta_2} S_2(t)^{1-\beta_2} I(t < \tau) I(t < T_{c^*}^{ratio})\right]$$

is bounded by $e^{-rt} c^{*\beta_2} \mathrm{E}\left[S_2(t) I(t < \tau)\right]$, which tends to 0 as t tends to ∞. Therefore, as t tends to ∞, we have

$$\left(\frac{S_1(0)}{S_2(0)}\right)^{\beta_2} S_2(0) = c^{*\beta_2} \mathrm{E}\left[e^{-rT_{c^*}^{ratio}} S_2(T_{c^*}^{ratio}) I(T_{c^*}^{ratio} < \tau)\right]$$

It follows from the representation of (4.3.4) and the condition $\frac{S_1(T_{c^*}^{ratio})}{S_2(T_{c^*}^{ratio})} = c^*$that, for $\frac{S_1(0)}{S_2(0)} < c^*$

$$\mathrm{E}\left[e^{-rT_{c^*}^{ratio}} S_2(T_{c^*}^{ratio}) [\frac{S_1(T_{c^*}^{ratio})}{S_2(T_{c^*}^{ratio})} - 1]_+ I(T_{c^*}^{ratio} < \tau)\right] = \left(\frac{S_1(0)}{S_2(0)}\right)^{\beta_2} \frac{S_2(0)}{c^{*\beta_2}} (c^* - 1) \tag{4.3.6}$$

4.3.2 Exchange option with a slanted barrier

In this subsection, we consider the pricing problem of an exchange option with a slanted barrier, which is the second part of (4.3.3). We would like to calculate

$$\mathrm{E}\left[e^{-r\tau}[S_1(\tau) - S_2(\tau)]_+ I(T_{c^*}^{ratio} > \tau)\right] \tag{4.3.7}$$

The expectation (4.3.7) is equivalent to

$$\begin{aligned}
&\mathrm{E}\left[e^{-r\tau} S_2(\tau)[\frac{S_1(\tau)}{S_2(\tau)} - 1]_+ I(T_{c^*}^{ratio} > \tau)\right] \\
=&\int_0^\infty e^{-rt}\mathrm{E}\left[S_2(t)[\frac{S_1(t)}{S_2(t)} - 1]_+ I(T_{c^*}^{ratio} > t)\right] f_\tau(t)\mathrm{d}t
\end{aligned}$$

Since

$$\begin{aligned}
&\mathrm{E}\left[S_2(t)[\frac{S_1(t)}{S_2(t)} - 1]_+ I(T_{c^*}^{ratio} > t)\right] \\
=&\mathrm{E}\left[S_2(t)\right] \times \mathrm{E}\left[[\frac{S_1(t)}{S_2(t)} - 1]_+ I(T_{c^*}^{ratio} > t); \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]
\end{aligned}$$

and

$$\mathrm{E}\left[S_2(t)\right] = S_2(0)e^{(r-\delta_2)t}$$

The expectation (4.3.7) is equal to

$$\mathrm{E}\left[e^{-\delta_2\tau}[S_1(0)e^{X_1(\tau)-X_2(\tau)} - S_2(0)]_+ I(T_{c^*}^{ratio} > \tau); \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \tag{4.3.8}$$

The above can be thought as the value of an up-and-out call option for asset $S_1(0)e^{X_1(t)-X_2(t)}$ with strike price $S_2(0)$ and the force of interest δ_2 .

Since

$$\mathrm{Var}\left[X_1(t) - X_2(t); \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2,$$

which we write as σ^2, and

$$\mathrm{E}\left[X_1(t) - X_2(t); \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \delta_2 - \delta_1 - \frac{1}{2}\sigma^2$$

Following (A.3) in Gerber, Shiu and Yang (2012), if $S_1(0) < S_2(0)$, (4.3.8) equals to

$$\begin{aligned} &\left(\frac{S_1(0)}{S_2(0)}\right)^{\beta_2} \left(\frac{\kappa_2 S_2(0)}{\beta_2(\beta_2 - 1)}\right) \\ &+ \left(\frac{\lambda}{\lambda + \delta_2} S_2(0) - \frac{\lambda}{\lambda + \delta_1} c^* S_2(0) - \frac{\kappa_2 S_2(0)^{(1-\alpha_2)} \left(c^* S_2(0)\right)^{\alpha_2}}{\alpha_2(\alpha_2 - 1)}\right) \left(\frac{S_1(0)}{c^* S_2(0)}\right)^{\beta_2} \end{aligned} \tag{4.3.9}$$

If $1 \leq \frac{S_1(0)}{S_2(0)} < c^*$, (4.3.8) equals to

$$\begin{aligned} &S_1(0)\frac{\lambda}{\lambda + \delta_1} - S_2(0)\frac{\lambda}{\lambda + \delta_2} + \left(\frac{S_1(0)}{S_2(0)}\right)^{\alpha_2} \left(\frac{\kappa_2 S_2(0)}{\alpha_2(\alpha_2 - 1)}\right) \\ &+ \left(\frac{\lambda}{\lambda + \delta_2} S_2(0) - \frac{\lambda}{\lambda + \delta_1} c^* S_2(0) - \frac{\kappa_2 S_2(0)^{(1-\alpha_2)} \left(c^* S_2(0)\right)^{\alpha_2}}{\alpha_2(\alpha_2 - 1)}\right) \left(\frac{S_1(0)}{c^* S_2(0)}\right)^{\beta_2} \end{aligned} \tag{4.3.10}$$

where $\kappa_2 = \frac{\lambda}{\frac{1}{2}\sigma^2(\beta_2 - \alpha_2)}$.

Remark 4.3.1. The first part of (4.3.9) is the price of an out-of-the-money option to exchange S_2 for S_1 at the exponential distributed expiry date, which means if $S_1(0) \leq S_2(0)$,

$$\mathrm{E}\left[e^{-r\tau}[S_1(\tau) - S_2(\tau)]_+\right] = \left(\frac{S_1(0)}{S_2(0)}\right)^{\beta_2} \left(\frac{\kappa_2 S_2(0)}{\beta_2(\beta_2 - 1)}\right) \tag{4.3.11}$$

Remark 4.3.2. The first line of (4.3.10) is the price of an in-the-money option to exchange S_2 for S_1 at the exponential distributed expiry date. To evaluate it, we first calculate the price of an out-of-the-money option to exchange S_1 for S_2. Due to the symmetry, if $S_2(0) \leq S_1(0)$, we have

$$\mathrm{E}\left[e^{-r\tau}[S_2(\tau) - S_1(\tau)]_+\right] = \left(\frac{S_2(0)}{S_1(0)}\right)^{\beta_1} \left(\frac{\kappa_1 S_1(0)}{\beta_1(\beta_1 - 1)}\right)$$

where $\alpha_1 < 0$, $\beta_1 > 0$ are the two roots of the equation (4.3.12), and $\kappa_1 = \frac{\lambda}{\frac{1}{2}\sigma^2(\beta_1 - \alpha_1)}$

$$\frac{\sigma^2}{2}\theta^2 + (\delta_1 - \delta_2 - \frac{\sigma^2}{2})\theta - (\delta_1 + \lambda) = 0 \tag{4.3.12}$$

Applying put-call parity, if $S_2(0) < S_1(0)$, the price of an in-the-money option to exchange S_2 for S_1 equals

$$\begin{aligned} &\mathrm{E}\left[e^{-r\tau}[S_1(\tau) - S_2(\tau)]_+\right] \\ &= S_1(0)\frac{\lambda}{\lambda + \delta_1} - S_2(0)\frac{\lambda}{\lambda + \delta_2} + \left(\frac{S_2(0)}{S_1(0)}\right)^{\beta_1} \left(\frac{\kappa_1 S_1(0)}{\beta_1(\beta_1 - 1)}\right) \end{aligned} \tag{4.3.13}$$

Remark 4.3.3. We have the following relationship for the roots of the equations (4.3.5) and (4.3.12),

$$\alpha_1 + \beta_2 = 1 \qquad\qquad \beta_1 + \alpha_2 = 1$$

Following the above, we have$\kappa_1 = \kappa_2$. Therefore, another representation of the price of an out-of-the-money option to exchange S_1 for S_2 is

$$\mathrm{E}\left[e^{-r\tau}[S_2(\tau) - S_1(\tau)]_+\right] = \left(\frac{S_1(0)}{S_2(0)}\right)^{\alpha_2} \left(\frac{\kappa_2 S_2(0)}{\alpha_2(\alpha_2 - 1)}\right)$$

4.3.3 First methodology

From (4.3.3), the value of an American exchange option with exponentially distributed expiry date equals to the sum of the value of a rebate option and a barrier exchange option. According to the calculation in the previous two subsections, if

$S_1(0) < S_2(0)$, the value of an American option to exchange S_2 for S_1 at the exponential distributed expiry date is

$$\left(\frac{S_1(0)}{S_2(0)}\right)^{\beta_2}\left(\frac{\kappa_2 S_2(0)}{\beta_2(\beta_2-1)}\right)$$
$$-\left(\frac{\lambda}{\lambda+\delta_1}c^* - \frac{\lambda}{\lambda+\delta_2} + \frac{\kappa_2 c^{*\alpha_2}}{\alpha_2(\alpha_2-1)} - (c^*-1)\right)\left(\frac{S_1(0)}{c^* S_2(0)}\right)^{\beta_2} S_2(0)$$

if $1 \leq \frac{S_1(0)}{S_2(0)} < c^*$, it equals

$$S_1(0)\frac{\lambda}{\lambda+\delta_1} - S_2(0)\frac{\lambda}{\lambda+\delta_2} + \left(\frac{S_1(0)}{S_2(0)}\right)^{\alpha_2}\left(\frac{\kappa_2 S_2(0)}{\alpha_2(\alpha_2-1)}\right)$$
$$-\left(\frac{\lambda}{\lambda+\delta_1}c^* - \frac{\lambda}{\lambda+\delta_2} + \frac{\kappa_2 c^{*\alpha_2}}{\alpha_2(\alpha_2-1)} - (c^*-1)\right)\left(\frac{S_1(0)}{c^* S_2(0)}\right)^{\beta_2} S_2(0)$$

For the case of $\frac{S_1(0)}{S_2(0)} \geq c^*$, the option should be optimally exercised. The exercise value equals $S_1(0) - S_2(0)$. To determine the optimal exercise ratio c^*, we select c^* to maximize the following

$$-\left(\frac{\lambda}{\lambda+\delta_1}c^* - \frac{\lambda}{\lambda+\delta_2} + \frac{\kappa_2 c^{*\alpha_2}}{\alpha_2(\alpha_2-1)} - (c^*-1)\right)\left(\frac{S_1(0)}{c^* S_2(0)}\right)^{\beta_2} S_2(0)$$

Taking derivative with respect to c^* and setting it equal to zero, we have

$$\frac{\delta_2\beta_2}{\lambda+\delta_2}c^{*-\beta_2} + \frac{\delta_1\alpha_1}{\lambda+\delta_1}c^{*\alpha_1} + \frac{\lambda}{\lambda+\delta_2}\frac{\beta_2}{\beta_1}c^{*(\alpha_1-\beta_1)} = 0 \tag{4.3.14}$$

The value of c^* can be numerically calculated.

4.3.4 Second methodology

Similar to the put option, the second approach for determining the value of an American exchange option is to decompose its value into the sum of the price of a European exchange option and the early exercise premium. The value of a European exchange option has been studied in (4.3.11) and (4.3.13). We now calculate the early exercise premium. To compensate the option holders who lose the right of early exercise, cash flow $[\delta_1 S_1(t) - S_2(t)\delta_2]dt$ should be paid when the option would have been exercised optimally. The exercise condition is the following: the ratio of the two stocks is larger than the optimal exercise boundary c^*. Therefore, the early exercise premium has the following representation

$$\delta_1 \mathrm{E}\int_0^\tau e^{-rt} S_1(t) I(\frac{S_1(t)}{S_2(t)} > c^*)\mathrm{d}t - \delta_2 \mathrm{E}\int_0^\tau e^{-rt} S_2(t) I(\frac{S_1(t)}{S_2(t)} > c^*)\mathrm{d}t \qquad (4.3.15)$$

The above equals

$$\begin{aligned}
&\frac{\delta_1}{\lambda}\mathrm{E}\left[e^{-r\tau}S_1(\tau)I(\frac{S_1(\tau)}{S_2(\tau)} > c^*);\right] - \frac{\delta_2}{\lambda}\mathrm{E}\left[e^{-r\tau}S_2(\tau)I(\frac{S_1(\tau)}{S_2(\tau)} > c^*)\right] \\
&= \frac{\delta_1}{\lambda}\int_0^\infty \lambda e^{-\lambda t}\mathrm{E}\left[e^{-rt}S_1(t)\right] \times \mathrm{E}\left[I(\frac{S_1(t)}{S_2(t)} > c^*); \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] \mathrm{d}t \\
&\quad - \frac{\delta_2}{\lambda}\int_0^\infty \lambda e^{-\lambda t}\mathrm{E}\left[e^{-rt}S_2(t)\right] \times \mathrm{E}\left[I(\frac{S_1(t)}{S_2(t)} > c^*); \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \mathrm{d}t \\
&= \frac{\delta_1}{\lambda}\mathrm{E}\left[e^{-\delta_1\tau}S_1(0)I(\frac{S_1(\tau)}{S_2(\tau)} > c^*); \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] - \frac{\delta_2}{\lambda}\mathrm{E}\left[e^{-\delta_2\tau}S_2(0)I(\frac{S_1(\tau)}{S_2(\tau)} > c^*); \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]
\end{aligned}$$

If $\frac{S_1(0)}{S_2(0)} < c^*$, the above equals

$$\frac{\delta_1}{\lambda}S_1(0)\left(\frac{S_1(0)}{S_2(0)c^*}\right)^{-\alpha_1}\frac{\kappa_1}{-\alpha_1} - \frac{\delta_2}{\lambda}S_2(0)\left(\frac{S_1(0)}{S_2(0)c^*}\right)^{\beta_2}\frac{\kappa_2}{\beta_2}$$

Therefore, the formula for the price of an American option to exchange S_2 for S_1 equals

$$\begin{cases} \left(\frac{S_1(0)}{S_2(0)}\right)^{\beta_2}\left(\frac{\kappa_2 S_2(0)}{\beta_2(\beta_2-1)}\right) \\ +\frac{\delta_1}{\lambda}S_1(0)\left(\frac{S_1(0)}{S_2(0)c^*}\right)^{-\alpha_1}\frac{\kappa_1}{-\alpha_1}-\frac{\delta_2}{\lambda}S_2(0)\left(\frac{S_1(0)}{S_2(0)c^*}\right)^{\beta_2}\frac{\kappa_2}{\beta_2} & \text{if } \frac{S_1(0)}{S_2(0)}<1 \\ \\ S_1(0)\frac{\lambda}{\lambda+\delta_1}-S_2(0)\frac{\lambda}{\lambda+\delta_2}+\left(\frac{S_2(0)}{S_1(0)}\right)^{\beta_1}\left(\frac{\kappa_1 S_1(0)}{\beta_1(\beta_1-1)}\right) \\ +\frac{\delta_1}{\lambda}S_1(0)\left(\frac{S_1(0)}{S_2(0)c^*}\right)^{-\alpha_1}\frac{\kappa_1}{-\alpha_1}-\frac{\delta_2}{\lambda}S_2(0)\left(\frac{S_1(0)}{S_2(0)c^*}\right)^{\beta_2}\frac{\kappa_2}{\beta_2} & \text{if } 1\le\frac{S_1(0)}{S_2(0)}<c^* \\ \\ S_1(0)-S_2(0) & \text{if } c^*\le\frac{S_1(0)}{S_2(0)} \end{cases} \tag{4.3.16}$$

Similar to the case of put option, to determine optimal exercise boundary c^*, which is a ratio of two stocks, we impose the continuity condition in (4.3.16) at the optimal exercise boundary c^*, which means the value of American exchange option equal to its exercise value at the optimal exercise boundary c^*. We can numerically solve the following equation for c^*

$$c^*\frac{\lambda}{\lambda+\delta_1}-\frac{\lambda}{\lambda+\delta_2}+\frac{1}{c^{*\beta_1}}\left(\frac{\kappa_1 c^*}{\beta_1(\beta_1-1)}\right)+\frac{\delta_1}{\lambda}c^*\frac{\kappa_1}{-\alpha_1}-\frac{\delta_2}{\lambda}\frac{\kappa_2}{\beta_2}=c^*-1$$

The above can be simplified as

$$c^{*\alpha_1}\frac{\delta_1\alpha_1}{\lambda+\delta_1}+\frac{\delta_2\beta_2}{\lambda+\delta_2}c^{*-\beta_2}+\frac{\lambda}{\lambda+\delta_1}\frac{\alpha_1}{\alpha_2}c^{*(\alpha_1-\beta_1)}=0$$

which is equivalent to equation (4.3.14).

4.4 Maximum option

4.4.1 First methodology

In this section, I consider the valuation problem of American maximum options with exponentially distributed expiry date. Similar to the examples of American put options and American exchange options, it can also be priced by two methodologies. For $0 < \utilde{b}^* < \frac{S_1(0)}{S_2(0)} < \utilde{c}^*$, the value of an American exchange option with exponentially distributed expiry date can be represented as

$$\sup_{\utilde{c}^*,\utilde{b}^*} \mathrm{E}\left[e^{-r(T^{ratio}_{\utilde{c}^*,\utilde{b}^*}\wedge\tau)}\max[S_1(T^{ratio}_{\utilde{c}^*,\utilde{b}^*}\wedge\tau),\ S_2(T^{ratio}_{\utilde{c}^*,\utilde{b}^*}\wedge\tau)]\right], \tag{4.4.1}$$

where $T^{ratio}_{\utilde{c}^*,\utilde{b}^*}$ is defined as

$$T^{ratio}_{\utilde{c}^*,\utilde{b}^*} = \inf\left\{t \geq 0 \left| \frac{S_1(t)}{S_2(t)} = \utilde{c}^* \text{or} \frac{S_1(t)}{S_2(t)} = \utilde{b}^*\right.\right\} \tag{4.4.2}$$

Here, it is sufficient to consider $0 < \utilde{b}^* \leq 1 \leq \utilde{c}^*$. The option holder will exercise the option if the ratio of two stocks is either higher than $\utilde{c}^*$ or lower than $\utilde{b}^*$. If we use c^* and b^* to denote the optimal ratios to make (4.4.1) maximized, depending on whether the first passage time $T^{ratio}_{c^*,b^*}$ is earlier than the expiry date τ or not, the first methodology calculates the price of the maximum option as the following

$$\begin{aligned}&\mathrm{E}\left[e^{-rT^{ratio}_{c^*,b^*}}\max[S_1(T^{ratio}_{c^*,b^*}),\ S_2(T^{ratio}_{c^*,b^*})]I(T^{ratio}_{c^*,b^*}\leq\tau)\right]\\&+\mathrm{E}\left[e^{-r\tau}\max[S_1(\tau),\ S_2(\tau)]I(T^{ratio}_{c^*,b^*}>\tau)\right]\end{aligned} \tag{4.4.3}$$

The first expectation of (4.4.3) can be written as,

$$\begin{aligned}\mathrm{E}\Bigg\{&e^{-rT^{ratio}_{c^*,b^*}}\max[S_1(T^{ratio}_{c^*,b^*}),\ S_2(T^{ratio}_{c^*,b^*})]I(T^{ratio}_{c^*,b^*}\leq\tau)\\&\times\left[I(\frac{S_1(T^{ratio}_{c^*,b^*})}{S_2(T^{ratio}_{c^*,b^*})}=c^*)+I(\frac{S_1(T^{ratio}_{c^*,b^*})}{S_2(T^{ratio}_{c^*,b^*})}=b^*)\right]\Bigg\}\end{aligned} \tag{4.4.4}$$

With the definitions

$$\gamma = \mathrm{E}\left[e^{-rT^{ratio}_{c^*,b^*}} S_2(T^{ratio}_{c^*,b^*}) I(T^{ratio}_{c^*,b^*} \le \tau) I(\frac{S_1(T^{ratio}_{c^*,b^*})}{S_2(T^{ratio}_{c^*,b^*})} = c^*)\right]$$

and

$$\eta = \mathrm{E}\left[e^{-rT^{ratio}_{c^*,b^*}} S_2(T^{ratio}_{c^*,b^*}) I(T^{ratio}_{c^*,b^*} \le \tau) I(\frac{S_1(T^{ratio}_{c^*,b^*})}{S_2(T^{ratio}_{c^*,b^*})} = b^*)\right]$$

we can rewrite the (4.4.4) as

$$\gamma c^* + \eta = \begin{pmatrix} c^* & 1 \end{pmatrix} \begin{pmatrix} \gamma \\ \eta \end{pmatrix}$$

To determine the expectation γ and η, we apply the similar argument in section (4.3.1) to the following two martingales

$$\left\{e^{-rt} S_1(t)^{\beta_2} S_2(t)^{1-\beta_2} I(t < \tau),\, t > 0\right\}$$

and

$$\left\{e^{-rt} S_1(t)^{\alpha_2} S_2(t)^{1-\alpha_2} I(t < \tau),\, t > 0\right\}$$

Applying the optional sampling theorem, if $b^* < \frac{S_1(0)}{S_2(0)} < c^*$, we obtain the equations

$$\left(\frac{S_1(0)}{S_2(0)}\right)^{\beta_2} S_2(0) = \gamma c^{*\beta_2} + \eta b^{*\beta_2}$$

and

$$\left(\frac{S_1(0)}{S_2(0)}\right)^{\alpha_2} S_2(0) = \gamma c^{*\alpha_2} + \eta b^{*\alpha_2}$$

whose solution is

$$\begin{pmatrix} \gamma \\ \eta \end{pmatrix} = \begin{pmatrix} c^{*\beta_2} & b^{*\beta_2} \\ c^{*\alpha_2} & b^{*\alpha_2} \end{pmatrix}^{-1} \begin{pmatrix} \left(\frac{S_1(0)}{S_2(0)}\right)^{\beta_2} S_2(0) \\ \left(\frac{S_1(0)}{S_2(0)}\right)^{\alpha_2} S_2(0) \end{pmatrix}$$

Therefore (4.4.4) equals

$$\begin{pmatrix} c^* & 1 \end{pmatrix} \begin{pmatrix} c^{*\beta_2} & b^{*\beta_2} \\ c^{*\alpha_2} & b^{*\alpha_2} \end{pmatrix}^{-1} \begin{pmatrix} \left(\frac{S_1(0)}{S_2(0)}\right)^{\beta_2} S_2(0) \\ \left(\frac{S_1(0)}{S_2(0)}\right)^{\alpha_2} S_2(0) \end{pmatrix} \tag{4.4.5}$$

The second expectation of (4.4.3)

$$\mathrm{E}\left[e^{-r\tau}\max[S_1(\tau),\ S_2(\tau)] I(T^{ratio}_{c^*,b^*} > \tau)\right] \tag{4.4.6}$$

can be written as

$$\begin{aligned} &\mathrm{E}\left[e^{-r\tau}\max[S_1(\tau),\ S_2(\tau)]\right] \\ &- \mathrm{E}\left\{e^{-r\tau}\max[S_1(\tau),\ S_2(\tau)] I(T^{ratio}_{c^*,b^*} \le \tau)\left[I(\frac{S_1(T^{ratio}_{c^*,b^*})}{S_2(T^{ratio}_{c^*,b^*})} = c^*) + I(\frac{S_1(T^{ratio}_{c^*,b^*})}{S_2(T^{ratio}_{c^*,b^*})} = b^*)\right]\right\} \end{aligned} \tag{4.4.7}$$

Motivated by (4.2.14), the second expectation of (4.4.7) can be written as

$$\begin{aligned} &\mathrm{E}_{T^{ratio}_{c^*,b^*}}[e^{-r\left(\tau - T^{ratio}_{c^*,b^*}\right)}\frac{1}{S_2(0)}\max[S_1(\tau),\ S_2(\tau)] | \frac{S_1(T^{ratio}_{c^*,b^*})}{S_2(T^{ratio}_{c^*,b^*})} = c^*] \\ &\times \mathrm{E}[e^{-rT^{ratio}_{c^*,b^*}} S_2(T^{ratio}_{c^*,b^*}) I(T^{ratio}_{c^*,b^*} < \tau) I(\frac{S_1(T^{ratio}_{c^*,b^*})}{S_2(T^{ratio}_{c^*,b^*})} = c^*)] \\ &+ \mathrm{E}_{T^{ratio}_{c^*,b^*}}[e^{-r\left(\tau - T^{ratio}_{c^*,b^*}\right)}\frac{1}{S_2(0)}\max[S_1(\tau),\ S_2(\tau)] | \frac{S_1(T^{ratio}_{c^*,b^*})}{S_2(T^{ratio}_{c^*,b^*})} = b^*] \\ &\times \mathrm{E}[e^{-rT^{ratio}_{c^*,b^*}} S_2(T^{ratio}_{c^*,b^*}) I(T^{ratio}_{c^*,b^*} < \tau) I(\frac{S_1(T^{ratio}_{c^*,b^*})}{S_2(T^{ratio}_{c^*,b^*})} = b^*)] \end{aligned}$$

Due to the memoryless property of the exponential distribution and the strong Markov property of the stock process, the above equals

$$\begin{pmatrix} \mathrm{E}[e^{-r\tau}\frac{1}{S_2(0)}\max[S_1(\tau),\, S_2(\tau)]|\frac{S_1(0)}{S_2(0)}=c^*] \\ \mathrm{E}[e^{-r\tau}\frac{1}{S_2(0)}\max[S_1(\tau),\, S_2(\tau)]|\frac{S_1(0)}{S_2(0)}=b^*] \end{pmatrix}' \times \begin{pmatrix} c^{*\beta_2} & b^{*\beta_2} \\ c^{*\alpha_2} & b^{*\alpha_2} \end{pmatrix}^{-1} \begin{pmatrix} \left(\frac{S_1(0)}{S_2(0)}\right)^{\beta_2} S_2(0) \\ \left(\frac{S_1(0)}{S_2(0)}\right)^{\alpha_2} S_2(0) \end{pmatrix} \tag{4.4.8}$$

Substituting the price of European maximum option with exponential distributed expiry date, (4.4.8) equals

$$\begin{pmatrix} c^{*\alpha_2}\left(\frac{\kappa_1}{-\beta_1\alpha_2}\right)+\frac{\lambda c^*}{\lambda+\delta_1} \\ b^{*\beta_2}\left(\frac{\kappa_2}{-\beta_2\alpha_1}\right)+\frac{\lambda}{\lambda+\delta_2} \end{pmatrix}' \begin{pmatrix} c^{*\beta_2} & b^{*\beta_2} \\ c^{*\alpha_2} & b^{*\alpha_2} \end{pmatrix}^{-1} \begin{pmatrix} \left(\frac{S_1(0)}{S_2(0)}\right)^{\beta_2} S_2(0) \\ \left(\frac{S_1(0)}{S_2(0)}\right)^{\alpha_2} S_2(0) \end{pmatrix} \tag{4.4.9}$$

To determine the optimal value of c^* and b^*, we need to maximize the following which is the difference of (4.4.5) and (4.4.9)

$$\begin{pmatrix} c^{*\alpha_2}\left(\frac{\kappa_1}{\beta_1\alpha_2}\right)+\frac{\delta_1 c^*}{\lambda+\delta_1} \\ b^{*\beta_2}\left(\frac{\kappa_2}{\beta_2\alpha_1}\right)+\frac{\delta_2}{\lambda+\delta_2} \end{pmatrix}' \begin{pmatrix} c^{*\beta_2} & b^{*\beta_2} \\ c^{*\alpha_2} & b^{*\alpha_2} \end{pmatrix}^{-1} \begin{pmatrix} \left(\frac{S_1(0)}{S_2(0)}\right)^{\beta_2} S_2(0) \\ \left(\frac{S_1(0)}{S_2(0)}\right)^{\alpha_2} S_2(0) \end{pmatrix}$$

Taking derivative with respect to c^* and b^* and setting them equal to zero, we obtain

$$\frac{\beta_1\delta_1 c^*}{(\lambda+\delta_1)}\left(\frac{b^*}{c^*}\right)^{\beta_2}+\frac{\lambda}{\lambda+\delta_2}\frac{\alpha_2}{\alpha_1}b^{*\beta_2}=-\frac{\alpha_2\delta_2}{\lambda+\delta_2} \tag{4.4.10}$$

and

$$\frac{\lambda}{\lambda+\delta_2}\frac{\beta_2}{\beta_1}\left(\frac{1}{c^*}\right)^{\beta_1}+\frac{\beta_2\delta_2}{\lambda+\delta_2}\frac{1}{c^*}\left(\frac{b^*}{c^*}\right)^{-\alpha_2}=-\frac{\alpha_1\delta_1}{\lambda+\delta_1} \tag{4.4.11}$$

The optimal exercise ratio c^* and b^* can be numerically calculated.

4.4.2 Second methodology

Similar to the case of the put options and the exchange options, the price of an American maximum option has another representation. The price can be calculated as the sum of the price of a European maximum option and the early exercise premium. The early exercise premium for a maximum option equals

$$\delta_1 \mathrm{E} \int_0^\tau e^{-rt} S_1(t) I(\frac{S_1(t)}{S_2(t)} > c^*) dt + \delta_2 \mathrm{E} \int_0^\tau e^{-rt} S_2(t) I(\frac{S_1(t)}{S_2(t)} < b^*) dt$$
$$= \frac{\delta_1}{\lambda} \mathrm{E} \left[e^{-\delta_1 \tau} S_1(0) I(\frac{S_1(\tau)}{S_2(\tau)} > c^*); \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] + \frac{\delta_2}{\lambda} \mathrm{E} \left[e^{-\delta_2 \tau} S_2(0) I(\frac{S_1(\tau)}{S_2(\tau)} < b^*); \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

If the ratio of the two stocks is between b^* and c^*, $b^* < \frac{S_1(0)}{S_2(0)} \leq c^*$, the above equals

$$\frac{\delta_1}{\lambda} S_1(0) \left(\frac{S_1(0)}{S_2(0) c^*} \right)^{-\alpha_1} \frac{\kappa_1}{-\alpha_1} - \frac{\delta_2}{\lambda} S_2(0) \left(\frac{S_1(0)}{S_2(0) b^*} \right)^{\alpha_2} \frac{\kappa_2}{\alpha_2} \tag{4.4.12}$$

If $S_1(0) < S_2(0)$, the price of a European maximum option with an exponentially distributed expiration date equals

$$\begin{aligned} \mathrm{E}\left[e^{-r\tau} \max\left[S_1(\tau),\, S_2(\tau) \right] \right] &= \mathrm{E}\left[e^{-r\tau} \max\left[S_1(\tau) - S_2(\tau),\, 0 \right] \right] + \mathrm{E}\left[e^{-r\tau} S_2(\tau) \right] \\ &= \left(\frac{S_1(0)}{S_2(0)} \right)^{\beta_2} \left(\frac{\kappa_2 S_2(0)}{\beta_2(\beta_2 - 1)} \right) + S_2(0) \frac{\lambda}{\lambda + \delta_2} \end{aligned} \tag{4.4.13}$$

If $S_2(0) < S_1(0)$, it equals

$$\begin{aligned} \mathrm{E}\left[e^{-r\tau} \max\left[S_1(\tau),\, S_2(\tau) \right] \right] &= \mathrm{E}\left[e^{-r\tau} \max\left[S_2(\tau) - S_1(\tau),\, 0 \right] \right] + \mathrm{E}\left[e^{-r\tau} S_1(\tau) \right] \\ &= \left(\frac{S_2(0)}{S_1(0)} \right)^{\beta_1} \left(\frac{\kappa_1 S_1(0)}{\beta_1(\beta_1 - 1)} \right) + S_1(0) \frac{\lambda}{\lambda + \delta_1} \end{aligned} \tag{4.4.14}$$

When the ratio of the two stocks is lower than b^* or higher than c^*, the option should be optimally exercised and the option holders will be paid either $S_2(0)$ or $S_1(0)$. Therefore, we have the following formula for the price of an American maximum option with exponentially distributed expiry date

$$\begin{cases} S_2(0) & \text{if } \frac{S_1(0)}{S_2(0)} \le b^* \\ \left(\frac{S_1(0)}{S_2(0)}\right)^{\beta_2} \left(\frac{\kappa_2 S_2(0)}{\beta_2(\beta_2-1)}\right) + S_2(0)\frac{\lambda}{\lambda+\delta_2} & \\ + \frac{\delta_1}{\lambda} S_1(0) \left(\frac{S_1(0)}{S_2(0)c^*}\right)^{-\alpha_1} \frac{\kappa_1}{-\alpha_1} - \frac{\delta_2}{\lambda} S_2(0) \left(\frac{S_1(0)}{S_2(0)b^*}\right)^{\alpha_2} \frac{\kappa_2}{\alpha_2} & \text{if } b^* < \frac{S_1(0)}{S_2(0)} \le 1 \\ \left(\frac{S_2(0)}{S_1(0)}\right)^{\beta_1} \left(\frac{\kappa_1 S_1(0)}{\beta_1(\beta_1-1)}\right) + S_1(0)\frac{\lambda}{\lambda+\delta_1} & \\ + \frac{\delta_1}{\lambda} S_1(0) \left(\frac{S_1(0)}{S_2(0)c^*}\right)^{-\alpha_1} \frac{\kappa_1}{-\alpha_1} - \frac{\delta_2}{\lambda} S_2(0) \left(\frac{S_1(0)}{S_2(0)b^*}\right)^{\alpha_2} \frac{\kappa_2}{\alpha_2} & \text{if } 1 < \frac{S_1(0)}{S_2(0)} \le c^* \\ S_1(0) & \text{if } \frac{S_1(0)}{S_2(0)} > c^* \end{cases}$$

To determine the optimal exercise ratio c^* and b^*, we impose the continuity condition at c^* and b^* and have the following two equations,

$$b^{*\beta_2} \left(\frac{\kappa_2}{\beta_2(\beta_2-1)}\right) + \frac{\lambda}{\lambda+\delta_2} + \frac{\delta_1}{\lambda} b^* \left(\frac{b^*}{c^*}\right)^{-\alpha_1} \frac{\kappa_1}{-\alpha_1} - \frac{\delta_2}{\lambda}\frac{\kappa_2}{\alpha_2} = 1 \tag{4.4.15}$$

and

$$c^{*-\beta_1} \left(\frac{\kappa_1}{\beta_1(\beta_1-1)}\right) + \frac{\lambda}{\lambda+\delta_1} + \frac{\delta_1}{\lambda}\frac{\kappa_1}{-\alpha_1} - \frac{\delta_2}{\lambda}\frac{1}{c^*} \left(\frac{c^*}{b^*}\right)^{\alpha_2} \frac{\kappa_2}{\alpha_2} = 1 \tag{4.4.16}$$

Through algebra calculation, we can verify equations (4.4.15) and (4.4.16) are equivalent to (4.4.10) and (4.4.11).

4.5 Appendix

4.5.1 Derivation of (4.2.10)

In Gerber, Shiu and Yang (2012), they provide a derivation for the discounted density function $f^r_{X(\tau),m(\tau)}(x, y)$. Here we give an alternative derivation.

To simplify the derivation, we assume $X(t)$ has the following density function

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi t}\sigma} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}$$

We first use the reflection principle to derive the joint probability density function $f_{X(t),\, M(t)}(x,\, y)$[1]

$$\begin{aligned}
f_{X(t),\, M(t)}(x,\, y) &= -\frac{\partial^2}{\partial y \partial x} \Pr\left[X(t) \leq x \,\&\, M(t) > y\right] \\
&= -\frac{\partial^2}{\partial y \partial x} e^{Ry} \Pr\left[X(t) \leq x - 2y\right] \\
&= -\frac{\partial}{\partial y} e^{Ry} f_{X(t)}(x - 2y) \\
&= -e^{Ry} \left(R + \frac{\partial}{\partial y}\right) f_{X(t)}(x - 2y)
\end{aligned}$$

Through algebra calculation, the joint probability density function $f_{X(t),\, M(t)}(x,\, y)$ equals

$$f_{X(t),\, M(t)}(x,\, y) = \frac{2(2y - x)}{\sigma^3 \sqrt{2\pi t^3}} \exp\left\{ \frac{\mu x - \frac{1}{2}\mu^2 t - \frac{(2y-x)^2}{2t}}{\sigma^2} \right\}$$

Now we consider the discounted density function of $X(\tau)$ and $M(\tau)$, which is defined as

$$f^r_{X(\tau),\, M(\tau)}(x,\, y) = \int_0^\infty e^{-rt} f_{X(t),\, M(t)}(x,\, y) f_\tau(t) \mathrm{d}t, \qquad y \geq \max(x,\, 0) \tag{4.5.2}$$

[1]The last equality is because of the *exponential shift formula*

$$\frac{\partial}{\partial z}[e^{az} g(z)] = a e^{az} g(z) + e^{az} \frac{\partial}{\partial z} g(z) = e^{az}(a + \frac{\partial}{\partial z}) g(z) \tag{4.5.1}$$

It can also be derived by using the product rule.

Substituting the joint probability density function $f_{X(t),\,M(t)}(x,\,y)$ and the density function of τ into (4.5.2),

$$
\begin{aligned}
& f^r_{X(\tau),\,M(\tau)}(x,\,y) \\
&= \int_0^\infty e^{-rt}\frac{2(2y-x)}{\sigma^3\sqrt{2\pi t^3}}\exp\left\{\frac{\mu x-\frac{1}{2}\mu^2 t-\frac{(2y-x)^2}{2t}}{\sigma^2}\right\}\lambda e^{\lambda t}\mathrm{d}t \\
&= \frac{2\lambda}{\sigma^2}\exp(\frac{\mu x}{\sigma^2})\times\int_0^\infty \exp\left\{-\frac{\frac{1}{2}\mu^2+\sigma^2\,(r-\lambda)}{\sigma^2}t\right\}\frac{(2y-x)}{\sigma\sqrt{2\pi t^3}}\exp\left\{\frac{-(2y-x)^2}{2t\sigma^2}\right\}\mathrm{d}t
\end{aligned}
$$

Because of the identity

$$
\int_0^\infty e^{-\zeta t}\frac{ae^{-\frac{a^2}{2t}}}{\sqrt{2\pi t^3}}dt = e^{-a\sqrt{2\zeta}}
$$

which is the Laplace transform of the probability density function for the first passage time of a standard Brownian motion at the level a, the discounted joint density function $f^r_{X(\tau),\,M(\tau)}(x,\,y)$ equals

$$
\begin{aligned}
f^r_{X(\tau),\,M(\tau)}(x,\,y) &= \exp(-\frac{2y-x}{\sigma}\sqrt{2\frac{\frac{1}{2}\mu^2+\sigma^2\,(r-\lambda)}{\sigma^2}})\times\frac{2\lambda}{\sigma^2}\exp(\frac{\mu x}{\sigma^2}) \\
&= \frac{2\lambda}{\sigma^2}\exp(\frac{\mu x}{\sigma^2}-\frac{2y-x}{\sigma^2}\sqrt{2\left(\frac{1}{2}\mu^2+\sigma^2\,(r-\lambda)\right)}) \qquad (4.5.3)
\end{aligned}
$$

Let $\mu = r-\delta-\frac{1}{2}\sigma^2$, the above equals

$$
\frac{\lambda}{\frac{1}{2}\sigma^2}e^{-\alpha x-(\beta-\alpha)y} \qquad (4.5.4)
$$

where $\alpha < 0$, $\beta > 0$ are the two roots of the following quadratic equation.

$$
\frac{1}{2}\sigma^2\theta^2+(r-\delta-\frac{1}{2}\sigma^2)\theta-(r+\lambda)=0
$$

To derive the discounted joint density function of $X(\tau)$ and $m(\tau)$, since

$$
m(t) = -\max\{-X(s);\,0\le s\le t\}
$$

we can use the previous result with $M(t)$ replaced by $-m(t)$ and $\left(r-\delta-\frac{1}{2}\sigma^2\right)$ is to be replaced by $-\left(r-\delta-\frac{1}{2}\sigma^2\right)$. The positive and negative roots of the equation

$$\frac{1}{2}\sigma^2\theta^2-(r-\delta-\frac{1}{2}\sigma^2)\theta-(r+\lambda)=0$$

equals $-\alpha$ and $-\beta$. Therefore the discounted joint density function of $X(\tau)$ and $m(\tau)$ equals

$$f^r_{X(\tau),\,m(\tau)}(x,\,y)=\frac{\lambda}{\frac{1}{2}\sigma^2}e^{-\beta x+(\beta-\alpha)y},\qquad y\leq\min(x,\,0).$$

Here in (4.5.4), we replace x with $-x$, y with $-y$, α with $-\beta$ and β with $-\alpha$.

CHAPTER 5

AMERICAN OPTION WITH ERLANG EXPIRY DATE

5.1 Introduction

In this chapter, the expiry date of the option is extended from the exponential distribution to the *Erlang* distribution. The *Erlang* distribution is a two-parameter distribution, and the exponential can be seen as a special case of the *Erlang* distribution. Similar to the exponential case, we calculate the price of American options with an *Erlang* distributed expiry date as the sum of the price of a corresponding European option and the early exercise premium. But the calculation here is not as simple as the exponential case. The tedious calculation is mainly due to the form of the optimal exercise boundary. The optimal exercise boundary for an American option with an *Erlang* distributed expiry date takes the form of a staircase. This is because the *Erlang* distribution with shape parameter n follows the same distribution as the sum of n independent exponential distributions, and for the exponential distributed expiry date, the optimal exercise boundary is flat. To determine the optimal exercise boundary, we recursively impose the "value matching" condition for the price of the option at the optimal exercise boundary. If we fix the mean of the *Erlang* distribution, and let n go to infinity, the *Erlang* distribution will converge to a fixed point. Following that, the price of the American option with an *Erlang* distributed expiry date will converge to the price of the American option with a fixed date.

The structure of this chapter is as follows. In section 5.2, I derive the discounted density function of $X(T_n)$. Section 5.3 considers the valuation problem of European style options with the *Erlang* distributed expiry date. Section 5.4 derives the formulas for the price of the American put option whose expiry date is *Erlang* distributed. Section 5.4 numerically approximates the price of the American put option with a fixed expiry date by the price of American put options with the *Erlang* distributed expiry date. Section 5.6 includes the valuation of American exchange options with the *Erlang* distributed expiry date .

5.2 Discounted density function of $X(T_n)$

Let $\{X(t) = \mu t + \sigma B(t),\, t \geq 0\}$, where $B(t)$ is a standard Brownian motion (Wiener process), and μ and $\sigma > 0$ are constants. We also assume T_n follows *Erlang*(n, λ) distribution and it is independent of the process $\{X(t)\}$. We are now interested in the distribution of $X(T_n)$. i.e., we want to find the probability density function of $X(T_n)$. One way to calculate the density function $f_{X(T_n)}(x)$ is to integrate the product of the conditional density function of $f_{X(T_n)|T_n}(x \mid t)$ and the density function $f_{T_n}(t)$ with respect to t. It equals to

$$f_{X(T_n)}(x) = \int_0^\infty f_{X(T_n)|T_n}(x \mid t) f_{T_n}(t) \mathrm{d}t$$

Because of the independence of T_n and $\{X(t)\}$, we have $f_{X(T_n)|T_n}(x \mid t) = f_{X(t)}(x)$. Therefore, the above equals

$$f_{X(T_n)}(x) = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}} \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} \mathrm{d}t$$

The integration seems difficult. To avoid this difficulty, we would calculate the moment generating function of $X(T_n)$ and then invert it to find $f_{X(T_n)}(x)$. The moment generating function of $X(T_n)$ equals

$$M_{X(T_n)}(t) = \mathrm{E}[e^{tX(T_n)}] = \mathrm{E}[\mathrm{E}[e^{tX(T_n)}|T_n]]$$

Because of the independence of T_n and $\{X(t)\}$, the above equals

$$(\frac{\lambda}{\lambda - \Psi(t)})^n$$

where $\Psi(t) = \mu t + \frac{1}{2}\sigma^2 t^2$. Let $\alpha^* < 0$ and $\beta^* > 0$ be the roots of the quadratic equation

$$\frac{1}{2}\sigma^2 x^2 + \mu x - \lambda = 0$$

The moment generating function of $X(T_n)$ is

$$M_{X(T_n)}(t) = \left(\frac{\lambda}{-\frac{1}{2}\sigma^2(t-\alpha^*)(t-\beta^*)}\right)^n$$

Thus, the probability density function of $X(T_n)$ is[1]

$$\begin{aligned} f_{X(T_n)}(x) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-zx} M_{X(T_n)}(z)\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-zx}\left(\frac{\lambda}{-\frac{1}{2}\sigma^2(z-\alpha^*)(z-\beta^*)}\right)^n \mathrm{d}z \end{aligned}$$

If we define $f(z) = e^{-zx}M_{X(T_n)}(z)$ and make the substitution $z = Re^{i\theta}$, we have

$$f(z)\mathrm{d}z = f(Re^{i\theta})Re^{i\theta}i\mathrm{d}\theta$$

For any $x > 0$, $R > \beta^* + 1$, since

$$|f(Re^{i\theta})Re^{i\theta}i| = |e^{-Re^{i\theta}x}M_{X(T_n)}(Re^{i\theta})Re^{i\theta}i| \leq Ce^{-Rx\cos(\theta)}\frac{1}{R^{n-1}(R-\beta^*)^n} \leq C$$

[1] The moment generating function is

$$M_{X(T_n)}(t) = \int_{-\infty}^{\infty} e^{tx} f_{X(T_n)}(x)\mathrm{d}x$$

By the Fourier inversion formula, the probability density function $f_{X(T_n)}(x)$ is

$$f_{X(T_n)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} M_{X(T_n)}(it)\mathrm{d}t$$

Then make the change of variable $z = it$

$$f_{X(T_n)}(x) = \frac{1}{2\pi}\int_{-i\infty}^{i\infty} e^{-zx} M_{X(T_n)}(z)\mathrm{d}\left(-iz\right)$$

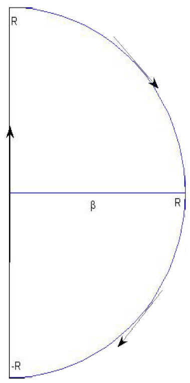
and

$$\lim_{R\to\infty} f(Re^{i\theta})Re^{i\theta}i = 0$$

according to the dominated convergence theorem, we have

$$\lim_{R\to\infty} \int_{\theta=\frac{\pi}{2}}^{\theta=-\frac{\pi}{2}} f(Re^{i\theta})\mathrm{d}\left(Re^{i\theta}\right) = 0$$

Figure 5.1: Semicircular contour in the right half-plane



We integrate $f(z)$ with respect to the semicircular contour in the right half-plane (Figure 5.1), because of the Residue Theorem, we have

$$\frac{1}{2\pi i}\left(\int_{-iR}^{iR} f(z)\mathrm{d}z + \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} f(Re^{i\theta})Re^{i\theta}i\mathrm{d}\theta\right) = (-1)\mathrm{Res}(f,\, \beta^*)$$

where $\mathrm{Res}(f,\, \beta^*)$ denotes the residual of f at β^*. Since when R goes to infinity, the second part of the above goes to zero. We have for any $x > 0$,

$$f_{X(T_n)}(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-zx} M_{X(T_n)}(z)\mathrm{d}z = (-1)\mathrm{Res}(f,\, \beta^*)$$

and since β^* is a pole of order n, we have

$$\begin{aligned}\text{Res}(f,\,\beta^*) &= \frac{1}{(n-1)!}\lim_{z\to\beta^*}(\frac{\partial}{\partial z})^{n-1}[(z-\beta^*)^n e^{-zx}(\frac{\lambda}{-\frac{1}{2}\sigma^2(z-\alpha^*)(z-\beta^*)})^n] \\ &= \frac{1}{(n-1)!}(\frac{-\lambda}{\frac{1}{2}\sigma^2})^n \lim_{z\to\beta^*}(\frac{\partial}{\partial z})^{n-1}[e^{-zx}(\frac{1}{(z-\alpha^*)})^n] \end{aligned} \quad (5.2.1)$$

To evaluate the $(n-1)$th derivative in (5.2.1), we apply the *exponential shift formula (4.5.1)*. By induction, we have

$$\frac{\partial}{\partial z}^m[e^{az}g(z)] = e^{az}(a+\frac{\partial}{\partial z})^m g(z)$$

Making the substitution $m = n-1$, $a = -x$ and $g(z) = (z-\alpha^*)^{-n}$ in the above formula, we have

$$\begin{aligned}&(\frac{\partial}{\partial z})^{n-1}[e^{-zx}(z-\alpha^*)^{-n}] \\ &= e^{-zx}(-x+\frac{\partial}{\partial z})^{n-1}[(z-\alpha^*)^{-n}] \\ &= e^{-zx}\sum_{k=0}^{n-1}\binom{n-1}{k}(-x)^k(\frac{\partial}{\partial z})^{n-1-k}(z-\alpha^*)^{-n} \\ &= e^{-zx}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^{n-1}x^k\frac{(2n-2-k)!}{(n-1)!}(z-\alpha^*)^{-n-(n-1-k)}\end{aligned}$$

Thus, (5.2.1) equals

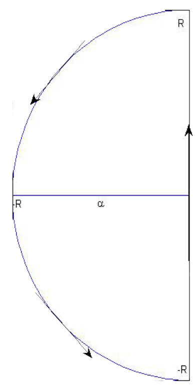
$$\begin{aligned}&\frac{1}{(n-1)!}(\frac{-\lambda}{\frac{1}{2}\sigma^2})^n e^{-\beta^* x}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^{n-1}x^k\frac{(2n-2-k)!}{(n-1)!}(\beta^*-\alpha^*)^{-2n+k+1} \\ &= \frac{-e^{-\beta^* x}}{(n-1)!}(\frac{\lambda}{\frac{1}{2}\sigma^2(\beta^*-\alpha^*)})^n \times \sum_{k=1}^{n}\binom{n-1}{k-1}x^{k-1}\frac{(2n-1-k)!}{(n-1)!}(\beta^*-\alpha^*)^{-n+k} \\ &= -(\frac{\lambda}{\frac{1}{2}\sigma^2(\beta^*-\alpha^*)})^n e^{-\beta^* x}\sum_{k=1}^{n}\frac{\binom{2n-k-1}{n-k}}{(k-1)!(\beta^*-\alpha^*)^{n-k}}x^{k-1}\end{aligned}$$

Therefore, for $x > 0$, the probability density function of $X(T_n)$ is

$$f_{X(T_n)}(x) = (-1)\text{Res}(f,\,\beta^*) = (\frac{\lambda}{\frac{1}{2}\sigma^2(\beta^*-\alpha^*)})^n e^{-\beta^* x}\sum_{k=1}^{n}\frac{\binom{2n-k-1}{n-k}}{(k-1)!(\beta^*-\alpha^*)^{n-k}}x^{k-1}$$

Similarly, to find $f_{X(T_n)}(x)$ for any $x < 0$, we integrate $f(z)$ with respect to the semicircular contour in the left half-plane (Figure 5.2), we have

Figure 5.2: Semicircular contour in the left half-plane



$$f_{X(T_n)}(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-zx} M_{X(T_n)}(z)\mathrm{d}z = \mathrm{Res}(f,\, \alpha^*)$$

and

$$\mathrm{Res}(f,\, \alpha^*) = \frac{1}{(n-1)!}(\frac{-\lambda}{\frac{1}{2}\sigma^2})^n \lim_{z\to\alpha^*}(\frac{\partial}{\partial z})^{n-1}[e^{-zx}(\frac{1}{(z-\beta^*)})^n]$$

Therefore,

$$f_{X(T_n)}(x) = \mathrm{Res}(f,\, \alpha^*) = (\frac{\lambda}{\frac{1}{2}\sigma^2(\beta^*-\alpha^*)})^n e^{-\alpha^* x}\sum_{k=1}^{n}\frac{\binom{2n-k-1}{n-k}}{(k-1)!(\beta^*-\alpha^*)^{n-k}}(-x)^{k-1}$$

We summarize the above as the following Proposition.

Proposition 5.2.1. *Let $\{X(t)\}$ be a Brownian motion with drift parameter μ and diffusion parameter σ^2, and T_n be an Erlang(λ, n) random variable independent of $\{X(t)\}$. The density function of $X(T_n)$ is*

$$f_{X(T_n)}(x) = \begin{cases} \kappa^{*n} e^{-\alpha^* x}\sum_{k=1}^{n}\frac{\binom{2n-k-1}{n-k}}{(k-1)!(\beta^*-\alpha^*)^{n-k}}(-x)^{k-1}, & x \le 0 \\ \kappa^{*n} e^{-\beta^* x}\sum_{k=1}^{n}\frac{\binom{2n-k-1}{n-k}}{(k-1)!(\beta^*-\alpha^*)^{n-k}}x^{k-1}, & x > 0 \end{cases}$$

where $\kappa^* = \frac{\lambda}{\frac{1}{2}\sigma^2(\beta^*-\alpha^*)}$, and $\alpha^* < 0$ and $\beta^* > 0$ are the roots of the quadratic *equation*

$$\frac{1}{2}\sigma^2 x^2 + \mu x - \lambda = 0.$$

Now we are interested in the discounted density function, which is defined as

$$f^r_{X(T_n)}(x) = \int_0^\infty e^{-rt} f_{X(t)}(x) f_{T_n}(t) dt.$$

The discounted density function is useful when we calculate the price of options. *Lemma* 5.2.1 is a factorization formula, and *Lemma* 5.2.2 can be thought as a generalization of Proposition 5.2.1.

Lemma 5.2.1. *Let $\{X(t)\}$ be a Brownian motion with drift parameter μ and diffusion parameter σ^2, and T_n be an Erlang(λ, n) random variable independent of $\{X(t)\}$. For a given function $h(x)$, we have the following identity*

$$\mathrm{E}\left[e^{-rT_n} h\left(X(T_n)\right)\right] = \mathrm{E}\left[e^{-rT_n}\right] \mathrm{E}^*\left[h\left(X(T_n)\right)\right] \tag{5.2.2}$$

where the asterisk signifies that there is a change of probability measure such that T_n follows the distribution Erlang($\lambda + r$, n).

Proof. We start with the left hand side of (5.2.2)

$$\begin{aligned}
\mathrm{E}\left[e^{-rT_n} h\left(X(T_n)\right)\right] &= \int_0^\infty e^{-rt} h\left(X(t)\right) f_{T_n}(t) \mathrm{d}t \\
&= \int_0^\infty e^{-rt} h\left(X(t)\right) \frac{\lambda^n}{(n-1)!} e^{-\lambda t} t^{n-1} \mathrm{d}t \\
&= \frac{\lambda^n}{(\lambda + r)^n} \int_0^\infty h\left(X(t)\right) \frac{(\lambda + r)^n}{(n-1)!} e^{-(\lambda + r)t} t^{n-1} \mathrm{d}t \\
&= \mathrm{E}\left[e^{-rT_n}\right] \mathrm{E}^*\left[h\left(X(T_n)\right)\right]
\end{aligned}$$

□

As an application of *Lemma* 5.2.1, the quantity of the discounted density function is given in *Lemma* 5.2.2.

Lemma 5.2.2. *Let $\{X(t)\}$ be a Brownian motion with drift parameter μ and diffusion parameter σ^2, and T_n be an Erlang(λ, n) random variable independent of $\{X(t)\}$. The discounted density function equals*

$$f^r_{X(T_n)}(x) = \begin{cases} \kappa^n e^{-\alpha x} \sum\limits_{k=1}^{n} \frac{\binom{2n-k-1}{n-k}}{(k-1)!(\beta-\alpha)^{n-k}} (-x)^{k-1}, & x \leq 0 \\ \kappa^n e^{-\beta x} \sum\limits_{k=1}^{n} \frac{\binom{2n-k-1}{n-k}}{(k-1)!(\beta-\alpha)^{n-k}} x^{k-1}, & x > 0 \end{cases}$$

where $\kappa = \frac{\lambda}{\frac{1}{2}\sigma^2(\beta-\alpha)}$, and $\alpha < 0$ and $\beta > 0$ are the roots of the quadratic equation

$$\frac{1}{2}\sigma^2 x^2 + \mu x - (\lambda + r) = 0. \tag{5.2.3}$$

The above formula is the formula (2.36) in Gerber, Shiu and Yang (2012), which was not derived by the method of complex variables.

5.3 European style options

To consider the pricing problem, we need to do the calculation under risk neutral measure. We consider the drift parameter μ equals to $r - \delta - \frac{1}{2}\sigma^2$, which means $X(t) = (r - \delta - \frac{1}{2}\sigma^2)t + \sigma B(t)$, $t > 0$. Similar to the previous chapters, the time t stock price is also modeled as $S(t) = S(0)e^{X(t)}$, $t \geq 0$. The following lemma is useful to derive the price of an out-of-money European put option.

Lemma 5.3.1. *Let T_n be an Erlang(n, λ) random variable independent of the stock price process and $L < S(0)$. For each $m > \alpha$, the price of the out-of-the-money all-or-nothing put option equals*

$$\begin{aligned} &\mathrm{E}\left[e^{-rT_n}[S(T_n)]^m I(S(T_n) < L)\right] \\ =&S(0)^m \left(\frac{L}{S(0)}\right)^{m-\alpha} \frac{\kappa^n}{(m-\alpha)^n} \sum_{i=0}^{n-1} \frac{\left(\ln\frac{S(0)}{L}\right)^i (m-\alpha)^i}{i!} \sum_{l=0}^{n-i-1} \frac{(m-\alpha)^l}{(\beta-\alpha)^l} \binom{n-1+l}{l} \end{aligned}$$

where $\kappa = \frac{\lambda}{\frac{1}{2}\sigma^2(\beta-\alpha)}$, and α and β are the roots of the quadratic equation (5.2.3) with $\mu = r - \delta - \frac{1}{2}\sigma^2$.

Proof. The expectation of $e^{-rT_n}[S(T_n)]^m I(S(T_n) < L)$ is the double integration, with respect to x and t, of

$$e^{-rt}[S(0)e^x]^m I(S(0)e^x < L) f_{X(t)}(x) f_{T_n}(t)$$

By defining the discounted density function

$$f^r_{X(T_n)}(x) = \int_0^\infty e^{-rt} f_{X(t)}(x) f_{T_n}(t) \mathrm{d}t,$$

the expectation of $e^{-rT_n}[S(T_n)]^m I(S(T_n) < L)$ equals

$$\int_{-\infty}^{\infty} [S(0)e^x]^m I(S(0)e^x < L) f^r_{X(T_n)}(x) \mathrm{d}x$$

Substituting the discounted density function with the form we derived in the last section, the above equals

$$\begin{aligned}
&\int_{-\infty}^{\ln \frac{L}{S(0)}} [S(0)e^x]^m \kappa^n e^{-\alpha x} \sum_{j=1}^{n} \frac{\binom{2n-j-1}{n-j}}{(j-1)!\,(\beta-\alpha)^{n-j}} (-x)^{j-1} \,\mathrm{d}x \\
&= S(0)^m \kappa^n \sum_{j=1}^{n} \frac{\binom{2n-j-1}{n-j}}{(j-1)!\,(\beta-\alpha)^{n-j}} \int_{\ln \frac{S(0)}{L}}^{\infty} e^{-(m-\alpha)x} x^{j-1} \mathrm{d}x
\end{aligned}$$

Multiplying and dividing $(m-\alpha)^n$, the above equals

$$S(0)^m \frac{\kappa^n}{(m-\alpha)^n} \sum_{j=1}^{n} \frac{\binom{2n-j-1}{n-j}}{(j-1)!\,(\beta-\alpha)^{n-j}} (m-\alpha)^{n-j} \int_{\ln \frac{S(0)}{L}}^{\infty} e^{-(m-\alpha)x} (m-\alpha)^j x^{j-1} \mathrm{d}x$$

Because of the identity (2.3.2), we have the following

$$\int_{\ln \frac{S(0)}{L}}^{\infty} e^{-(m-\alpha)x} (m-\alpha)^j x^{j-1} \mathrm{d}x = (j-1)! e^{(\alpha-m)\ln \frac{S(0)}{L}} \sum_{i=0}^{j-1} \frac{\left((m-\alpha)\ln \frac{S(0)}{L}\right)^i}{i!}$$

Therefore, the expectation of $e^{-rT_n}[S(T_n)]^m I(S(T_n) < L)$ equals

$$\frac{S(0)^m \kappa^n}{(m-\alpha)^n} \sum_{j=1}^{n} \frac{\binom{2n-j-1}{n-j}}{(j-1)!\,(\beta-\alpha)^{n-j}} (m-\alpha)^{n-j} (j-1)! e^{(\alpha-m)\ln\frac{S(0)}{L}} \sum_{i=0}^{j-1} \frac{\left((m-\alpha)\ln\frac{S(0)}{L}\right)^i}{i!}$$

$$= S(0)^m \left(\frac{S(0)}{L}\right)^{\alpha-m} \frac{\kappa^n}{(m-\alpha)^n} \sum_{j=1}^{n} \frac{\binom{2n-j-1}{n-j}}{(\beta-\alpha)^{n-j}} (m-\alpha)^{n-j} \sum_{i=0}^{j-1} \frac{\left((m-\alpha)\ln\frac{S(0)}{L}\right)^i}{i!}$$

If we define $n-j=l$, the above equals to

$$S(0)^m \left(\frac{S(0)}{L}\right)^{\alpha-m} \frac{\kappa^n}{(m-\alpha)^n} \sum_{l=n-1}^{0} \frac{\binom{n-1+l}{l}}{(\beta-\alpha)^{l}} (m-\alpha)^{l} \sum_{i=0}^{n-l-1} \frac{\left((m-\alpha)\ln\frac{S(0)}{L}\right)^i}{i!}$$

Interchanging the order of sum, the above equals

$$S(0)^m \left(\frac{L}{S(0)}\right)^{m-\alpha} \frac{\kappa^n}{(m-\alpha)^n} \sum_{i=0}^{n-1} \frac{\left((m-\alpha)\ln\frac{S(0)}{L}\right)^i}{i!} \sum_{l=0}^{n-i-1} \frac{\binom{n-1+l}{l}}{(\beta-\alpha)^{l}} (m-\alpha)^{l}$$

□

The formula for the value of an out-of-the-money put option is given in Corollary 5.3.1 below.

Corollary 5.3.1. *If $S(0) > K$, The value of an out-of-the-money European put option,*

$$\begin{aligned}
&\mathrm{E}[e^{-rT_n}[K-S(T_n)]_+ | S(0) > K] \\
&= K \left(\frac{K}{S(0)}\right)^{-\alpha} \frac{\kappa^n}{(-\alpha)^n} \sum_{i=0}^{n-1} \frac{\left(-\alpha\ln\frac{S(0)}{K}\right)^i}{i!} \sum_{l=0}^{n-i-1} \frac{\binom{n-1+l}{l}}{(\beta-\alpha)^{l}} (-\alpha)^{l} \\
&- S(0) \left(\frac{K}{S(0)}\right)^{1-\alpha} \frac{\kappa^n}{(1-\alpha)^n} \sum_{i=0}^{n-1} \frac{\left((1-\alpha)\ln\frac{S(0)}{K}\right)^i}{i!} \sum_{l=0}^{n-i-1} \frac{\binom{n-1+l}{l}}{(\beta-\alpha)^{l}} (1-\alpha)^{l} \qquad (5.3.1)
\end{aligned}$$

Proof. The expectation of $e^{-rT_n}[K - S(T_n)]_+$ can be written as

$$\begin{aligned}&\mathrm{E}[e^{-rT_n} K I(S(T_n) < K)] - \mathrm{E}[e^{-rT_n} S(T_n) I(S(T_n) < K)] \\ &= K\mathrm{E}[e^{-rT_n}[S(T_n)]^0 I(S(T_n) < K)] - \mathrm{E}[e^{-rT_n} S(T_n) I(S(T_n) < K)]\end{aligned}$$

If $S(0) > K$, according to *Lemma* 5.3.1, we have

$$\begin{aligned}&= K\left(\frac{K}{S(0)}\right)^{-\alpha} \frac{\kappa^n}{(-\alpha)^n} \sum_{i=0}^{n-1} \frac{\left(-\alpha \ln \frac{S(0)}{K}\right)^i}{i!} \sum_{l=0}^{n-i-1} \frac{\binom{n-1+l}{l}}{(\beta-\alpha)^l} (-\alpha)^l \\ &- S(0)\left(\frac{K}{S(0)}\right)^{1-\alpha} \frac{\kappa^n}{(1-\alpha)^n} \sum_{i=0}^{n-1} \frac{\left((1-\alpha) \ln \frac{S(0)}{K}\right)^i}{i!} \sum_{l=0}^{n-i-1} \frac{\binom{n-1+l}{l}}{(\beta-\alpha)^l} (1-\alpha)^l\end{aligned}$$

□

Remark 5.3.1. Following Carr (1998), we define

$$p = \tfrac{-\alpha}{\beta-\alpha}; \quad q = \tfrac{\beta}{\beta-\alpha}; \quad \hat{p} = \tfrac{1-\alpha}{\beta-\alpha}; \quad \hat{q} = \tfrac{\beta-1}{\beta-\alpha}$$

$$R = \tfrac{\lambda}{\lambda+r}; \quad 2\epsilon = \beta - \alpha$$

for the case of that the dividend rate δ equals to zero, we have

$$\tfrac{\kappa}{-\alpha} = qR \;\text{ and }\; \tfrac{\kappa}{1-\alpha} = \hat{q}$$

The value of an out-of-the-money put option is

$$\begin{aligned}&K\left(\frac{S(0)}{K}\right)^{\alpha} (qR)^n \sum_{i=0}^{n-1} \frac{\left(\ln \frac{S(0)}{K}\right)^i (2\epsilon)^i}{i!} \sum_{l=0}^{n-i-1} \binom{n-1+l}{l} p^{i+l} \\ &- K\left(\frac{S(0)}{K}\right)^{\alpha} \hat{q}^n \sum_{i=0}^{n-1} \frac{\left(\ln \frac{S(0)}{K}\right)^i (2\epsilon)^i}{i!} \sum_{l=0}^{n-i-1} \binom{n-1+l}{l} \hat{p}^{i+l}\end{aligned}$$

which is formula (43) in Carr (1998).

Lemma 5.3.2. *Let T_n be an Erlang(n, λ) random variable independent of the* *stock price process. If $S(0) < H$, it can be shown that the price of out-of-the-money all-or-nothing call option equals*

$$\begin{aligned}
&\mathrm{E}\left[e^{-rT_n}[S(T_n)]^m I(S(T_n) > H)\right] \\
&= S(0)^m \left(\frac{S(0)}{H}\right)^{\beta-m} \frac{\kappa^n}{(\beta-m)^n} \sum_{i=0}^{n-1} \frac{\left((\beta-m)\ln\frac{H}{S(0)}\right)^i}{i!} \sum_{l=0}^{n-i-1} \frac{(\beta-m)^l}{(\beta-\alpha)^l} \binom{n-1+l}{l}
\end{aligned}$$

where κ, α and β are defined in Lemma 5.3.1.

Proof. Same as the proof in *Lemma* 5.3.1, the expectation of

$$e^{-rT_n}[S(T_n)]^m I(S(T_n) > H)$$

equals

$$\begin{aligned}
&\int_{\ln\frac{H}{S(0)}}^{\infty} [S(0)e^x]^m \kappa^n e^{-\beta x} \sum_{j=1}^{n} \frac{\binom{2n-j-1}{n-j}}{(j-1)!(\beta-\alpha)^{n-j}} x^{j-1} \mathrm{d}x \\
&= S(0)^m \kappa^n \sum_{j=1}^{n} \frac{\binom{2n-j-1}{n-j}}{(j-1)!(\beta-\alpha)^{n-j}} \int_{\ln\frac{H}{S(0)}}^{\infty} e^{(m-\beta)x} x^{j-1} \mathrm{d}x \\
&= S(0)^m \frac{\kappa^n}{(\beta-m)^n} \sum_{j=1}^{n} \frac{\binom{2n-j-1}{n-j}}{(j-1)!(\beta-\alpha)^{n-j}} (\beta-m)^{n-j} \int_{\ln\frac{H}{S(0)}}^{\infty} e^{(m-\beta)x} (\beta-m)^j x^{j-1} \mathrm{d}x
\end{aligned}$$

Because of the identity (2.3.2), we have the following

$$\int_{\ln\frac{H}{S(0)}}^{\infty} e^{(m-\beta)x} (\beta-m)^j x^{j-1} \mathrm{d}x = (j-1)! e^{(m-\beta)\ln\frac{H}{S(0)}} \sum_{i=0}^{j-1} \frac{\left(-(m-\beta)\ln\frac{H}{S(0)}\right)^i}{i!}$$

Therefore, the expectation of $e^{-rT_n}[S(T_n)]^m I(S(T_n) > H)$ equals

$$\frac{S(0)^m \kappa^n}{(\beta-m)^n} \sum_{j=1}^{n} \frac{\binom{2n-j-1}{n-j}}{(j-1)!(\beta-\alpha)^{n-j}} (\beta-m)^{n-j} (j-1)! e^{(m-\beta)\ln\frac{H}{S(0)}} \sum_{i=0}^{j-1} \frac{\left(-(m-\beta)\ln\frac{H}{S(0)}\right)^i}{i!}$$

$$= S(0)^m \left(\frac{H}{S(0)}\right)^{m-\beta} \frac{\kappa^n}{(\beta-m)^n} \sum_{j=1}^{n} \frac{\binom{2n-j-1}{n-j}}{(\beta-\alpha)^{n-j}} (\beta-m)^{n-j} \sum_{i=0}^{j-1} \frac{\left(-(m-\beta)\ln\frac{H}{S(0)}\right)^i}{i!}$$

If we define $n-j=l$, the above equals to

$$S(0)^m \left(\frac{H}{S(0)}\right)^{m-\beta} \frac{\kappa^n}{(\beta-m)^n} \sum_{l=n-1}^{0} \frac{\binom{n-1+l}{l}}{(\beta-\alpha)^l}(\beta-m)^l \sum_{i=0}^{n-l-1} \frac{\left(-(m-\beta)\ln\frac{H}{S(0)}\right)^i}{i!}$$

$$= S(0)^m \left(\frac{S(0)}{H}\right)^{\beta-m} \frac{\kappa^n}{(\beta-m)^n} \sum_{i=0}^{n-1} \frac{\left((\beta-m)\ln\frac{H}{S(0)}\right)^i}{i!} \sum_{l=0}^{n-i-1} \frac{\binom{n-1+l}{l}}{(\beta-\alpha)^l}(\beta-m)^l$$

□

Corollary 5.3.2. *If $S(0) < K$, The value of an out-of-the-money European call option is*

$$\begin{aligned}
&\mathrm{E}[e^{-rT_n}[S(T_n)-K]_+|S(0)<K] \\
&= S(0)\left(\frac{S(0)}{K}\right)^{\beta-1} \frac{\kappa^n}{(\beta-1)^n} \sum_{i=0}^{n-1} \frac{\left((\beta-1)\ln\frac{K}{S(0)}\right)^i}{i!} \sum_{l=0}^{n-i-1} \frac{\binom{n-1+l}{l}}{(\beta-\alpha)^l}(\beta-1)^l \\
&- K\left(\frac{S(0)}{K}\right)^{\beta} \frac{\kappa^n}{\beta^n} \sum_{i=0}^{n-1} \frac{\left(\beta\ln\frac{K}{S(0)}\right)^i}{i!} \sum_{l=0}^{n-i-1} \frac{\binom{n-1+l}{l}}{(\beta-\alpha)^l}\beta^l
\end{aligned} \tag{5.3.2}$$

Proof. The proof is similar as *Corollary 5.3.1.* □

Remark 5.3.2. For the case of that the dividend rate δ equals to zero, we have

$$\tfrac{\kappa}{\beta-1} = \hat{p}; \text{ and } \quad \tfrac{\kappa}{\beta} = pR$$

The value of an out-of-the-money call option is

$$S(0)\left(\frac{S(0)}{K}\right)^{\beta-1} \hat{p}^n \sum_{i=0}^{n-1} \frac{\left(\ln\frac{K}{S(0)}\right)^i (2\epsilon)^i}{i!} \sum_{l=0}^{n-i-1} \binom{n-1+l}{l} \hat{q}^{i+l}$$

$$-K\left(\frac{S(0)}{K}\right)^{\beta} (pR)^n \sum_{i=0}^{n-1} \frac{\left(\ln\frac{K}{S(0)}\right)^i (2\epsilon)^i}{i!} \sum_{l=0}^{n-i-1} \binom{n-1+l}{l} q^{i+l}$$

which is same as formula (44) in Carr (1998).

5.4 American put option

In this section, I shall derive the formula for the price of an American put option whose expiry date is *Erlang* distributed. We first provide following lemmas which are useful to calculate the price of an American put option. Here, we define T_n, $n \geq 1$ according to (2.3.1), and they are independent of the stock price process.

Lemma 5.4.1. *For non-negative functions $g(x)$, we have the following identity*

$$\mathrm{E}[\int_{T_{k-1}}^{T_k} e^{-rt} g(S(t))dt] = \frac{1}{\lambda}\mathrm{E}[e^{-rT_k} g(S(T_k))] \tag{5.4.1}$$

Proof. We start from left-hand side of (5.4.1)

$$\mathrm{E}[\int_{T_{k-1}}^{T_k} e^{-rt} g(S(t))dt] = \mathrm{E}[\int_0^\infty I(T_{k-1} \leq t \leq T_k) e^{-rt} g(S(t))\mathrm{d}t]$$

Since $I(T_{k-1} \leq t \leq T_k)e^{-rt}g(S(t))$ is non-negative, we could interchange the order of integration and expectation, the right-hand side of the above equals

$$\int_0^\infty \mathrm{E}[I(T_{k-1} \leq t \leq T_k) e^{-rt} g(S(t))]\mathrm{d}t \tag{5.4.2}$$

Because of the independence of T_n, $n \geq 1$ and the stock price process and (2.3.4), (5.4.2) equals

$$\begin{aligned} &\int_0^\infty e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!} \mathrm{E}[e^{-rt} g(S(t))]\mathrm{d}t \\ &= \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} \frac{\lambda^k t^{k-1}}{\Gamma(k)} \mathrm{E}[e^{-rt} g(S(t))]\mathrm{d}t \end{aligned}$$

Since $e^{-\lambda t}\frac{\lambda^k t^{k-1}}{\Gamma(k)}$ is the density function of *Erlang*(k, λ) distribution, the above could be written as

$$\frac{1}{\lambda}\mathrm{E}[e^{-rT_k} g(S(T_k))]$$

which is the right-hand side of (5.4.1). □

Lemma 5.4.2. *The price of the payoff* $\int_0^{T_n} e^{-rt}S(t)\delta\, dt$ *equals*

$$\mathrm{E}\left\{\int_0^{T_n} e^{-rt}S(t)\delta\, dt\right\} = S(0) - \mathrm{E}\left\{e^{-rT_n}S(T_n)\right\} \tag{5.4.3}$$

Proof. we can rewrite the left-hand side of (5.4.3) as an iterated expectation

$$\mathrm{E}\left\{\int_0^{T_n} e^{-rt}S(t)\delta \mathrm{d}t\right\} = \mathrm{E}\left[\mathrm{E}\left\{\int_0^{T_n} e^{-rt}S(t)\delta \mathrm{d}t\right\}|T_n\right]$$

Because of the independence of T_n and the stock price process, the right-hand side of the above equals

$$\int_0^{\infty} \mathrm{E}\left[\int_0^{t} e^{-ru}S(u)\delta du\right] f_{T_n}(t)\mathrm{d}t \tag{5.4.4}$$

where $f_{T_n}(t)$ is the density function of T_n. According to the Fundamental Theorem Asset Pricing,

$$\mathrm{E}\left[\int_0^{t} e^{-ru}S(u)\delta du\right] = S(0) - \mathrm{E}\left[S(t)e^{-rt}\right]$$

Therefore (5.4.4) equals

$$\begin{aligned}&\int_0^{\infty} \left(S(0) - \mathrm{E}\left[S(t)e^{-rt}\right]\right) f_{T_n}(t)\mathrm{d}t\\ &= S(0) - \mathrm{E}\left\{e^{-rT_n}S(T_n)\right\}\end{aligned}$$

□

Now, we are ready to calculate the value of an American put option whose expiry date is *Erlang* distributed. Similar to the exponential case, an American put option with *Erlang* distributed expiry date can be calculated as the sum of the price of a European put option and the early exercise premium. If we assume the expiry date T_n follows *Erlang*(n, λ), the early exercise premium can be represented as

$$\mathrm{E}\left[\int_0^{T_n} e^{-rt}(Kr - \delta S(t))I(S(t) < \underline{S}_t)\mathrm{d}t\right] \tag{5.4.5}$$

Here $\underline{S}_t$ is the optimal exercise boundary. In the exponential case, the optimal exercise boundary $\underline{S}_t$ is flat. Since an *Erlang*(n, λ) distribution can be viewed as the sum of the n exponential distributions, i.e. $T_n = \sum_{i=1}^{n} \tau_i$, where τ_i, $n = 1, 2, \ldots$, are independent and identically distributed exponential random variables with rate parameter λ, the optimal exercise boundary takes the form of a piece wise constant. We assume

$$\underline{S}_n < \underline{S}_{n-1} < \underline{S}_{n-2} < \ldots\ldots < \underline{S}_1 < S_0 = K$$

is the optimal exercise boundary in each sub-period. Then, if the initial stock price is larger than exercise boundary $\underline{S}_n$, the price of an American put option can be represented as the expectation of

$$e^{-rT_n}[K - S(T_n)]_+ + \sum_{k=1}^{n}[\int_{T_{k-1}}^{T_k} e^{-rt}[Kr - S(t)\delta]I(S(t) < \underline{S}_{n+1-k})\mathrm{d}t] \tag{5.4.6}$$

The first term of (5.4.6) can be written as

$$e^{-rT_n}[K - S(T_n)]_+ = e^{-rT_n}[S(T_n) - K]_+ + e^{-rT_n}[K - S(T_n)]$$

by put-call parity. Because of $I(S(t) < \underline{S}_{n+1-k}) = 1 - I(S(t) \geq \underline{S}_{n+1-k})$, the second term of (5.4.6) can be written as

$$\begin{aligned}
&\sum_{k=1}^{n}[\int_{T_{k-1}}^{T_k} e^{-rt}[Kr - S(t)\delta]I(S(t) < \underline{S}_{n+1-k})\mathrm{d}t] \\
&= \sum_{k=1}^{m}[\int_{T_{k-1}}^{T_k} e^{-rt}[Kr - S(t)\delta]I(S(t) < \underline{S}_{n+1-k})\mathrm{d}t] \\
&\quad + \sum_{k=m+1}^{n}[\int_{T_{k-1}}^{T_k} e^{-rt}[Kr - S(t)\delta](1 - I(S(t) \geq \underline{S}_{n+1-k}))\mathrm{d}t]
\end{aligned}$$

Also, we can see that the expectation of

$$\sum_{k=m+1}^{n}[\int_{T_{k-1}}^{T_k} e^{-rt}[Kr - S(t)\delta]\mathrm{d}t]$$

is the same as that of[2]

$$-e^{-rT_n}[K - S(T_n)] + e^{-rT_m}[K - S(T_m)]$$

Therefore, the price of an American put option equals to the expectation of

$$\begin{aligned}
&e^{-rT_n}[S(T_n) - K]_+ + e^{-rT_m}[K - S(T_m)] \\
&+ \sum_{k=1}^{m}\left[\int_{T_{k-1}}^{T_k} e^{-rt}[Kr - S(t)\delta]I(S(t) < \underline{S}_{n+1-k})\mathrm{d}t\right] \\
&- \sum_{k=m+1}^{n}\left[\int_{T_{k-1}}^{T_k} e^{-rt}[Kr - S(t)\delta]I(S(t) > \underline{S}_{n+1-k})\mathrm{d}t\right]
\end{aligned} \tag{5.4.8}$$

Theorem 5.4.1. *The price of the American put equals*

$$\begin{cases}
p^n(S(0)) + \sum_{k=1}^{n} b_k(S(0)) & S(0) > K \\
c^n(S(0)) + v_i^n(S(0)) + \sum_{k=1}^{n-i+1} b_k(S(0)) - \sum_{k=n-i+2}^{n} a_k(S(0)) & S(0) \in (\underline{S}_i,\ \underline{S}_{i-1}] \\
K - S(0) & S(0) \le \underline{S}_n
\end{cases} \tag{5.4.9}$$

where $p^n(S(0))$ is defined as (5.3.1) which is the price of an out-of-the-money

---

[2]

$$\begin{aligned}
&\mathrm{E}\left\{\sum_{k=m+1}^{n}[\int_{T_{k-1}}^{T_k} e^{-rt}[Kr - S(t)\delta]\mathrm{d}t]\right\} \\
&= \mathrm{E}\left\{\int_{T_m}^{T_n} e^{-rt}[Kr - S(t)\delta]\mathrm{d}t\right\} \\
&= \mathrm{E}\left\{\int_{T_m}^{T_n} e^{-rt}Kr\mathrm{d}t\right\} - \mathrm{E}\left\{\int_{T_m}^{T_n} e^{-rt}S(t)\delta\mathrm{d}t\right\}
\end{aligned} \tag{5.4.7}$$

The first integral on the right hand side of (5.4.7)

$$\int_{T_m}^{T_n} e^{-rt}Kr\mathrm{d}t = -K[e^{-rT_n} - e^{-rT_m}]$$

According to *Lemma* 5.4.2, the last expectation in (5.4.7) is

$$\mathrm{E}\left\{\int_{T_m}^{T_n} e^{-rt}S(t)\delta\mathrm{d}t\right\} = \mathrm{E}\left\{e^{-rT_m}S(T_m)\right\} - \mathrm{E}\left\{e^{-rT_n}S(T_n)\right\}$$

European put option, $c^n(S(0))$ *is defined as (5.3.2) which is the price of an out-of-the-money European call option ,* $b_k(S(0))$ *is defined as*

$$b_k(s) =$$

$$\frac{Kr}{\lambda}\left(\frac{\underline{S}_{n+1-k}}{s}\right)^{-\alpha}\frac{\kappa^k}{(-\alpha)^k}\sum_{j=0}^{k-1}\frac{\left(\ln\frac{s}{\underline{S}_{n+1-k}}\right)^j(-\alpha)^j}{j!}\sum_{l=0}^{k-j-1}\frac{(-\alpha)^l}{(\beta-\alpha)^l}\binom{k-1+l}{l}$$
$$-\frac{\delta}{\lambda}s\left(\frac{\underline{S}_{n+1-k}}{s}\right)^{1-\alpha}\frac{\kappa^k}{(1-\alpha)^k}\sum_{j=0}^{k-1}\frac{\left(\ln\frac{s}{\underline{S}_{n+1-k}}\right)^j(1-\alpha)^j}{j!}\sum_{l=0}^{k-j-1}\frac{(1-\alpha)^l}{(\beta-\alpha)^l}\binom{k-1+l}{l}$$

which can be interpreted as the present value of the cash flow received for the kth period if the initial price $S(0)$ *is larger than* $\underline{S}_{n+1-k}$*,* $a_k(S(0))$ *is defined as*

$$a_k(s) =$$

$$\frac{Kr}{\lambda}\left(\frac{s}{\underline{S}_{n+1-k}}\right)^{\beta}\frac{\kappa^k}{\beta^k}\sum_{j=0}^{k-1}\frac{\beta^j\left(\ln\frac{\underline{S}_{n+1-k}}{s}\right)^j}{j!}\sum_{l=0}^{k-j-1}\frac{\beta^l}{(\beta-\alpha)^l}\binom{k-1+l}{l}$$
$$-\frac{\delta}{\lambda}s\left(\frac{s}{\underline{S}_{n+1-k}}\right)^{\beta-1}\frac{\kappa^k}{(\beta-1)^k}\sum_{j=0}^{k-1}\frac{\left(\ln\frac{\underline{S}_{n+1-k}}{s}\right)^j(\beta-1)^j}{j!}\sum_{l=0}^{k-j-1}\frac{(\beta-1)^l}{(\beta-\alpha)^l}\binom{k-1+l}{l}$$

which can be interpreted as the present value of the cash flow paid for the kth period if the initial price $S(0)$ *is smaller than* $\underline{S}_{n+1-k}$ *and* $v_i^n(S(0))$ *is defined as*

$$v_i^n(S(0)) = K\left(\frac{\lambda}{\lambda+r}\right)^{n-i+1} - S(0)\left(\frac{\lambda}{\lambda+\delta}\right)^{n-i+1}$$

which equals to price of the payoff $K - S(T_{n-i+1})$*.*

Proof. Recall we have the inequality

$$\underline{S}_n < \underline{S}_{n-1} < \underline{S}_{n-2} < \ldots\ldots < \underline{S}_1 < S_0 = K$$

For the case when $S(0) > K$, we use *Corollary 5.3.1* to calculate the expectation of the first term of (5.4.6), which is the price of an out-of-the-money put option. The expectation of the second term of (5.4.6) can be calculated using *Lemma* 5.4.1

and *Lemma* 5.3.1. According to *Lemma* 5.4.1.

$$\mathrm{E}\left[\int_{T_{k-1}}^{T_k} e^{-rt}[Kr - S(t)\delta]I(S(t) < \underline{S}_{n+1-k})\mathrm{d}t\right]$$

equals

$$\frac{1}{\lambda}\mathrm{E}\left[e^{-rT_k}[Kr - S(T_k)\delta]I(S(T_k) < \underline{S}_{n+1-k})\right]$$

The above equals to

$$\begin{aligned}
&\frac{Kr}{\lambda}\left(\frac{\underline{S}_{n+1-k}}{S(0)}\right)^{-\alpha}\frac{\kappa^k}{(-\alpha)^k}\sum_{j=0}^{k-1}\frac{\left(\ln\frac{S(0)}{\underline{S}_{n+1-k}}\right)^j(-\alpha)^j}{j!}\sum_{l=0}^{k-j-1}\frac{(-\alpha)^l}{(\beta-\alpha)^l}\binom{k-1+l}{l} \\
&-\frac{\delta S(0)}{\lambda}\left(\frac{\underline{S}_{n+1-k}}{S(0)}\right)^{1-\alpha}\frac{\kappa^k}{(1-\alpha)^k}\sum_{j=0}^{k-1}\frac{\left(\ln\frac{S(0)}{\underline{S}_{n+1-k}}\right)^j(1-\alpha)^j}{j!}\sum_{l=0}^{k-j-1}\frac{(1-\alpha)^l}{(\beta-\alpha)^l}\binom{k-1+l}{l}
\end{aligned}$$

Since (5.4.8) can be written as

$$\begin{aligned}
&e^{-rT_n}[S(T_n) - K]_+ + e^{-rT_m}[K - S(T_m)] \\
&+\sum_{k=1}^{m}\left[\int_{T_{k-1}}^{T_k} e^{-rt}[Kr - S(t)\delta]I\left(S(t) < \underline{S}_{n+1-k}\right)\right. \\
&\qquad\left.\times\left[I\left(S(0) < \underline{S}_{n+1-k}\right) + I\left(S(0) > \underline{S}_{n+1-k}\right)\right]\mathrm{d}t\right] \\
&-\sum_{k=m+1}^{n}\left[\int_{T_{k-1}}^{T_k} e^{-rt}[Kr - S(t)\delta]I(S(t) > \underline{S}_{n+1-k})\right. \\
&\qquad\left.\times\left[I(S(0) < \underline{S}_{n+1-k}) + I(S(0) > \underline{S}_{n+1-k})\right]\mathrm{d}t\right] \qquad (5.4.10)
\end{aligned}$$

For the case of $S(0) \in (\underline{S}_i,\ \underline{S}_{i-1}]$, we choose $m = n - i + 1$. We can use *Corollary 5.3.2* to derive the first term of (5.4.10), which is the price of an out-of-the-money call option. Similarly to the calculation of the second term of (5.4.6), the expectation of the second line of (5.4.10) can be derived with *Lemma* 5.4.1 and *Lemma* 5.3.1, and the third line of (5.4.10) can be derived with *Lemma* 5.4.1 and *Lemma* 5.3.2. The second part of the (5.4.10) can be calculated by the law of iterated expectations,

$$
\begin{aligned}
&\mathrm{E}\left[e^{-rT_{n-i+1}}[K - S(T_{n-i+1})]\right] \\
&= \mathrm{E}\left\{\mathrm{E}\left[e^{-rT_{n-i+1}}[K - S(T_{n-i+1})]\right] |T_{n-i+1}\right\} \\
&= \mathrm{E}\left\{Ke^{-rT_{n-i+1}} - S(0)e^{-\delta T_{n-i+1}}\right\} \\
&= K\left(\frac{\lambda}{\lambda + r}\right)^{n-i+1} - S(0)\left(\frac{\lambda}{\lambda + \delta}\right)^{n-i+1}
\end{aligned}
$$

For the case of $S(0) \leq \underline{S}_n$, we need to exercise the American option immediately, so the value of the American option equals to the exercise value. $\square$

The optimal exercise boundary can be determined by imposing the "smooth pasting" condition at the optimal exercise boundary. Continuity at the optimal exercise price for each $m = 1, 2, 3 \ldots n$ implies

$$
K - \underline{S}_m = c^m(\underline{S}_m) + v_m^m(\underline{S}_m) + b_1(\underline{S}_m) - \sum_{k=2}^{m} a_k(\underline{S}_m) \tag{5.4.11}
$$

We can solve $\underline{S}_m$ in the above equation recursively.

Remark 5.4.1. The formula (26) for the American put option with the *Erlang* distributed expiry date in Carr (1998) is

$$
\begin{cases}
p^n(S(0)) + \sum\limits_{k=1}^{n} b_k(S(0)) & S(0) > K \\
v_i^n(S(0)) + \sum\limits_{k=1}^{n-i+1} b_k(S(0)) + \sum\limits_{k=1}^{n-i+1} a_k(S(0)) & S(0) \in (\underline{S}_i,\, \underline{S}_{i-1}] \\
K - S(0) & S(0) \leq \underline{S}_n
\end{cases}
$$

Compared to the result we obtain, the above implies

$$
c^n(S(0)) = \sum_{k=1}^{n} a_k(S(0))
$$

Remark 5.4.2. Since the exponential distribution can be thought as a special case of the *Erlang* distribution with the shape parameter equal to 1, the pricing formula (5.4.9) with $n = 1$ is same to the formula (4.2.19).

5.5 Numerical results

If we fix $\lambda n = T$ and increase the value of n, the *Erlang*(n, λ) distributed expiration date will converge to a fixed point T. As a result, (5.4.9) will converge to the price of an American option with fixed expiration date T. However, with a large n, the calculation in (5.4.9) will not be efficient. Similarly to Carr (1998), we use Richardson extrapolation to improve computational efficiency. Here we use an example to illustrate how Richardson extrapolation works. We use $y(T, n)$ and $\hat{y}(T, n)$ to denote the accurate and approximate value of an American put option whose expiration date is *Erlang*$(n, \frac{T}{n})$ distributed. Substituting n by $\frac{1}{\Delta}$, we could define the function $P(T, \Delta)$ as $y(T, \frac{1}{\Delta})$ and $\hat{P}(T, \Delta)$ as $\hat{y}(T, \frac{1}{\Delta})$. Since a large n implies a small Δ, We could approximate $P(T, \Delta)$ by its Taylor expansion at $\Delta = 0$:

$$\hat{P}(T, \Delta) \approx \hat{P}(T, 0) + \hat{P}'(T, 0)\Delta + \frac{1}{2}\hat{P}''(T, 0)\Delta^2$$

Substituting in $\Delta = 1$, $\Delta = \frac{1}{2}$, and $\Delta = \frac{1}{3}$ leads to three equations in the three unknowns $\hat{P}(T, 0)$, $\hat{P}'(T, 0)$, and $\hat{P}''(T, 0)$. Inverting the system implies that the three-point extrapolation is given by

$$\hat{P}(T, 0) = \frac{1}{2}P(T, 1) - 4P(T, \frac{1}{2}) + \frac{9}{2}P(T, \frac{1}{3})$$

From Marchuk and Shaidurov (1983, p. 24), an $N-$point Richardson extrapolation $\hat{P}^N(T, 0)$ is the following weighted average of N values of American put options with *Erlang* distributed expiration dates:

$$\hat{P}^N(T, 0) = \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} P(T, \frac{1}{n}).$$

An $N-$point Richardson extrapolation for the optimal exercise boundary at time t can be obtained by the following sum:

$$\hat{\underline{S}}^N(t) = \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} \underline{S}_n(T-t)$$

where $\underline{S}_n(T-t)$ is the optimal exercise boundary determined by (5.4.11). Figure 5.3 shows the prices of American and European put options for different initial stock prices with five-point Richardson extrapolation. The parameters are the following: $K = 80$, $\sigma = 0.3$, $r = 0.1$, $T = 1$. Figure 5.4 shows the optimal exercise boundary with five-point Richardson extrapolation.

Figure 5.3: Price of American put option

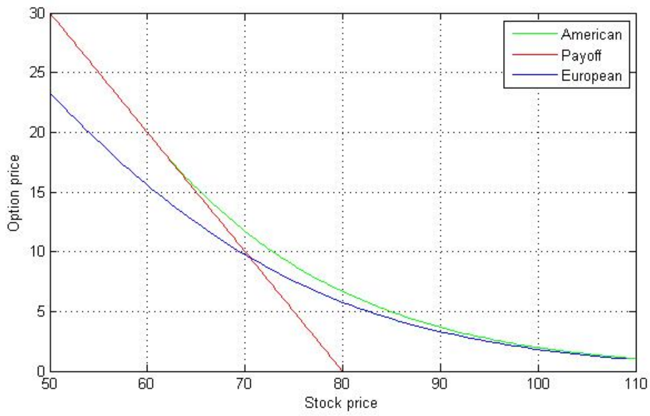
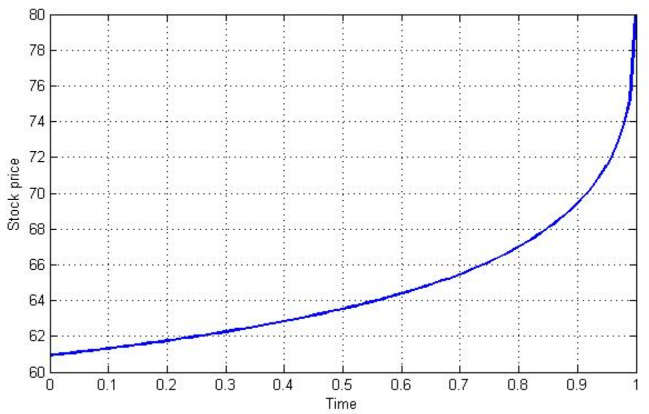


Figure 5.4: Optimal exercise boundary for American put option



5.6 American exchange option

In this section, I shall calculate the price of an American exchange option whose expiry date is *Erlang* distributed. Similar to the exponential case, the decision to exercise an option depends on the ratio of the two stock prices. We calculate the price of this option as the sum of the European style option and the early exercise premium. The early exercise premium can be represented as

$$\mathrm{E}\left[\int_0^{T_n} e^{-rt}(\delta_1 S_1(t) - \delta_2 S_2(t)) I(\frac{S_1(t)}{S_2(t)} > \underline{c}_t)\mathrm{d}t\right]$$

Here $\underline{c}_t$ is the optimal exercise boundary. Namely, an American exchange option should be optimally exercised when the price ratio of two stocks is larger than the optimal exercise boundary. Similarly to the put options which we discussed in last section , the optimal exercise boundary for an American exchange option takes the form of a piece wise constant . We assume

$$\underline{c}_n > \underline{c}_{n-1} > \underline{c}_{n-2} > \ldots\ldots > \underline{c}_1 > c_0 = 1$$

is the optimal exercise boundary in each sub-period. Then, if the initial ratio of two stock prices is smaller than exercise boundary $\underline{c}_n$, the expectation of the early exercise premium can be represented as

$$\sum_{k=1}^{n}\left[\int_{T_{k-1}}^{T_k} e^{-rt}\left(\delta_1 S_1(t) - \delta_2 S_2(t)\right) I(\frac{S_1(t)}{S_2(t)} > \underline{c}_{n+1-k})\mathrm{d}t\right]$$

The above equals to

$$\sum_{k=1}^{m}\left[\int_{T_{k-1}}^{T_k} e^{-rt}\left(\delta_1 S_1(t) - \delta_2 S_2(t)\right) I(\frac{S_1(t)}{S_2(t)} > \underline{c}_{n+1-k})\mathrm{d}t\right] + \sum_{k=m+1}^{n}\left[\int_{T_{k-1}}^{T_k} e^{-rt}\left(\delta_1 S_1(t) - \delta_2 S_2(t)\right)\left(1 - I(\frac{S_1(t)}{S_2(t)} < \underline{c}_{n+1-k})\right)\mathrm{d}t\right]$$

Also, we can see the expectation of

$$\sum_{k=m+1}^{n}[\int_{T_{k-1}}^{T_k} e^{-rt}(\delta_1 S_1(t) - \delta_2 S_2(t))\mathrm{d}t]$$

is the same as that of[3]

$$e^{-rT_m}[S_1(T_m) - S_2(T_m)] + e^{-rT_n}[S_2(T_n) - S_1(T_n)]$$

According to put-call parity, we have

$$[S_1(T_n) - S_2(T_n)]_+ + [S_2(T_n) - S_1(T_n)] = [S_2(T_n) - S_1(T_n)]_+$$

Therefore, the price of an American exchange option equals to the expectation of

$$\begin{aligned}
&e^{-rT_n}[S_2(T_n) - S_1(T_n)]_+ + e^{-rT_m}[S_1(T_m) - S_2(T_m)] \\
&+ \sum_{k=1}^{m}\left[\int_{T_{k-1}}^{T_k} e^{-rt}\left(\delta_1 S_1(t) - \delta_2 S_2(t)\right) I(\frac{S_1(t)}{S_2(t)} > \underline{c}_{n+1-k})\mathrm{d}t\right] \\
&- \sum_{k=m+1}^{n}\left[\int_{T_{k-1}}^{T_k} e^{-rt}\left(\delta_1 S_1(t) - \delta_2 S_2(t)\right) I(\frac{S_1(t)}{S_2(t)} < \underline{c}_{n+1-k})\mathrm{d}t\right]
\end{aligned} \quad (5.6.2)$$

[3]

$$\begin{aligned}
&\mathrm{E}\left\{\sum_{k=m+1}^{n}[\int_{T_{k-1}}^{T_k} e^{-rt}(\delta_1 S_1(t) - \delta_2 S_2(t))\mathrm{d}t]\right\} \\
&= \mathrm{E}\left\{\int_{T_m}^{T_n} e^{-rt}(\delta_1 S_1(t) - \delta_2 S_2(t))\mathrm{d}t\right\} \\
&= \mathrm{E}\left\{\int_{T_m}^{T_n} e^{-rt}\delta_1 S_1(t)\mathrm{d}t\right\} - \mathrm{E}\left\{\int_{T_m}^{T_n} e^{-rt}\delta_2 S_2(t)\mathrm{d}t\right\}
\end{aligned} \quad (5.6.1)$$

According to *Lemma* 5.4.2, , we have

$$\mathrm{E}\left\{\int_{T_m}^{T_n} e^{-rt}\delta_1 S_1(t)\mathrm{d}t\right\} = \mathrm{E}\left\{e^{-rT_m}S_1(T_m)\right\} - \mathrm{E}\left\{e^{-rT_n}S_1(T_n)\right\}$$

$$\mathrm{E}\left\{\int_{T_m}^{T_n} e^{-rt}\delta_2 S_2(t)\mathrm{d}t\right\} = \mathrm{E}\left\{e^{-rT_m}S_2(T_m)\right\} - \mathrm{E}\left\{e^{-rT_n}S_2(T_n)\right\}$$

Before we calculate the price of an American exchange option with an *Erlang* distributed expiration date, we provide the following Lemmas.

Lemma 5.6.1. *If* $\frac{S_1(0)}{S_2(0)} < h$*, we have the following identity*

$$\mathrm{E}\left[\int_{T_{k-1}}^{T_k} e^{-rt}(\delta_1 S_1(t) - \delta_2 S_2(t)) I(\frac{S_1(t)}{S_2(t)} > h) dt\right] = a_k\left(S_1(0),\, S_2(0)\right) \tag{5.6.3}$$

and

$$\begin{aligned} &a_k(s_1,\, s_2) = \\ &\frac{s_1\delta_1}{\lambda}\left(\frac{s_1/s_2}{h}\right)^{-\alpha_1} \frac{\kappa_1^k}{(-\alpha_1)^k} \times \sum_{j=0}^{k-1} \frac{(-\alpha_1)^j \left(\ln\frac{h}{s_1/s_2}\right)^j}{j!} \sum_{l=0}^{k-j-1} \frac{(-\alpha_1)^l}{(-\alpha_1+\beta_1)^l} \binom{k-1+l}{l} \\ &- \frac{s_2\delta_2}{\lambda}\left(\frac{s_1/s_2}{h}\right)^{\beta_2} \frac{\kappa_2^k}{\beta_2^k} \times \sum_{j=0}^{k-1} \frac{\beta_2^j \left(\ln\frac{h}{s_1/s_2}\right)^j}{j!} \sum_{l=0}^{k-j-1} \frac{\beta_2^l}{(\beta_2-\alpha_2)^l} \binom{k-1+l}{l} \end{aligned} \tag{5.6.4}$$

where $\beta_1 > 0,\, \alpha_1 < 0$ *are the roots of the quadratic equation (4.3.12) and* $\beta_2 > 0,\, \alpha_2 < 0$ *are the roots of the quadratic equation (4.3.5) .*

Proof. The left-hand side of (5.6.3) can be calculated using *Lemma 5.4.1*

$$\begin{aligned} &\mathrm{E}\left[\int_{T_{k-1}}^{T_k} e^{-rt}(\delta_1 S_1(t) - \delta_2 S_2(t)) I(\frac{S_1(t)}{S_2(t)} > h)\mathrm{d}t\right] \\ &= \frac{1}{\lambda}\mathrm{E}\left[e^{-rT_k}(\delta_1 S_1(T_k) - \delta_2 S_2(T_k)) I(\frac{S_1(T_k)}{S_2(T_k)} > h)\right] \end{aligned}$$

The above equals to

$$
\begin{aligned}
&\frac{\delta_1}{\lambda}\int_0^\infty f(t)\mathrm{E}\left[e^{-rt}S_1(t)\right]\times\mathrm{E}\left[I(\frac{S_1(t)}{S_2(t)}>h);\begin{pmatrix}1\\0\end{pmatrix}\right]\mathrm{d}t\\
&-\frac{\delta_2}{\lambda}\int_0^\infty f(t)\mathrm{E}\left[e^{-rt}S_2(t)\right]\times\mathrm{E}\left[I(\frac{S_1(t)}{S_2(t)}>h);\begin{pmatrix}0\\1\end{pmatrix}\right]\mathrm{d}t\\
&=\frac{\delta_1}{\lambda}\mathrm{E}\left[e^{-\delta_1 T_k}S_1(0)I(\frac{S_1(T_k)}{S_2(T_k)}>h);\begin{pmatrix}1\\0\end{pmatrix}\right]\\
&-\frac{\delta_2}{\lambda}\mathrm{E}\left[e^{-\delta_2 T_k}S_2(0)I(\frac{S_1(T_k)}{S_2(T_k)}>h);\begin{pmatrix}0\\1\end{pmatrix}\right]
\end{aligned}
$$

To calculate the above, we use Lemma 5.3.2 for the asset $\frac{S_1(0)}{S_2(0)}e^{X_1(t)-X_2(t)}$ under the different measures and the discount rates. If $\frac{S_1(0)}{S_2(0)} < h$, the above equals $a_k\left(S_1(0),\, S_2(0)\right)$ where $a_k(s_1,\, s_2)$ is defined as (5.6.4). □

Similarly to the Lemma 5.6.1, we have Lemma 5.6.2 for the reverse inequality. The proof is similar to the proof of Lemma 5.6.1.

Lemma 5.6.2. *If* $\frac{S_1(0)}{S_2(0)} > l$*, we have the following identity*

$$
\mathrm{E}\left[\int_{T_{k-1}}^{T_k} e^{-rt}(\delta_1 S_1(t)-\delta_2 S_2(t))I(\frac{S_1(t)}{S_2(t)}<l)dt\right]=b_k\left(S_1(0),\, S_2(0)\right) \tag{5.6.5}
$$

and

$$
\begin{aligned}
&b_k(s_1,\, s_2)\\
&=\frac{s_1\delta_1}{\lambda}\left(\frac{l}{s_1/s_2}\right)^{\beta_1}\frac{\kappa_1^k}{\beta_1^k}\sum_{j=0}^{k-1}\frac{\left(\ln\frac{s_1/s_2}{l}\right)^j\beta_1^j}{j!}\sum_{l=0}^{k-j-1}\frac{\beta_1^l}{(\beta_1-\alpha_1)^l}\binom{k-1+l}{l}\\
&-\frac{s_2\delta_2}{\lambda}\left(\frac{l}{s_1/s_2}\right)^{-\alpha_2}\frac{\kappa_2^k}{(-\alpha_2)^k}\sum_{j=0}^{k-1}\frac{\left(\ln\frac{s_1/s_2}{l}\right)^j(-\alpha_2)^j}{j!}\sum_{l=0}^{k-j-1}\frac{(-\alpha_2)^l}{(\beta_2-\alpha_2)^l}\binom{k-1+l}{l}
\end{aligned}
$$

Lemma 5.6.3 and Lemma 5.6.4 calculate the price of an out-of-the-money European exchange option whose expiry date is *Erlang* distributed.

Lemma 5.6.3. *If* $\frac{S_1(0)}{S_2(0)} > 1$*, the price of an out-of-the-money European exchange option* S_2 *for* S_1 *equals*

$$\mathrm{E}\left[e^{-rT_n}[S_2(T_n) - S_1(T_n)]_+\right] = e_2^n(S_1(0)\, S_2(0))$$

where

$$\begin{aligned} e_2^n(s_1\, s_2) =& s_2 \left(\frac{s_2}{s_1}\right)^{\beta_1 - 1} \frac{\kappa_1^n}{(\beta_1 - 1)^n} \sum_{i=0}^{n-1} \frac{\left((\beta_1 - 1)\ln\frac{s_1}{s_2}\right)^i}{i!} \sum_{l=0}^{n-i-1} \frac{\binom{n-1+l}{l}}{(\beta_1 - \alpha_1)^l} (\beta_1 - 1)^l \\ &- s_1 \left(\frac{s_2}{s_1}\right)^{\beta_1} \frac{\kappa_1^n}{\beta_1^n} \sum_{i=0}^{n-1} \frac{\left(\beta_1 \ln\frac{s_1}{s_2}\right)^i}{i!} \sum_{l=0}^{n-i-1} \frac{\binom{n-1+l}{l}}{(\beta_1 - \alpha_1)^l} \beta_1^l \end{aligned} \tag{5.6.6}$$

Proof. The price of an out-of-the-money European exchange option S_2 for S_1 equals

$$\begin{aligned} &\mathrm{E}\left[e^{-rT_n}[S_2(T_n) - S_1(T_n)]_+\right] \\ =&\mathrm{E}\left[e^{-rT_n} S_1(T_n)[\frac{S_2(T_n)}{S_1(T_n)} - 1]_+\right] \\ =&\int_0^\infty f(t)\mathrm{E}\left[e^{-rt} S_1(t)\right] \times \mathrm{E}\left[[\frac{S_2(t)}{S_1(t)} - 1]_+; \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] \mathrm{d}t \\ =&\mathrm{E}\left[e^{-\delta_1 T_n} S_1(0)[\frac{S_2(t)}{S_1(t)} - 1]_+; \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] \end{aligned}$$

If $\frac{S_1(0)}{S_2(0)} > 1$, the above is the price of an out-of-the-money European call option for asset $\frac{S_2(0)}{S_1(0)} e^{X_2(t) - X_1(t)}$ under the different measure and the discount rate δ_1. Using *Lemma* 5.3.2, the above equals to $e_2^n(S_1(0)\, S_2(0))$, where $e_2^n(s_1\, s_2)$ is defined as (5.6.6). □

Similar to the calculation of the exchange option S_2 for S_1, the price of an out-of-the-money European exchange option S_1 for S_2 is given in Lemma 5.6.4. The proof is similar as the proof of Lemma 5.6.3.

Lemma 5.6.4. *If* $\frac{S_1(0)}{S_2(0)} < 1$*, the price of an out-of-the-money European exchange* *option* S_1 *for* S_2 *equals*

$$\mathrm{E}\left[e^{-rT_n}[S_1(T_n) - S_2(T_n)]_+\right] = e_1^n(S_1(0)\, S_2(0))$$

where

$$\begin{aligned} e_1^n(s_1,\, s_2) =& s_1 \left(\frac{s_1}{s_2}\right)^{\beta_2 - 1} \frac{\kappa_2^n}{(\beta_2 - 1)^n} \sum_{i=0}^{n-1} \frac{\left((\beta_2 - 1)\ln\frac{s_2}{s_1}\right)^i}{i!} \sum_{l=0}^{n-i-1} \frac{\binom{n-1+l}{l}}{(\beta_2 - \alpha_2)^l} (\beta_2 - 1)^l \\ & - s_2 \left(\frac{s_1}{s_2}\right)^{\beta_2} \frac{\kappa_2^n}{\beta_2^n} \sum_{i=0}^{n-1} \frac{\left(\beta_2 \ln\frac{s_2}{s_1}\right)^i}{i!} \sum_{l=0}^{n-i-1} \frac{\binom{n-1+l}{l}}{(\beta_2 - \alpha_2)^l} \beta_2^l \end{aligned} \tag{5.6.7}$$

The price of the American exchange option with *Erlang* distributed expiry date is given in the following theorem.

Theorem 5.6.1. *The price of the American exchange option with Erlang distributed expiry date equals*

$$\begin{cases} e_1^n\left(S_1(0),\, S_2(0)\right) + \sum_{k=1}^{n} a_k\left(S_1(0),\, S_2(0)\right) & \frac{S_1(0)}{S_2(0)} < 1 \\ e_2^n\left(S_1(0),\, S_2(0)\right) + v_i^n\left(S_1(0),\, S_2(0)\right) & \\ \quad + \sum_{k=1}^{n-i+1} a_k\left(S_1(0),\, S_2(0)\right) - \sum_{k=n-i+2}^{n} b_k\left(S_1(0),\, S_2(0)\right) & \frac{S_1(0)}{S_2(0)} \in (\underline{c}_{i-1},\, \underline{c}_i] \\ S_1(0) - S_2(0) & \frac{S_1(0)}{S_2(0)} \geq \underline{c}_n \end{cases} \tag{5.6.8}$$

where $e_1^n(s_1,\, s_2)$*,* $e_2^n(s_1,\, s_2)$*,* $a_k^n(s_1,\, s_2)$*,* $b_k^n(s_1,\, s_2)$*, is defined above, and* $v_i^n(s_1,\, s_2)$ *is defined as*

$$v_i^n(s_1,\, s_2) = s_1 \left(\frac{\lambda}{\lambda + \delta_1}\right)^{n-i+1} - s_2 \left(\frac{\lambda}{\lambda + \delta_2}\right)^{n-i+1}$$

Proof. For the case of $\frac{S_1(0)}{S_2(0)} \geq \underline{c}_n$, we need to exercise the option immediately. Therefore the price of the American exchange option with *Erlang* distributed expiry date equals to its exercise value $S_1(0) - S_2(0)$. For the case of $\frac{S_1(0)}{S_2(0)} \in (\underline{c}_{i-1},\, \underline{c}_i]$, we need to calculate the expected value of (5.6.2). Similar to the proof of

Theorem 5.4.1, we choose $m = n-i+1$. The expectation of $e^{-rT_{n-i+1}}[S_1(T_{n-i+1}) - S_2(T_{n-i+1})]$ equals

$$S_1(0)\left(\frac{\lambda}{\lambda+\delta_1}\right)^{n-i+1} - S_2(0)\left(\frac{\lambda}{\lambda+\delta_2}\right)^{n-i+1}$$

The expectation of the remaining part of (5.6.2) is calculated in Lemma 5.6.1, Lemma 5.6.2 and Lemma 5.6.3. For the case of $\frac{S_1(0)}{S_2(0)} < 1$, we calculate the expectation of

$$e^{-rT_n}[S_1(T_n) - S_2(T_n)]_+ + \sum_{k=1}^{n}\left[\int_{T_{k-1}}^{T_k} e^{-rt}\left(\delta_1 S_1(t) - \delta_2 S_2(t)\right) I\left(\frac{S_1(t)}{S_2(t)} > \underline{c}_{n+1-k}\right)\mathrm{d}t\right].$$

Using Lemma 5.6.1 and Lemma 5.6.4, we can obtain the first line of (5.6.8). □

Remark 5.6.1. A put option could been thought as a special case of the exchange option. If stock one is a fixed number, pricing formula (5.6.8) will be degenerated to (5.4.9).

The optimal exercise boundary $\underline{c}_i$, $i = 1, 2 \ldots$ can be determined by imposing the “value matching” condition at the optimal exercise boundary.

CHAPTER 6

FITTING LIFE DISTRIBUTION

6.1 Introduction

A key idea used in my thesis is that the distribution of $T(x)$, the time-until-death random variable, can be approximated by combinations of exponential distributions or mixtures of *Erlang* distributions. In this Chapter, I would first propose a method to illustrate how to approximate the density function of T_x with a linear combination of exponential densities. Then I extend this method to estimate the linear coefficients, the shape parameters, and the rate parameter of mixtures of *Erlang* distributions with the common rate parameter.

As shown in Dufresne (2007a) and Ko and Ng (2007), combinations of exponential distributions are a (weakly) dense subset in the space of all probability distributions with support R_+. Hence any positive distribution can be approximated by combinations of exponential distributions. Note that the linear coefficients are not restricted to be non-negative. Dufresne (2007a) also proposes a non-statistical method to estimate the parameters. By introducing Jacobian polynomials, the parameters are obtained by integrating various polynomials. The efficiency of Dufresne's method highly depends on the several parameters to be specified in advance, but he does not provide a scheme for selecting the parameters.

Because the exponential distribution is a one-parameter distribution, and its coefficient of variation is one, to approximate well a distribution with a coefficient of variation that is far from unity, we need a large number of exponential distributions. In contrast, the *Erlang* distribution is a two-parameter distribution with larger degree of freedom. Fewer *Erlang* distributions are needed to achieve the same accuracy. Similarly to combinations of exponential distributions, it is shown in Tijms (1994, p.163) that mixtures of *Erlang* distributions with the same rate parameter are dense in the space of positive distributions. Tijms (1994, p.163) also provides a mathematical method of parameter estimation, but the estimation is not satisfactory, because to fit the data well, a large number of *Erlang* distributions must be used, which results in slow convergence and overfitting problem. In other words, it is not practical to directly use the approximation in Tijms (1994, p.163). Following Tijms's work, Lee and Lin (2010) iteratively use the EM algorithm in Dempster, A. P., Laird, N. M., & Rubin, D. B. (1977) to estimate the parameters of a mixture of *Erlang* distributions with a common rate parameter. Lee and Lin (2010) also propose an adjustment and diagnosis procedure to identify the shape parameters of *Erlang* distributions. Though

In section 6.2, I propose a method to estimate the linear coefficients and rate parameters of combinations of exponential distributions. Through splitting the original problem into two sub-problems, the linear optimization and nonlinear optimization, the results are more robust to the initial guess. In section 6.3, I apply the adjustment procedure provided in Lee and Lin (2010) to extend this method to estimate the parameters of mixtures of *Erlang* distributions. In section 6.4, I numerically fit the life table data to combinations of exponential distributions and mixtures of *Erlang* distributions and evaluate the fitting results.

6.2 Combinations of exponential distributions

The survival distribution of the class of approximating distributions, combinations of exponential distributions, has the following representation,

$$ {}_tp_x = \sum_{j=1}^{n} \alpha_j e^{-\lambda_j t},\ t \geq 0,\ \lambda_j > 0,\ j = 1, \dots, n,\ 1 \leq n < \infty $$

To fit combinations of exponential distributions to the distribution of $T(x)$, we first calculate the survival probability through the life table,

$$ {}_k\hat{p}_x = P(T(x) > k) = \frac{l_{x+k}}{l_x} $$

Here l_x is the number of living people at age x. Mathematically, the fitting problem can be formalized by the following. First we make a choice for n, the number of exponential distributions. Then we seek the parameters $\alpha_1, \cdots, \alpha_n, \lambda_1, \cdots, \lambda_n$, which minimize the weighted sum of squares,

$$\sum_{k\geq 1} w_k \left[{}_k p_x - \sum_{j=1}^{n} \alpha_j e^{-\lambda_j k} \right]^2, \tag{6.2.1}$$

subject to

$$\sum_{j=1}^{n} \alpha_j = 1$$

and $\lambda_1 > 0, \cdots, \lambda_n > 0$. The above fitting problem (6.2.1) can be split into two sub-problems, the linear and nonlinear problems. The linear problem is to solve the following optimization:

$$\boldsymbol{\alpha^{opt}}(\lambda_1, \cdots, \lambda_n) = \underset{\alpha_1, \cdots, \alpha_n}{\arg\min} \sum_{k\geq 1} w_k \left[{}_k p_x - \sum_{j=1}^{n} \alpha_j e^{-\lambda_j k} \right]^2, \tag{6.2.2}$$

Here we use $\boldsymbol{\alpha^{opt}}$ to denote the optimized vector for $(\alpha_1, \cdots, \alpha_n)$. Once the rate parameters λ_is are given, the above is a linear programming, which has a unique global minimizer. We replace the linear coefficients of function with the optimal value $\boldsymbol{\alpha^{opt}} = (\alpha_1^{opt}, \cdots, \alpha_n^{opt})$, and obtain the following:

$$f(\lambda_1, \cdots, \lambda_n, k\,; \boldsymbol{\alpha^{opt}}) = \sum_{j=1}^{n} \alpha_j^{opt} e^{-\lambda_j k} \tag{6.2.3}$$

The non-linear optimization is to seek the rate parameters $\lambda_1, \cdots, \lambda_n$ to minimize the following:

$$\sum_{k\geq 1} w_k \left[{}_k p_x - f(\lambda_1, \cdots, \lambda_n, k\,; \boldsymbol{\alpha^{opt}}) \right]^2.$$

The above optimization is equivalent to the the fitting problem (6.2.1). It is a nonlinear optimization problem, and furthermore, it is not a convex optimization problem. Therefore, only the local minimizer can be obtained. We use the trust region method, which is a numerical optimization method tailored for non-linear optimization, to numerically calculate the optimal λ_is. Compared to directly solving the optimization problem (6.2.1), the split problem has two more advantages: 1)It is more robust to the initial guess. 2)It is more likely to obtain the global minimizer.

Remark 6.2.1. If we would like to value life-contingent options that will expire at a fixed time T, we could approximate

$$ {}_tp_x I(t < T) $$

by combinations of exponential distributions, and use the results in Chapter 4. However, we need a large number of exponential functions to obtain a reasonably good approximation. This is because that the above function has a big jump from ${}_Tp_x$ to 0 at $t = T$. Here, we suggest the method presented in section 10 of Gerber, Shiu and Yang (2012). Using the memoryless property of exponential distribution, it derives the analytic pricing formula for the options that will expire at a fixed time T.

6.3 Mixture of the *Erlang* distributions

The density function of m mixtures of *Erlang* distributions with common rate parameter λ is defined as

$$ f(x) = \sum_{i=1}^{m} \alpha_i e^{-\lambda x} \frac{\lambda^{r_i} x^{r_i - 1}}{(r_i - 1)!} $$

with $\alpha > 0$, and $\sum_{i=1}^{m} \alpha_i = 1$. Integrating the density function from x to ∞, we have the survival distribution

$$ \bar{F}(x) = \sum_{i=1}^{m} \alpha_i \int_x^{\infty} e^{-\lambda t} \frac{\lambda^{r_i} t^{r_i - 1}}{(r_i - 1)!} dt \tag{6.3.1} $$

It is shown in Tijms (1994, p. 163) that the probability distribution of any positive random variable can be approximated by a mixture of *Erlang* distributions with the same rate parameter. Detailed proof will be provided in the Appendix. Mathematically, the fitting problem could be characterized as follows. We seek the parameters $\alpha_1, \cdots, \alpha_m, r_1, \cdots, r_m, \lambda$, which minimize the weighted sum of squares,

$$\sum_{k\geq 1} w_k \left[{}_k p_x - \sum_{i=1}^{m} \alpha_i \int_x^\infty e^{-\lambda t} \frac{\lambda^{r_i} t^{r_i - 1}}{(r_i - 1)!} dt \right]^2,$$

subject to

$$\sum_{j=1}^{m} \alpha_j = 1$$

and $\alpha_1 > 0, \cdots, \alpha_m > 0$, $r_1, r_2 \cdots, r_m$ are integers and $\lambda > 0$. For a fixed set of shape parameters, we would use the algorithm provided in the last section. The difference here is that the rate parameter for each *Erlang* distribution is the same.

After applying the algorithm we provided above, the rate parameter and the linear coefficient locally minimize the distance between the empirical distribution and the desired distribution (mixtures of *Erlang* distributions). It only locally minimizes the distance because the shape parameters are not altered by the algorithm. We need to consider the shape parameters in a larger set. Since it is impossible to consider the whole set of the natural numbers, we use the following procedure to expand the selection of the parameters. This procedure is motivated by the work of Lee and Lin (2010).

1) Initially, the algorithm is run for the set of shape parameters $\{r_1, r_2, \cdots, r_m\}$.

2) The algorithm will be run again for the set $\{r_1, r_2, \cdots, r_m + 1\}$. If the new distance between the empirical distribution and the desired distribution is lower, the new parameter set replaces the old one. Otherwise, we go to the next step. This step is repeated until the distance between the empirical distribution and the desired distribution is not improved.

3) This procedure is then applied to the $(m-1)$th shape and so forth until all the shapes are treated.

4) Repeat steps 2 and 3 in a similar fashion, but instead of increasing the value of the shape parameters, we will decrease them. We will run the algorithm starting from the shapes $\{r_1, r_2, \cdots, r_m - 1\}$, the final estimates of the previous execution of the algorithm.

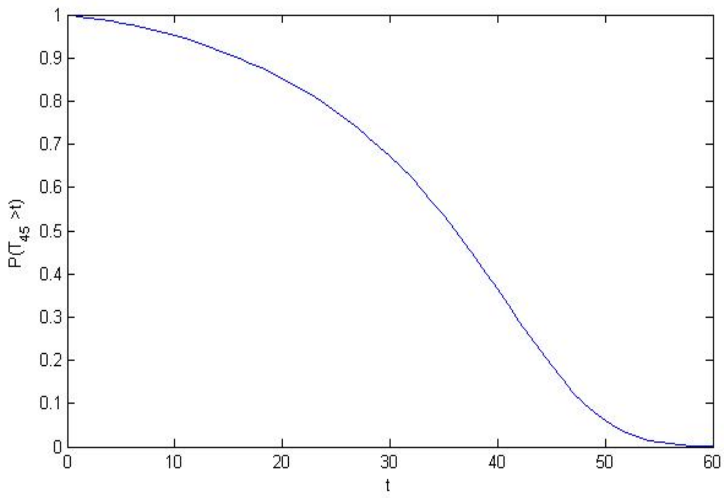
Another point I need to mention is the initialization of the parameters. A good initialization of the parameters can improve the computational speed. Similar to Lee and Lin (2010), our initialization is also based on the method provided by Tijm (1994, p. 163) which ensures good starting values and fast convergence. First we choose an m, the number of *Erlang* distributions. Based on the Tijm's approximation, the shape parameters of each *Erlang* distribution are $r_i = i$, $i = 1, 2, \ldots m$. The common rate parameter λ is chosen such that $\frac{m}{\lambda}$ is approximately equal to the maximum data point. Notice that the *Erlang* distributions do not need to share the same rate parameter. If the rate parameter can be different among the *Erlang* distributions, we may have a better approximation and with fewer terms.

6.4 Numerical results

In this section we use an example to evaluate the fitting results. We use the life table for males published by the US Social Security Department in 2013. The data set is from http://www.ssa.gov/oact/STATS/table4c6.html. All computations are performed with MATLAB, and the description of the functions is provided in the Appendix. Figure 6.1 shows the empirical survival distribution of $T(45)$.

Figure 6.2 shows the fitting results using 4, 6, 8 and 10-terms of exponential distributions. Here, the weights have been set equally. Since the linear coefficients are not restricted to be positive, the fitting distributions may be negative in place. The fitting result is poor when we use 4 terms, but the 8-term approximation does better. The 4-term approximation and 8-term approximation of the survival

Figure 6.1: Empirical value for $P(T(45) > t)$



distribution is given below

$$\bar{F}_4^A(t) = 10^7 \times (-8.766e^{-0.1487t} + 4.068e^{-0.1485t} + 4.72e^{-0.1489t} - 0.025e^{-0.1525t})$$

and

$$\begin{aligned}\bar{F}_8^A(t) =& 10^{12} \times (0.1981e^{-0.0721t} - 0.092e^{-0.0736t} + 2.4 \times 10^{-5}e^{-0.067t} - 0.1654e^{-0.072t} \\ & +2.157e^{-0.0733t} - 2.098e^{-0.0734t} + 4.1 \times 10^{-9}e^{-0.1534t} - 8.29 \times 10^{-12}e^{-0.2865t})\end{aligned}$$

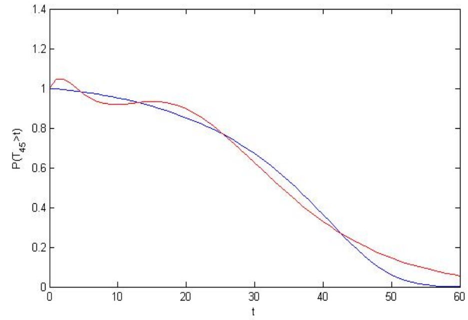
The precision is given below,

$$\|\bar{F} - \bar{F}_4^A\| = 0.153, \quad \|\bar{F} - \bar{F}_6^A\| = 0.015, \quad \|\bar{F} - \bar{F}_8^A\| = 0.0011, \quad \|\bar{F} - \bar{F}_{10}^A\| = 5.79 \times 10^{-4}$$

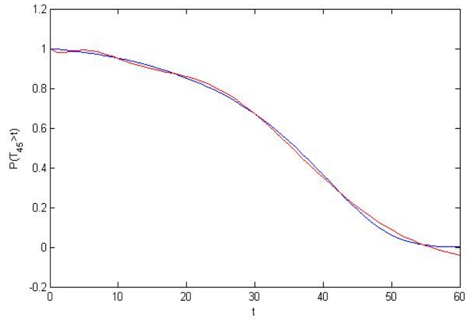
Figure 6.3 shows the approximation of mixtures of 3, 4, 5 and 6 terms *Erlang* distributions. Similar to the approximation using exponentials, the weights have been set equally. Since the *Erlang* distribution is a two-parameter distribution, fewer terms is needed to fit the same distribution. Also, since the linear coefficients for the mixture distribution are restricted to be positive, the fitting distributions are positive everywhere. The 3-term approximation and 6-term approximation of

Figure 6.2: Exponential fitting for $P(T_{45} > t)$

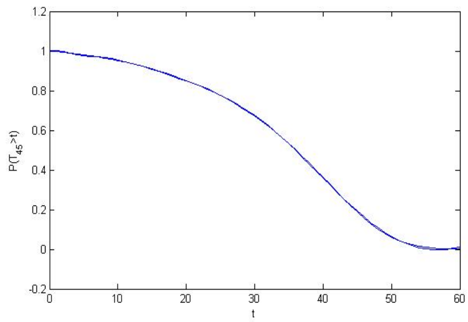
(a) 4 terms



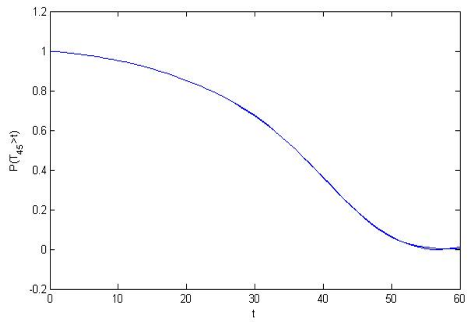
(b) 6 terms



(c) 8 terms



(d) 10 terms



the probability density functions are given below

$$f_3^A(t) = 0.0128e^{-0.37t}0.37 + 0.1273e^{-0.37t}\frac{0.37^5t^4}{4!} + 0.86e^{-0.37t}\frac{0.37^{14}t^{13}}{13!}$$

and

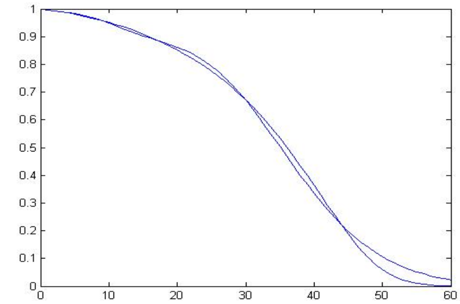
$$\begin{aligned} f_6^A(t) =& 0.0044e^{-0.716t}0.716 + 0.0186e^{-0.716t}\frac{0.716^3t^2}{2!} + 0.0448e^{-0.716t}\frac{0.716^8t^7}{7!} \\ &+ 0.0260e^{-0.716t}\frac{0.716^9t^8}{8!} + 0.2085e^{-0.716t}\frac{0.716^{17}t^{16}}{16!} + 0.6977e^{-0.716t}\frac{0.716^{29}t^{28}}{28!} \end{aligned}$$

The precision is given below,

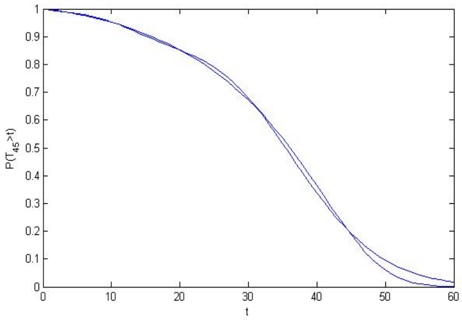
$$\|\bar{F}-\bar{F}_3^A\| = 0.0327, \quad \|\bar{F}-\bar{F}_4^A\| = 0.0199, \quad \|\bar{F}-\bar{F}_5^A\| = 0.0094, \quad \|\bar{F}-\bar{F}_6^A\| = 0.0038$$

Figure 6.3: *Erlang* fitting for $P(T(45) > t)$

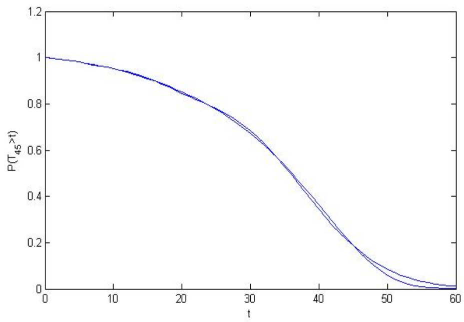
(a) 3 terms



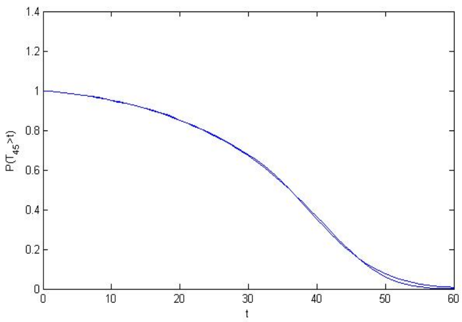
(b) 4 terms



(c) 5 terms



(d) 6 terms



6.5 Appendix

6.5.1 Matlab functions

Syntax: y = empirical_distribution(t, data, x)

Description: y = empirical_distribution() returns an array of empirical survival distribution. Specifically, it calculates the quantity ${}_tp_x$, which is the probability that (x) survives to age x+t.

Input arguments:

- data: a column vector represents the number of surviving people at each age. The data is from the life table.
- x: a number represents the age of a person.
- t: a vector represents how many more years (x) will survive.

Syntax: [lambda,linear_coeff] = fit_nonlinear(lambda_0, t, data, x, lb)

Description: [lambda,linear_coeff] = fit_nonlinear() returns the linear coefficients and the exponential coefficients for the approximate distribution.

Input arguments:

- lambda_0 : a vector represents the initial guess of λ.
- data: a column vector represents the number of surviving people at each age. The data is from the life table.
- x: a number represents the age of a person.
- t: a vector represents how many more years (x) will survive.
- lb: a number represents the lower bound of lambdas. The default value for lb is zero.

Syntax: [shape, rate, linear_coeff, fitted, norm] = fit_erlang(t,rate0, data,x,n_shape, max_shape)

Description: [shape, rate, linear_coeff, fitted, norm] = fit_erlang() returns the linear coefficients, the shape parameters and the rate parameters for the approximate distribution.

Output arguments:

- shape: a vector represents the shape parameters of the mixture of Erlang distribution.
- rate: a vector represents the rate parameters of the mixture of Erlang distribution.
- linear_coeff: a vector represents the linear coefficients of the mixture of Erlang distribution.
- fitted: a vector represents the fitted values.
- norm: a number represents the distance between the target distribution and the fitted distribution.

Input arguments:

- rate0 : a number represents the initial guess of the common λ.
- data: a column vector represents the number of surviving people at each age. The data is from the life table.
- x: a number represents the age of a person.
- t: a vector represents how many more years (x) will survive.
- n_shape: a number represents the number of the mixture distribution.
- max_shape: a number represents the largest value of the shape parameters considered.

6.5.2 Approximation by a mixtures of the *Erlang* distribution

The probability distribution of any positive random variable can be arbitrarily closely approximated by a mixture of *Erlang* distributions with the same scale parameter. The theoretical basis for the use of mixtures of *Erlang* distributions is provided in Tijms (1994, p.163).

Theorem 6.5.1. *Let $F(t)$ be the probability distribution function of a positive random variable. For fixed $\Delta > 0$ define the probability distribution function $F_\Delta(x)$ by*

$$F_\Delta(x) = \sum_{j=1}^{\infty} p_j(\Delta) \left\{ 1 - \sum_{k=0}^{j-1} e^{-\frac{x}{\Delta}} \frac{\left(\frac{x}{\Delta}\right)^k}{k!} \right\}, \qquad x \geq 0,$$

where $p_j(\Delta) = F(j\Delta) - F((j-1)\Delta)$, $j = 1, 2, \ldots$. Then

$$\lim_{\Delta \to 0} F_\Delta(x) = F(x)$$

for each continuity point x of $F(t)$.

Proof. For fixed Δ, $x > 0$, let $Y_{\Delta,x}$ be a Poisson distributed random variable with

$$\mathrm{P}(Y_{\Delta,x} = k\Delta) = e^{-\frac{x}{\Delta}} \frac{\left(\frac{x}{\Delta}\right)^k}{k!}, \qquad k = 0, 1, \ldots.$$

The mean and variance of $Y_{\Delta,x}$ equal

$$\mathrm{E}(Y_{\Delta,x}) = x \quad \text{and} \quad \mathrm{Var}(Y_{\Delta,x}) = x\Delta, \text{ respectively.}$$

Let $g(t)$ be a bounded function. We now prove that

$$\lim_{\Delta\to 0} \mathrm{E}(g(Y_{\Delta,x})) = g(x) \tag{6.5.1}$$

for each continuity point x of $g(t)$. To see this, fix $\epsilon > 0$ and a continuity point x of $g(t)$. According to the definition of continuity, there exists a number $\delta > 0$ such that $|g(t) - g(x)| \le \frac{\epsilon}{2}$ for all t with $|t - x| \le \delta$. Also, let $M > 0$ be such that $|g(t)| \le \frac{M}{2}$ for all t. Then

$$\begin{aligned}
|\mathrm{E}(g(Y_{\Delta,x})) - g(x)| &\le \sum_{k=0}^{\infty} |g(k\Delta) - g(x)| \mathrm{P}(Y_{\Delta,x} = k\Delta) \\
&\le \frac{\epsilon}{2} + M \sum_{k:|k\Delta - x| > \delta} \mathrm{P}(Y_{\Delta,x} = k\Delta) \\
&= \frac{\epsilon}{2} + M\mathrm{P}\left\{|Y_{\Delta,x} - \mathrm{E}(Y_{\Delta,x})| > \delta\right\}
\end{aligned}$$

By Tschebyshev's inequality,

$$\mathrm{P}\left\{|Y_{\Delta,x} - \mathrm{E}(Y_{\Delta,x})| > \delta\right\} \le \frac{x\Delta}{\delta^2}$$

For enough small Δ, we have $M\frac{x\Delta}{\delta^2} < \frac{\epsilon}{2}$. This prove (6.5.1). Now we apply (6.5.1) with $g(t) = F(t)$. For each continuity point x of $F(t)$,

$$\begin{aligned}
F(x) = \lim_{\Delta\to 0} \mathrm{E}(F(Y_{\Delta,x})) &= \lim_{\Delta\to 0} \sum_{k=0}^{\infty} F(k\Delta) e^{-\frac{x}{\Delta}} \frac{\left(\frac{x}{\Delta}\right)^k}{k!} \\
&= \lim_{\Delta\to 0} \sum_{k=0}^{\infty} e^{-\frac{x}{\Delta}} \frac{\left(\frac{x}{\Delta}\right)^k}{k!} \sum_{j=1}^{k} p_j(\Delta).
\end{aligned}$$

where the last equality uses that $F(0) = 0$. Interchanging the order of summation, we obtain

$$F(x) = \lim_{\Delta \to 0} \sum_{j=1}^{\infty} p_j(\Delta) \left\{ \sum_{k=j}^{\infty} e^{-\frac{x}{\Delta}} \frac{\left(\frac{x}{\Delta}\right)^k}{k!} \right\}$$

$\square$

If F_Δ, F denote the probability distribution functions of the measures P_Δ, P respectively, the above theorem essentially shows measure P_Δ converges weakly to P.

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