Discussion on “Capital Forbearance, Ex Ante Life Insurance Guaranty Schemes, and Interest Rate Uncertainty,” by Ya-Wen Hwang, Shih-Chieh Chang and Yang-Che Wu, Volume 19(2)

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1 Introduction

Professors Hwang, Chang and Wu are to be congratulated for this interesting paper. The authors study ex ante insurance guaranty schemes. In particular, capital forbearance mechanism and stochastic interest rate are incorporated in their model, and an explicit formula for the risk-based premium of the insurance guaranty fund is derived. I am especially interested in Section 2.3 and Appendix A. In this discussion, I shall provide an alternative derivation of the fair premium $P(0)$ given by formula (13) in the paper using the actuarial method of Esscher transform (Gerber and Shiu 1994, 1996; Shiryaev 1999; Bingham and Kiesel 2013).

The paper decomposes $P(0)$ into three parts: the premium for the audit window component P^a, the premium for the capital forbearance component P^c, and the premium for the grace period component P^ϵ. It follows from formulas (5) to (8) in the paper that $P(0)$ is given by

$$P(0) = P^a + P^c + P^\epsilon,$$

with

$$P^a := \mathrm{E}^Q\left[\exp\left(-\int_0^\tau r(t)\mathrm{d}t\right)(\gamma L(\tau) - A(\tau))\, I_{\{\tau<T\}}\right], \tag{D1}$$

$$P^c := \mathrm{E}^Q\left[\exp\left(-\int_0^T r(t)\mathrm{d}t\right)(\gamma L(T) - A(T))\, I_{\{\tau>T,\ \beta L(T)>A(T)\}}\right], \tag{D2}$$

$$P^\epsilon := \mathrm{E}^Q\left[\exp\left(-\int_0^{T+\epsilon} r(t)\mathrm{d}t\right)(\gamma L(T+\epsilon) - A(T+\epsilon))_+\, I_{\{\tau>T,\ \beta L(T)<A(T)<\alpha L(T)\}}\right]. \tag{D3}$$

Here, $\tau = \inf\{t > 0 | A(t) < \eta L(t)\}$ is the default time, T is the audit time, ϵ is the length of the grace period, $A(t)$ and $L(t)$ are the time-t values of the insurer’s assets and liabilities, respectively. Note from (D.1) that the time of payment τ, which is the default time, occurs before the audit time T; so we must wonder how τ can be determined without auditing.

2 Financial framework and derivation of the risk-neutral measure

In this section, we shall briefly discuss the financial model in the paper and show in detail how a risk-neutral measure is found. Moreover, we shall relate this framework to a geomeric Brownian motion model, which will lead to our alternative derivation in the next section.

The authors consider the Vasicek interest rate model:

$$\mathrm{d}r(t) = \kappa\big(\theta - r(t)\big)\mathrm{d}t + \sigma_r \mathrm{d}W_r(t). \tag{D4}$$

Vasicek (1977) assumes that, in setup (D.4), the market price of interest rate risk for all bonds is a constant, which is denoted by λ_r in this paper. Then a change of measure can be made such that, under the new measure, denoted by Q,

$$W_r^Q(t) := W_r(t) + \lambda_r t \tag{D5}$$

becomes a standard Brownian motion.

In addition to fixed-income securities, a stock index fund is introduced and its dynamic is assumed to be of the form

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu(t)\mathrm{d}t + \sigma_1 \mathrm{d}W_r(t) + \sigma_2 \mathrm{d}W_S(t), \tag{D6}$$

where $(W_r(t), W_S(t))'$ is a standard two-dimensional Brownian motion under the physical measure P.

Defining $W_S^Q(t)$ by

$$W_S^Q(t) := W_S(t) + \int_0^t \frac{\mu(s) - r(s) - \sigma_1\lambda_r}{\sigma_2}\mathrm{d}s, \tag{D7}$$

we see that (D.6) can be rewritten as

$$\frac{\mathrm{d}S(t)}{S(t)} = r(t)\mathrm{d}t + \sigma_1 \mathrm{d}W_r^Q(t) + \sigma_2 \mathrm{d}W_S^Q(t), \tag{D8}$$

where $\left(W_r^Q(t), W_S^Q(t)\right)'$ becomes a standard two-dimensional Brownian motion under the risk-neutral measure Q.

Briefly speaking, the interpretation of (D.8) is that, in a risk-neutral world, all risky securities are expected to earn at the same instantaneous rate as the risk-free cash bond.

By the Girsanov Theorem, the Radon-Nikodym derivative for constructing the measure Q is

$$\left.\frac{\mathrm{d}Q}{\mathrm{d}P}\right|_{\tilde{T}} = \exp\left(-\frac{1}{2}\int_0^{\tilde{T}} \boldsymbol{\lambda}(t)'\boldsymbol{\lambda}(t)\mathrm{d}t - \int_0^{\tilde{T}} \boldsymbol{\lambda}(t)'\mathrm{d}\boldsymbol{W}(t)\right),$$

with $\tilde{T}$ being a sufficiently long time horizon, $\boldsymbol{\lambda}(t) := \left(\lambda_r, \frac{\mu(t)-r(t)-\sigma_1\lambda_r}{\sigma_2}\right)'$ and $\boldsymbol{W}(t) := (W_r(t), W_S(t))'$.

It is worthwhile to mention that the measure Q depends on the value of λ_r, which is exogenously specified. Hence, there are infinitely many risk-neutral measures.

The paper assumes that the insurer invests its assets in a stock index fund, a rolling-horizon bond with fixed horizon date R, and a risk-free cash bond in fixed proportions:

$$\frac{\mathrm{d}A(t)}{A(t)} = w_1\frac{\mathrm{d}S(t)}{S(t)} + w_2\frac{\mathrm{d}B_R(t)}{B_R(t)} + w_3 r(t)\mathrm{d}t, \tag{D9}$$

where $w_i, i = 1, 2, 3$, represent the proportions of assets invested in the corresponding securities.
Obviously, $w_1 + w_2 + w_3 = 1$. It is assumed that all w_i are fixed numbers so that $\{X(t)\}$ defined below in (D.12) is a (linear) Brownian motion.
Combining (D.8), (D.9) and equation (3) in the paper, the risk-neutral process of the asset value is

$$\frac{\mathrm{d}A(t)}{A(t)} = r(t)\mathrm{d}t + \sigma_{A,r}\mathrm{d}W_r^Q(t) + \sigma_{A,S}\mathrm{d}W_S^Q(t), \tag{D10}$$

with $\sigma_{A,r} := w_1\sigma_1 + w_2\sigma_R$ and $\sigma_{A,S} := w_1\sigma_2$.
Define $W_A^Q(t) := \frac{\sigma_{A,r}W_r^Q(t)+\sigma_{A,S}W_S^Q(t)}{\sigma_A}$ with $\sigma_A := \sqrt{\sigma_{A,r}^2 + \sigma_{A,S}^2}$, then

$$\frac{\mathrm{d}A(t)}{A(t)} = r(t)\mathrm{d}t + \sigma_A\mathrm{d}W_A^Q(t). \tag{D11}$$

Solving (D.11) yields

$$\exp\left(-\int_0^t r(u)\mathrm{d}u\right)A(t) = A(0)\exp\left(X(t)\right), \tag{D12}$$

with $X(t) := at + \sigma_A W_A^Q(t)$ and $a = -\frac{1}{2}\sigma_A^2$ as given on page 100.
The paper models the liability process in the form

$$L(t) = L(0)\exp\left(\int_0^t r(u)\mathrm{d}u\right). \tag{D13}$$

Remarks:
(1) The Vasicek interest rate model is not arbitrage-free in the sense that its time-0 bond prices cannot match the current bond prices in the market (as there are not enough parameters in the model). Also, interest rates can be negative.
(2) The expression of λ_S given at the top of page 100 in the paper seems incorrect: λ_S, and thus the risk-neutral measure Q defined on page 99, should be independent of the asset allocation (w_1, w_2).
(3) By Ito's Lemma, or directly from Rutkowski (1999), the volatility of the rolling bond $B_R(t)$ is calculated to be $-\frac{1-e^{-\kappa R}}{\kappa}\sigma_r := -\sigma_R$, where $\frac{1-e^{-\kappa R}}{\kappa}$ can be interpreted as the bond's duration. Therefore, we suggest that σ_R and λ_r in equation (3) in the paper be replaced by $-\sigma_R$ and $-\lambda_r$, repectively, although this change really makes no difference in the subsequent derivation.
(4) Due to the simple structure of the liability process, the generalization to stochastic interest rate does not add any technical difficulty in deriving the fair premium.

3 An alternative derivation of $P(0)$

Before giving the alternative derivations of P^a, P^c and P^ϵ, we shall first rewrite them in more recognizable forms. Note that $A(\tau) = \eta L(\tau)$ by the definition of τ, and that combining (D.12) and (D.13) yields $\frac{A(t)}{L(t)} = \frac{A(0)}{L(0)}e^{X(t)}$. Let $m(t)$ be the running minimum up to time t of $X(\cdot)$; then $\tau < T$ is equivalent to $m(T) \leqslant \ln\frac{\eta L(0)}{A(0)}$. Therefore, (D.1) to (D.3) can be rewritten as

$$
\begin{aligned}
P^a &= (\gamma-\eta)L(0)\mathrm{Pr}^Q\left(m(T)\leqslant B\right),\\
P^c &= \gamma L(0)\mathrm{Pr}^Q\left(m(T)>B, X(T)<B_2\right) - A(0)\mathrm{E}^Q\left[e^{X(T)}I_{\{m(T)>B,X(T)<B_2\}}\right],\\
P^\epsilon &= \gamma L(0)\mathrm{Pr}^Q\left(m(T)>B, B_2<X(T)<B_1, X(T+\epsilon)<B_3\right)\\
&\quad - A(0)\mathrm{E}^Q\left[e^{X(T+\epsilon)}I_{\{m(T)>B,B_2<X(T)<B_1,X(T+\epsilon)<B_3\}}\right],
\end{aligned}
$$

where $B := \ln\frac{\eta L(0)}{A(0)}$, $B_1 := \ln\frac{\alpha L(0)}{A(0)}$, $B_2 := \ln\frac{\beta L(0)}{A(0)}$, $B_3 := \ln\frac{\gamma L(0)}{A(0)}$.
We shall see that pricing $P(0)$ is reduced to evaluating the joint probability distribution of a linear Brownian motion and its running minimum, which can be obtained by the following proposition.

Proposition D1 Let $X_\mu(t) = \mu t + \sigma W(t)$, where $W(t)$ is a standard Brownian motion. Denote by $m_\mu(t)$ the running minimum of $X_\mu(\cdot)$ up to time t, i.e., $m_\mu(t) = \min\limits_{0\leqslant s\leqslant t} X_\mu(s)$. Then for real numbers z and $y \leqslant \min\{x, 0\}$, and for $0<t<t_1$,

$$
\mathrm{Pr}\left(m_\mu(t)\leqslant y, X_\mu(t)>x, X_\mu(t_1)<z\right) = e^{Ry}\mathrm{Pr}\left(X_\mu(t)>x-2y, X_\mu(t_1)<z-2y\right),
$$

where $R := \frac{2\mu}{\sigma^2}$.

Here, the symbol R is used because it is the usual notation for the adjustment coefficient in actuarial risk theory. Note that the "$<$" after $X_\mu(t_1)$ can be replaced by "$>$". When μ is zero, the result is reduced to the well-known Reflection Principle of Brownian motion. A proof of this proposition is provided at the end of the discussion.

3.1 Derivation of P^a

Directly applying Proposition D1 when $\mu = a$, $\sigma = \sigma_A$, $x = y = B$, and $z \to \infty$ gives the distribution of $m(T)$:

$$
\begin{aligned}
\mathrm{Pr}^Q\left(m(T)\leqslant B\right) &= \mathrm{Pr}^Q\left(X(T)\leqslant B\right) + \mathrm{Pr}^Q\left(m(T)\leqslant B, X(T)>B\right)\\
&= \mathrm{Pr}^Q\left(X(T)\leqslant B\right) + e^{RB}\mathrm{Pr}^Q\left(X(T)>-B\right)\\
&= N\left(\frac{B-aT}{\sigma_A\sqrt{T}}\right) + e^{RB}N\left(\frac{B+aT}{\sigma_A\sqrt{T}}\right),
\end{aligned}
$$

which is formula (A2) on page 110.

3.2 Derivation of P^c

The derivation of P^c is very similar to that of P^ϵ, and thus we omit this part for the sake of brevity.

3.3 Derivation of P^ϵ

We first derive the formula for $\mathrm{Pr}^Q\left(m(T)>B, B_2<X(T)<B_1, X(T+\varepsilon)<B_3\right)$. Notice that we can rewrite it as the difference of two probabilities:

$$
\mathrm{Pr}^Q\left(m(T)>B, X(T)>B_2, X(T+\epsilon)<B_3\right) - \mathrm{Pr}^Q\left(m(T)>B, X(T)>B_1, X(T+\epsilon)<B_3\right). \tag{D14}
$$

Each probability in (D.14) can be evaluated according to Proposition D1. The first probability is

$$
\begin{aligned}
&\Pr^Q\left(X(T)>B_2, X(T+\epsilon)<B_3\right)-\Pr^Q\left(m(T)\leqslant B, X(T)>B_2, X(T+\epsilon)<B_3\right)\\
&=\Pr^Q\left(X(T)>B_2, X(T+\epsilon)<B_3\right)-e^{RB}\Pr^Q\left(X(T)>B_2-2B, X(T+\epsilon)<B_3-2B\right),
\end{aligned}
$$

and the second probability is

$$
\Pr^Q\left(X(T)>B_1, X(T+\epsilon)<B_3\right)-e^{RB}\Pr^Q\left(X(T)>B_1-2B, X(T+\epsilon)<B_3-2B\right).
$$

It then follows that an explicit expression for $\Pr^Q\left(m(T)>B, B_2<X(T)<B_1, X(T+\varepsilon)<B_3\right)$ is

$$
\begin{aligned}
&\Pr^Q\left(X(T)<B_1, X(T+\varepsilon)<B_3\right)-\Pr^Q\left(X(T)<B_2, X(T+\epsilon)<B_3\right)\\
&-e^{RB}\left\{\Pr^Q\left(X(T)<B_1-2B, X(T+\epsilon)<B_3-2B\right)-\Pr^Q\left(X(T)<B_2-2B, X(T+\epsilon)<B_3-2B\right)\right\}\\
&=N\left(d_1,e_1,\rho\right)-N\left(d_5,e_1,\rho\right)-e^{RB}\left\{N\left(d_3,e_2,\rho\right)-N\left(d_6,e_2,\rho\right)\right\},
\end{aligned}
\tag{D15}
$$

where $\rho=\sqrt{\frac{T}{T+\epsilon}}$ is the correlation coefficient between $X(T)$ and $X(T+\epsilon)$, and $d_1, d_3, d_5, d_6, e_1, e_2$ are defined in Lemma 3 on page 112.

The derivation of $\mathrm{E}^Q\left[e^{X(T+\epsilon)}I_{\{m(T)>B, B_2<X(T)<B_1, X(T+\epsilon)<B_3\}}\right]$ can be significantly simplified by the factorization formula in the method of Esscher transform. Specifically,

$$
\begin{aligned}
&\mathrm{E}^Q\left[e^{X(T+\epsilon)}I_{\{m(T)>B, B_2<X(T)<B_1, X(T+\epsilon)<B_3\}}\right]\\
&=\mathrm{E}^Q\left[e^{X(T+\epsilon)}\right]\times\mathrm{E}^Q\left[I_{\{m(T)>B, B_2<X(T)<B_1, X(T+\epsilon)<B_3\}};1\right]\\
&=1\times\Pr^Q\left(m(T)>B, B_2<X(T)<B_1, X(T+\epsilon)<B_3;1\right).
\end{aligned}
$$

$\mathrm{E}^Q\left[e^{X(T+\epsilon)}\right]=1$ because $\{e^{X(t)}\}$ is a Q-martingale or simply by explicit calculation. Under the Esscher-transformed probability measure indexed by 1, $\{X(t)\}$ is still a linear Brownian motion, with the same volatility σ_A^2, but with a changed drift

$$
a+1\times\sigma_A^2=-a
$$

because $a=-\frac{1}{2}\sigma_A^2$. Therefore, a formula for $\mathrm{E}^Q\left[e^{X(T+\epsilon)}I_{\{m(T)>B, B_2<X(T)<B_1, X(T+\epsilon)<B_3\}}\right]$ is readily obtained from (D.15) by merely substituting all a in (D.15) with $-a$.

We shall conclude the discussion by sketching a proof of Proposition D1. Again, the method of Esscher transform plays a key role.

Proof of Proposition D1

First, let $\mu=0$ and we restate the Reflection Principle of Brownian motion in the following way:

By the strong Markov property of standard Brownian motion, the two events

$$
\{m_0(t)\leqslant y, X_0(t)>x, X_0(t_1)<z\}\ \text{ and }\ \{2y-X_0(t)>x, 2y-X_0(t_1)<z\}
$$

have the same probability.

Thus we are prompted to make the drift of $X_\mu(t)$ become zero. This can be achieved by considering a factorization based on the Esscher transform with index $-\frac{R}{2}$. For $0<t<t_1$,

$$\begin{aligned}
&\Pr\left(m_\mu(t)\leqslant y, X_\mu(t)>x, X_\mu(t_1)<z\right)\\
&=\mathrm{E}\left[e^{-\frac{R}{2}X_\mu(t_1)}\times e^{\frac{R}{2}X_\mu(t_1)}I_{\{m_\mu(t)\leqslant y, X_\mu(t)>x, X_\mu(t_1)<z\}}\right]\\
&=\mathrm{E}\left[e^{-\frac{R}{2}X_\mu(t_1)}\right]\times \mathrm{E}\left[e^{\frac{R}{2}X_\mu(t_1)}I_{\{m_\mu(t)\leqslant y, X_\mu(t)>x, X_\mu(t_1)<z\}};-\frac{R}{2}\right]\\
&=\mathrm{E}\left[e^{-\frac{R}{2}X_\mu(t_1)}\right]\times \mathrm{E}\left[e^{\frac{R}{2}X_0(t_1)}I_{\{m_0(t)\leqslant y, X_0(t)>x, X_0(t_1)<z\}}\right].
\end{aligned}$$

The Reflection Principle implies that the last expectation is

$$\begin{aligned}
&\mathrm{E}\left[e^{\frac{R}{2}X_0(t_1)}I_{\{m_0(t)\leqslant y, X_0(t)>x, X_0(t_1)<z\}}\right]\\
&=\mathrm{E}\left[e^{\frac{R}{2}(2y-X_0(t_1))}I_{\{2y-X_0(t)>x, 2y-X_0(t_1)<z\}}\right]\\
&=e^{Ry}\mathrm{E}\left[e^{-\frac{R}{2}X_0(t_1)}I_{\{X_0(t)<2y-x, X_0(t_1)>2y-z\}}\right]\\
&=e^{Ry}\mathrm{E}\left[e^{\frac{R}{2}X_0(t_1)}I_{\{X_0(t)>x-2y, X_0(t_1)<z-2y\}}\right],
\end{aligned}$$

where the last step is due to the fact that $X_0(\cdot)$ and $-X_0(\cdot)$ have the same distribution.

Finally, rewrite $\mathrm{E}\left[e^{\frac{R}{2}X_0(t_1)}I_{\{X_0(t)>x-2y, X_0(t_1)<z-2y\}}\right]$ as $\mathrm{E}\left[e^{\frac{R}{2}X_\mu(t_1)}I_{\{X_\mu(t)>x-2y, X_\mu(t_1)<z-2y\}};-\frac{R}{2}\right]$ and hence,

$$\begin{aligned}
&\Pr\left(m_\mu(t)\leqslant y, X_\mu(t)>x, X_\mu(t_1)<z\right)\\
&=\mathrm{E}\left[e^{-\frac{R}{2}X_\mu(t_1)}\right]\times e^{Ry}\mathrm{E}\left[e^{\frac{R}{2}X_\mu(t_1)}I_{\{X_\mu(t)>x-2y, X_\mu(t_1)<z-2y\}};-\frac{R}{2}\right]\\
&=e^{Ry}\Pr\left(X_\mu(t)>x-2y, X_\mu(t_1)<z-2y\right).
\end{aligned}$$

We reverse the factorization in the last step and the proof is complete.

Acknowledgments

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