Randomly Weighted Sums of Subexponential Random Variables with Application to Capital Allocation

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Abstract

We are interested in the tail behavior of the randomly weighted sum $\sum_{i=1}^{n} \theta_i X_i$, in which the primary random variables $X_1, \ldots, X_n$ are real valued, independent and subexponentially distributed, while the random weights $\theta_1, \ldots, \theta_n$ are nonnegative and arbitrarily dependent, but independent of $X_1, \ldots, X_n$. For various important cases, we prove that the tail probability of $\sum_{i=1}^{n} \theta_i X_i$ is asymptotically equivalent to the sum of the tail probabilities of $\theta_1 X_1, \ldots, \theta_n X_n$, which complies with the principle of a single big jump. An application to capital allocation is proposed.

Keywords: Asymptotics; Capital allocation; Matuszewska indices; Randomly weighted sum; Subexponentiality

MSC: Primary 62E20; Secondary 60G70

1 Introduction

Throughout the paper, all random variables are defined on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ unless otherwise stated. Let $X_1, \ldots, X_n$ be n real-valued independent random variables, called primary random variables, and let $\theta_1, \ldots, \theta_n$ be n nonnegative random variables, called random weights, independent of the primary random variables. The target of this study is the randomly weighted sum

$$S_n^{\theta} = \sum_{i=1}^{n} \theta_i X_i. \tag{1.1}$$

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It is a central theme crossing various applied areas of probability and statistics to model the dependence structure of multiple random variables of interest, say, $Z_1, \ldots, Z_n$. This has become especially imperative in insurance, finance and risk management due to the increasing complexity of insurance and financial products. In lieu of the prevailing copula approach, we propose the stochastic representation

$$Z_i = \theta_i X_i, \qquad i = 1, \ldots, n, \tag{1.2}$$

with the primary random variables $X_1, \ldots, X_n$ and the random weights $\theta_1, \ldots, \theta_n$ described above.

Our study will proceed in a general setting in which the random weights $\theta_1, \ldots, \theta_n$ are arbitrarily dependent and not necessarily bounded. In the mechanism through (1.1)–(1.2), we intend to use the primary random variables $X_1, \ldots, X_n$ to depict the magnitude of $Z_1, \ldots, Z_n$ while using the random weights $\theta_1, \ldots, \theta_n$ to capture the dependence. The independence between the two sequences $\{X_1, \ldots, X_n\}$ and $\{\theta_1, \ldots, \theta_n\}$ separates the magnitude and dependence, bringing us great convenience in both modeling and computation.

In addition to an application to capital allocation to be given in Section 4, we list here three examples in insurance, finance and risk management in which a randomly weighted sum appears naturally as a key quantity of interest.

In the first example, consider a portfolio consisting of n obligors that are subject to possible default over a given period. Denote by X_i the loss given default of obligor i, equal to the percentage loss given default multiplied by the exposure, and denote by θ_i the default indicator for obligor i, which is a Bernoulli random variable with $\theta_i = 1$ corresponding to the default of obligor i. Then the randomly weighted sum (1.1) represents the aggregate amount of losses. An important feature of credit risk is that the default indicators are strongly dependent due to both common macroeconomic factors and credit contagion. For detailed discussions, we refer the reader to McNeil et al. (2005), Das et al. (2007) and Hatchett and Kühn (2009), among many others.

In the second example, consider an insurer who makes both risk-free and risky investments. Over each period i, the net loss is denoted by a real-valued random variable X_i, equal to total claims minus total premiums, and the overall accumulation factor is denoted by a positive random variable A_i. With $\theta_i = \prod_{j=1}^{i} A_j^{-1}$, the randomly weighted sum (1.1) then represents the stochastic present value of the net losses over the first n periods. See, for example, Nyrhinen (2001) and Tang and Tsitsiashvili (2003a) for related discussions. Obviously, the random variables $\theta_1, \ldots, \theta_n$ should be strongly dependent.

The third example gives another interpretation of (1.1) in terms of an investment portfolio consisting of n risky assets over one period. Each asset i incurs a potential loss variable X_i at the terminal time while the corresponding stochastic discount factor over the period is θ_i. Then the randomly weighted sum (1.1) represents the total amount of discounted losses potentially incurred from the investment portfolio. See, for example, Björk (2009) for discussions on stochastic discount factors. The stochastic discount factors $\theta_1, \ldots, \theta_n$ should be dependent as they result from financial evolvements over the same time period.

We study the tail behavior of the randomly weighted sum (1.1) under the assumptions that the primary random variables are heavy tailed and that the random weights are dependent. These assumptions are highly relevant in the motivating examples above. Since many risk measures, such as Value at Risk (VaR) and Conditional Tail Expectation (CTE), focus on the tail area of a risk variable, the study of the tail behavior of randomly weighted sums has an immediate application to the computation of such risk measures.

Recently, randomly weighted sums have been an attractive research topic in the literature of applied probability. See Shen et al. (2009), Zhang et al. (2009), Gao and Wang (2010), Chen et al. (2011), Yi et al. (2011), Fougères and Mercadier (2012), Hazra and Maulik (2012) and Olvera-Cravioto (2012), to name a few very recent ones. See also Hashorva et al. (2010), who interpreted the stochastic representation (1.2) in terms of random contraction.

Tang and Tsitsiashvili (2003b) proved that, if the primary random variables $X_1, \ldots, X_n$ are independent and identically distributed by a subexponential distribution and the random weights $\theta_1, \ldots, \theta_n$ take values in the interval $[a, b]$ for some constants a and b, $0 < a \leq b < \infty$, then

$$\mathrm{P}\left(\bigvee_{i=1}^{n} S_i^{\theta} > x\right) \sim \mathrm{P}\left(S_n^{\theta} > x\right) \sim \mathrm{P}\left(\bigvee_{i=1}^{n} \theta_i X_i > x\right) \sim \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right). \tag{1.3}$$

This suggests that the heavy tails of the primary random variables dissolve the dependence of the two-sided bounded random weights. This complies with the principle of a single big jump in the presence of random weights. It can facilitate the computation of the tail probabilities, especially when there is tangled dependence among $\theta_1, \ldots, \theta_n$. However, the two-sided bound restriction on the random weights severely limits the merit of the result.

Our study is a revisit to the work of Tang and Tsitsiashvili (2003b). We devote ourselves to relaxing the two-sided bound restriction. The lower-bound restriction is removed in Theorem 3.1 while the upper-bound restriction is further removed in Theorems 3.2 and 3.3 under some additional conditions. A feature of our work is that the random weights are allowed to be arbitrarily dependent on each other.

The rest of the paper consists of five sections. In Section 2 we prepare some preliminaries on heavy-tailed distributions, in Section 3 we present our main results, in Section 4 we propose an application to capital allocation, and in Sections 5 and 6 we show the proofs.

2 Preliminaries

2.1 Heavy-tailed distributions

Throughout this paper, all limit relationships are according to $x \to \infty$ unless otherwise stated. For two positive functions $f(\cdot)$ and $g(\cdot)$, write $f(x) \lesssim g(x)$ or $g(x) \gtrsim f(x)$ if $\limsup f(x)/g(x) \le 1$, write $f(x) \sim g(x)$ if $\lim f(x)/g(x) = 1$, and write $f(x) \asymp g(x)$ if $f(\cdot)$ and $g(\cdot)$ are weakly equivalent, that is, $0 < \liminf f(x)/g(x) \le \limsup f(x)/g(x) < \infty$. For real numbers x, x_1, …, x_n, write $x^+ = x \vee 0$, $x^- = -(x \wedge 0)$ and $x_{(n)} = x_1 \vee \cdots \vee x_n$.

We shall assume that the primary random variables X_1, …, X_n are heavy tailed, that is, their moment generating functions are infinite on $(0, \infty)$. One of the most important classes of heavy-tailed distribution functions is the subexponential distribution class. A distribution function F on $\mathbb{R}_+ = [0, \infty)$ is said to be subexponential, written as $F \in \mathcal{S}$, if it has an ultimate tail (that is, $\overline{F}(x) > 0$ for all $x \ge 0$) and

$$\overline{F^{2*}}(x) \sim 2\overline{F}(x),$$

where F^{2*} denotes the two-fold convolution of F. More generally, a distribution function F on $\mathbb{R}$ is still said to be subexponential if the distribution function $F_+(x) = F(x)\mathbf{1}_{(x \ge 0)}$ is subexponential, where $\mathbf{1}_E$ is the indicator function of a set E. The class $\mathcal{S}$ contains a lot of important distributions such as Pareto, lognormal and heavy-tailed Weibull distributions. The reader is referred to the monographs Embrechts et al. (1997), Asmussen and Albrecher (2010) and Foss et al. (2011) for reviews of the class $\mathcal{S}$.

Among many distribution classes that are closely related to the class $\mathcal{S}$, we list a few here for our reference later. A distribution function F on $\mathbb{R}$ is said to be long tailed, written as $F \in \mathcal{L}$, if its (ultimate) right tail satisfies

$$\overline{F}(x + y) \sim \overline{F}(x) \qquad \text{for all } y \in \mathbb{R}. \tag{2.1}$$

Automatically, relation (2.1) holds uniformly on every compact set of y. Hence, it is easy to see that there is some positive function $l(\cdot)$, with $l(x) \le x/2$ and $l(x) \uparrow \infty$, such that relation (2.1) holds uniformly for $-l(x) \le y \le l(x)$. It is well known that every subexponential distribution is long tailed; see Lemma 2 of Chistyakov (1964) or Lemma 1.3.5(a) of Embrechts et al. (1997). By Theorem 2.1(a) of Klüppelberg (1988), a long-tailed distribution with a right tail weakly equivalent to a subexponential tail is subexponential.

A distribution function F on $\mathbb{R}$ is said to be dominatedly-varying tailed, written as $F \in \mathcal{D}$, if its right tail satisfies $\overline{F}(xy) = O\left(\overline{F}(x)\right)$ for all $0 < y < 1$. The intersection $\mathcal{L} \cap \mathcal{D}$ forms a useful subclass of $\mathcal{S}$; see, for example, Theorem 1 of Goldie (1978) or Proposition 1.4.4(a) of Embrechts et al. (1997). In particular, it covers the famous class $\mathcal{R}$ of distributions with a regularly-varying tail. By definition, for a distribution function F on $\mathbb{R}$, we write $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$ if its right tail is regularly varying with index $-\alpha$, that is, $\overline{F}(xy) \sim y^{-\alpha}\overline{F}(x)$ for all $y > 0$. In summary,

$$\mathcal{R} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L}.$$

In this paper, we need two more distribution classes. A distribution function F on $\mathbb{R}$ is said to belong to the class $\mathcal{A}$ if it is subexponential and its right tail satisfies

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < 1 \qquad \text{for some } y > 1. \tag{2.2}$$

Note that (2.2) is really a mild condition fulfilled by almost all useful distributions with an ultimate right tail. From this point of view, the class $\mathcal{A}$ almost takes up the entire class $\mathcal{S}$. Moreover, a distribution function F on $\mathbb{R}$ is said to be rapidly-varying tailed, written as $F \in \mathcal{R}_{-\infty}$, if $\overline{F}(xy) = o\left(\overline{F}(x)\right)$ for all $y > 1$. Note that $\mathcal{R}_{-\infty}$ is a very broad class containing both heavy-tailed and light-tailed distributions.

2.2 Matuszewska indices

The Matuszewska indices can provide useful information on the tail behavior of a distribution function. For a distribution function F with an ultimate right tail, define

$$M^*(F) = \inf\left\{-\frac{\log \overline{F}_*(y)}{\log y} : y > 1\right\} \quad \text{and} \quad M_*(F) = \sup\left\{-\frac{\log \overline{F}^*(y)}{\log y} : y > 1\right\},$$

where $\overline{F}_*(y) = \liminf \overline{F}(xy)/\overline{F}(x)$ and $\overline{F}^*(y) = \limsup \overline{F}(xy)/\overline{F}(x)$. From Corollary 2.1.6 and Theorem 2.1.5 of Bingham et al. (1987), we see that $M^*(F)$ and $M_*(F)$ are the upper and lower Matuszewska indices of the function $f = 1/\overline{F}$, respectively. Without any confusion we simply call them the upper and lower Matuszewska indices of F.

Let us recollect some simple and useful results related to the Matuszewska indices. Clearly, two distributions with weakly equivalent tails have the same Matuszewska indices. It is easy to see that $F \in \mathcal{D}$ if and only if $0 \le M^*(F) < \infty$, or, equivalently, the function $\overline{F}$ has bounded decrease. Note also that inequality (2.2) holds if and only if $0 < M_*(F) \le \infty$, or, equivalently, the function $\overline{F}$ has positive decrease. Thus, $F \in \mathcal{A}$ can be characterized as $F \in \mathcal{S}$ with a tail possessing positive decrease. See page 71 of Bingham et al. (1987) for the definitions of bounded decrease and positive decrease.

For a distribution function F, with some simple adjustments on Proposition 2.2.1 of Bingham et at. (1987), we see the following: if $0 \le M^*(F) < \infty$, then for every $\beta > M^*(F)$, there are some positive constants C_1 and x_1 such that the inequality

$$\frac{\overline{F}(x)}{\overline{F}(xy)} \le C_1 y^{\beta} \tag{2.3}$$

holds for all $xy \geq x \geq x_1$; while if $0 < M_*(F) \leq \infty$, then for every $0 < \gamma < M_*(F)$, there are some positive constants C_2 and x_2 such that the inequality

$$\frac{\overline{F}(xy)}{\overline{F}(x)} \leq C_2 y^{-\gamma} \tag{2.4}$$

holds for all $xy \geq x \geq x_2$.

From (2.3) and (2.4), it is easy to see that the relations

$$x^{-\beta} = o\left(\overline{F}(x)\right) \qquad \text{and} \qquad \overline{F}(x) = o\left(x^{-\gamma}\right) \tag{2.5}$$

hold for every $\beta > M^*(F)$ and $\gamma < M_*(F)$. Therefore, for a random variable X distributed by F, if $M_*(F) > 1$ then $\mathrm{E}\left[X^+\right] < \infty$, while if $\mathrm{E}\left[X^+\right] < \infty$ then $M^*(F) \geq 1$.

3 Main Results

Recall the randomly weighted sum (1.1), in which the primary random variables $X_1, \ldots, X_n$ are real valued, independent and distributed by $F_1, \ldots, F_n$, respectively, while the random weights $\theta_1, \ldots, \theta_n$ are nonnegative, not degenerate at 0, and arbitrarily dependent on each other, but independent of the primary random variables.

Here comes our first main result, which extends Theorem 3.1 of Tang and Tsitsiashvili (2003b) by squarely removing the lower-bound restriction on the random weights:

Theorem 3.1 *If $F_i \in \mathcal{L}$ and $\overline{F_i}(x) \asymp \overline{F}(x)$ for some $F \in \mathcal{S}$ and all $i = 1, \ldots, n$, and if $\theta_1, \ldots, \theta_n$ are bounded above, then the relations in (1.3) hold.*

Then we aim to further remove the upper-bound restriction on the random weights. In doing so, we need to confine the primary distributions to the class $\mathcal{A}$, which is slightly smaller than the class $\mathcal{S}$, as mentioned before.

Theorem 3.2 *If $F_i \in \mathcal{L}$ and $\overline{F_i}(x) \asymp \overline{F}(x)$ for some $F \in \mathcal{A}$ and all $i = 1, \ldots, n$, and if the relation*

$$\mathrm{P}\left(\theta_i > ux\right) = o(1)\mathrm{P}\left(\theta_i X_i > x\right), \qquad u > 0, \tag{3.1}$$

holds for all $i = 1, \ldots, n$, then the relations in (1.3) hold.

Lemma 3.2 of Tang (2006) shows that (3.1) is equivalent to the existence of a positive auxiliary function $a(\cdot)$, with $a(x) \uparrow \infty$ and $a(x) = o(x)$, such that

$$\mathrm{P}\left(\theta_i > a(x)\right) = o(1)\mathrm{P}\left(\theta_i X_i > x\right). \tag{3.2}$$

As shown in Corollary 2.1 of Tang (2006), either one of the following is sufficient for (3.1) to hold:

- $\mathrm{P}(\theta_i > xy) = o\left(\overline{F_i}(x)\right)$ for some $y > 0$;
- $\mathrm{P}(\theta_i > xy) = o(1)\mathrm{P}(\theta_i > x)$ for some $y > 1$;
- $0 < M_*(G_i) \le \infty$ and $\mathrm{E}\left[\left(X_i^+\right)^{\beta_i}\right] = \infty$ for some $0 < \beta_i < M_*(G_i)$, where G_i is the distribution function of θ_i.

In the following theorem, we confine the primary distributions to the class $\mathcal{L} \cap \mathcal{D}$ but do not assume weak equivalence for their tails. It slightly extends Theorem 2.1 of Wang and Tang (2006).

Theorem 3.3 *If $F_i \in \mathcal{L} \cap \mathcal{D}$ and $\mathrm{E}\left[\theta_i^{\beta_i}\right] < \infty$ for some $\beta_i > M^*(F_i)$ and all $i = 1, \ldots, n$, then the relations in (1.3) hold.*

We postpone the proofs of Theorems 3.1–3.3 to Section 5.

4 Application to Capital Allocation

Consider an investor who invests in n lines of business. Each line i generates a potential net loss variable Z_i in loss-profit form. These loss variables must depend on each other since the n lines operate in similar macroeconomic environments. We model them by (1.2) in which, as before, the primary random variables $X_1, \ldots, X_n$ are real valued, independent and distributed by $F_1, \ldots, F_n$, respectively, while the random weights $\theta_1, \ldots, \theta_n$ are nonnegative, not degenerate at 0, and arbitrarily dependent on each other, but independent of the primary random variables. Then the randomly weighted sum

$$S_n^\theta = \sum_{i=1}^n \theta_i X_i \tag{1.1}$$

represents the total loss.

For some regulated investors, such as banks or insurance companies, a risk capital is usually held as a cushion to protect them from large losses. Our interest is in allocating the risk capital to the individual lines. We refer the reader to Bauer and Zanjani (2014) for an overview of various allocation principles, many of which boil down to Euler's principle, also referred to as gradient allocation principle. Euler's principle allocates risk capital according to the risk contribution of each line and is known to provide right signals for performance measurement; see also Tasche (1999) for related discussions. Moreover, it is coherent if the underlying risk measure is coherent; see Denault (2001) for an introduction of coherent allocation principles. For more details and economic justifications of Euler's principle, we refer the reader to Myers and Read (2001), McNeil et al. (2005) and Dhaene et al. (2012), among others. See also Asimit et al. (2011) for an asymptotic analysis of capital allocation in the presence of extreme risks, to which our study in this section is similar in spirit.

Assume that each random variable in (1.1) has a finite mean and let CTE be the underlying risk measure. According to Euler's principle, the amount of capital allocated to line i is

$$\mathrm{AC}_i = \mathrm{E}\left[\theta_i X_i \middle| S_n^{\theta} > x\right] = \frac{\mathrm{E}\left[\theta_i X_i \mathbf{1}_{\left(S_n^{\theta} > x\right)}\right]}{\mathrm{P}\left(S_n^{\theta} > x\right)}, \qquad i = 1, \ldots, n, \tag{4.1}$$

where x is the VaR of S_n^{θ}, that is,

$$x = \mathrm{VaR}_q\left[S_n^{\theta}\right] = \inf\left\{y \in \mathbb{R} : \mathrm{P}\left(S_n^{\theta} \le y\right) \ge q\right\}, \qquad 0 < q < 1.$$

See Section 6.3.2 of McNeil et al. (2005) for this assertion. Note that AC_i given by (4.1) could be nonpositive, namely, no capital or even a negative amount of capital is allocated to line i in this case, meaning that such a line should be rewarded for risk capital. See Erel et al. (2013) for related discussions.

Typically, the expression given by (4.1) is difficult to evaluate. We aim at an asymptotic formula as $q \uparrow 1$. Under the corresponding conditions of each of Theorems 3.1–3.3, the asymptotics for the denominator in (4.1) has been given by (1.3), which consequently leads to

$$\mathrm{VaR}_q\left[S_n^{\theta}\right] \approx \inf\left\{x \in \mathbb{R} : \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right) \le 1 - q\right\}, \qquad q \uparrow 1.$$

Thus, only asymptotics for the numerator in (4.1) needs to be derived. We show the results for $i = 1$ only.

Theorem 4.1 *In addition to the conditions of Theorem 3.1, Theorem 3.2, or Theorem 3.3, assume that*

$$\mathrm{P}\left(\theta_i X_i > x\right) = O(1)\mathrm{P}\left(\theta_1 X_1 > x\right), \qquad i = 2, \ldots, n. \tag{4.2}$$

Then we have

$$\mathrm{E}\left[\theta_1 X_1 1_{\left(S_n^{\theta} > x\right)}\right] \sim \mathrm{E}\left[\theta_1 X_1 1_{\left(\theta_1 X_1 > x\right)}\right]. \tag{4.3}$$

Within the scope of Theorem 3.1 or Theorem 3.2, a sufficient condition for (4.2) is that $\mathrm{P}\left(\theta_i > t\right) = O\left(\mathrm{P}\left(\theta_1 > t\right)\right)$ as $t \uparrow \hat{t}_1$ for all $i = 2, \ldots, n$, where $0 < \hat{t}_1 \le \infty$ is the essential upper bound of θ_1; while within the scope of Theorem 3.3, a sufficient condition for (4.2) is that $\overline{F_2}(x) + \cdots + \overline{F_n}(x) = O\left(\overline{F_1}(x)\right)$.

In the next theorem for the $\mathcal{L} \cap \mathcal{D}$ case, we do not require condition (4.2) but slightly strengthen the moment conditions on the random weights.

Theorem 4.2 *If $F_i \in \mathcal{L} \cap \mathcal{D}$ and $\mathrm{E}\left[\theta_1^{\beta_1} + \theta_1 \theta_i^{\beta_i}\right] < \infty$ for some $\beta_i > M^*(F_i)$ and all $i = 2, \ldots, n$, then*

$$\mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(S_n^\theta > x\right)}\right] \sim \sum_{i=1}^n \mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{(\theta_i X_i > x)}\right] \quad \textit{and} \quad \mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(S_n^\theta > x\right)}\right] \sim \sum_{i=1}^n \mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{(\theta_i X_i > x)}\right],$$

where the second relation bears a meaning only when X_1 has a nontrivial negative part X_1^-.

The two asymptotic relations in Theorem 4.2 readily give that $\mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{\left(S_n^\theta > x\right)}\right] \sim \sum_{i=1}^n \mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_i X_i > x)}\right]$ under a mild technical requirement that

$$\liminf_{x \to \infty} \frac{\sum_{i=1}^n \mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{(\theta_i X_i > x)}\right]}{\sum_{i=1}^n \mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{(\theta_i X_i > x)}\right]} > 1 \quad \text{or} \quad \limsup_{x \to \infty} \frac{\sum_{i=1}^n \mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{(\theta_i X_i > x)}\right]}{\sum_{i=1}^n \mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{(\theta_i X_i > x)}\right]} < 1.$$

We make the right-hand side of (4.3) explicit in the next corollary by imposing some more conditions. In order to state the corollary below, we introduce two nonnegative random variables X and θ, which are independent of each other and independent of all other sources of randomness. Let X be distributed by F and denote by $\hat{t}$ the essential upper bound of θ. We show some cases below in which the asymptotics for the capital allocation becomes transparent:

Corollary 4.1 *Assume that, for each $i = 1, \ldots, n$, $\overline{F_i}(x) \sim c_i \overline{F}(x)$ for some constant $c_i > 0$.*

(a) For $F \in \mathcal{S} \cap \mathcal{R}_{-\infty}$ and $0 < \hat{t} < \infty$, assume that $\lim_{t \uparrow \hat{t}} \mathrm{P}(\theta_i > t) / \mathrm{P}(\theta > t) = d_i$ for some $d_1 > 0$, $d_2 \geq 0$, $\ldots$, $d_n \geq 0$. Then, as $q \uparrow 1$,

$$\mathrm{AC}_1 \sim \frac{c_1 d_1}{\sum_{i=1}^n c_i d_i} \mathrm{VaR}_{\tilde{q}}[\theta X] \qquad \textit{with } \tilde{q} = 1 - \frac{1-q}{\sum_{i=1}^n c_i d_i}. \tag{4.4}$$

(b) For $F \in \mathcal{S} \cap \mathcal{R}_{-\infty}$ and $\hat{t} = \infty$, assume that $\lim_{t \uparrow \hat{t}} \mathrm{P}(\theta_i > t) / \mathrm{P}(\theta > t) = d_i$ for some $d_1 > 0$, $d_2 \geq 0$, $\ldots$, $d_n \geq 0$ and that relation (3.1) holds for all $u > 0$ and all $i = 1, \ldots, n$. Then relation (4.4) still holds.

(c) For $F \in \mathcal{R}_{-\alpha}$ with $\alpha > 1$, assume that $\mathrm{E}\left[\theta_i^\beta\right] < \infty$ for some $\beta > \alpha$ and all $i = 1, \ldots, n$. Then

$$\mathrm{AC}_1 \sim \frac{\alpha}{\alpha - 1} \frac{c_1 \mathrm{E}\left[\theta_1^\alpha\right]}{\left(\sum_{i=1}^n c_i \mathrm{E}\left[\theta_i^\alpha\right]\right)^{1-1/\alpha}} \mathrm{VaR}_q[X], \qquad q \uparrow 1.$$

We postpone the proofs of Theorems 4.1–4.2 and Corollary 4.1 to Section 6.

5 Proofs of Theorems 3.1–3.3

For each of Theorems 3.1–3.3, the last relation in (1.3) can be easily proven by verifying

$$\sum_{1\le j\neq k\le n} \mathrm{P}\left(\theta_j X_j > x, \theta_k X_k > x\right) = o(1)\sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right). \tag{5.1}$$

For Theorem 3.1, relation (5.1) is trivial because the random weights are bounded. For Theorem 3.2, relation (5.1) can be verified by conditioning on θ_j and θ_k for each term on the left-hand side and utilizing relation (3.2). For Theorem 3.3, relation (5.1) follows from Lemma 5.7 below.

The rest of (1.3) amounts to the conjunction of

$$\mathrm{P}\left(S_n^{\theta} > x\right) \gtrsim \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right) \tag{5.2}$$

and

$$\mathrm{P}\left(\sum_{i=1}^{n} \theta_i X_i^{+} > x\right) \lesssim \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right). \tag{5.3}$$

For this reason, we shall focus on (5.2) and (5.3) only in the proofs of Theorems 3.1–3.3.

5.1 Proof of Theorem 3.1

Following the proof of Proposition 5.1 of Tang and Tsitsiashvili (2003b) with some obvious changes, we obtain

Lemma 5.1 *Let $X_1, \ldots, X_n$ be n real-valued independent random variables, each distributed by F_i satisfying $F_i \in \mathcal{L}$ and $\overline{F_i}(x) \asymp \overline{F}(x)$ for some $F \in \mathcal{S}$ and all $i = 1, \ldots, n$. Then for every fixed $0 < a \le b < \infty$, it holds uniformly for all $(c_1, \ldots, c_n) \in [a, b]^n$ that*

$$\mathrm{P}\left(\sum_{i=1}^{n} c_i X_i > x\right) \sim \sum_{i=1}^{n} \mathrm{P}\left(c_i X_i > x\right). \tag{5.4}$$

The following lemma can be easily verified by conditioning on θ and utilizing relation (3.2); see also Theorem 2.2(iii) of Cline and Samorodnitsky (1994):

Lemma 5.2 *Let X and θ be two nonnegative independent random variables. If X is long tailed, θ is not degenerate at 0, and $\mathrm{P}\left(\theta > ux\right) = o(1)\mathrm{P}\left(\theta X > x\right)$ for all $u > 0$, then the product θX is long tailed.*

Proof of Theorem 3.1. Without loss of generality, we assume that the random weights $\theta_1, \ldots, \theta_n$ are bounded above by 1.

To prove (5.2), first assume that the random variables $X_1, \ldots, X_n$ are nonnegative. For this case, relation (5.2) follows from the inequality $S_n^\theta \geq \bigvee_{i=1}^n \theta_i X_i$ and relation (5.1). Now consider the general case where $X_1, \ldots, X_n$ may be negative. For an arbitrary subset $I \subset \{1, \ldots, n\}$, write $I^c = \{1, \ldots, n\} \backslash I$ and

$$\Omega_I(X) = \{\omega : X_i \geq 0 \text{ for } i \in I \text{ and } X_j < 0 \text{ for } j \in I^c\}.$$

Since each random weight is bounded above by 1, it follows that

$$\mathrm{P}\left(S_n^\theta > x\right) \geq \sum_{\varnothing \neq I \subset \{1,\ldots,n\}} \mathrm{P}\left(\sum_{i \in I} \theta_i X_i + \sum_{j \in I^c} X_j > x, \Omega_I(X)\right), \tag{5.5}$$

where a sum over an empty set is equal to 0 by convention. Conditioning on X_j for $j \in I^c$ and applying relation (5.2) for the nonnegative case, we see that each probability on the right-hand side of (5.5) is asymptotically not smaller than

$$\sum_{i \in I} \mathrm{P}\left(\theta_i X_i + \sum_{j \in I^c} X_j > x, \Omega_I(X)\right) \sim \sum_{i \in I} \mathrm{P}\left(\theta_i X_i > x, \Omega_I(X)\right),$$

where we further applied the dominated convergence theorem and the fact that $\theta_i X_i$ is long tailed due to Lemma 5.2. Substituting this into (5.5) and interchanging the order of the summations yield relation (5.2).

Then we turn to establishing (5.3). First we assume that the random weights are positive. Let I and I^c be as before and write

$$\Omega_I^\varepsilon(\theta) = \{\omega : \theta_i > \varepsilon \text{ for } i \in I \text{ and } \theta_j \leq \varepsilon \text{ for } j \in I^c\}, \qquad 0 < \varepsilon < 1.$$

Clearly,

$$\mathrm{P}\left(\sum_{i=1}^n \theta_i X_i^+ > x\right) \leq \sum_{I \subset \{1,\ldots,n\}} \mathrm{P}\left(\sum_{i \in I} \theta_i X_i^+ + \sum_{j \in I^c} \varepsilon X_j^+ > x, \Omega_I^\varepsilon(\theta)\right). \tag{5.6}$$

By Lemma 5.1, each term on the right-hand side of (5.6) is asymptotically equivalent to

$$\begin{aligned}
&\sum_{i \in I} \mathrm{P}\left(\theta_i X_i > x, \Omega_I^\varepsilon(\theta)\right) + \sum_{j \in I^c} \mathrm{P}\left(\varepsilon X_j > x\right) \mathrm{P}\left(\Omega_I^\varepsilon(\theta)\right) \\
&= \sum_{i \in I} \mathrm{P}\left(\theta_i X_i > x, \Omega_I^\varepsilon(\theta)\right) + \sum_{j \in I^c} \mathrm{P}\left(\varepsilon X_j > x, \theta_j > \varepsilon\right) \frac{\mathrm{P}\left(\Omega_I^\varepsilon(\theta)\right)}{\mathrm{P}\left(\theta_j > \varepsilon\right)}.
\end{aligned}$$

Substituting this into (5.6) and interchanging the order of the summations, we obtain

$$\begin{aligned}
&\mathrm{P}\left(\sum_{i=1}^{n}\theta_i X_i^+ > x\right)\\
&\lesssim \sum_{i=1}^{n}\sum_{I:i\in I\subset\{1,\ldots,n\}}\mathrm{P}\left(\theta_i X_i > x, \Omega_I^\varepsilon(\theta)\right) + \sum_{j=1}^{n}\sum_{I:j\notin I\subset\{1,\ldots,n\}}\mathrm{P}\left(\theta_j X_j > x\right)\frac{\mathrm{P}\left(\Omega_I^\varepsilon(\theta)\right)}{\mathrm{P}\left(\theta_j > \varepsilon\right)}\\
&= \sum_{i=1}^{n}\mathrm{P}\left(\theta_i X_i > x, \theta_i > \varepsilon\right) + \sum_{j=1}^{n}\mathrm{P}\left(\theta_j X_j > x\right)\frac{\mathrm{P}\left(\theta_j \le \varepsilon\right)}{\mathrm{P}\left(\theta_j > \varepsilon\right)}\\
&\le \left(1 + \max_{1\le j\le n}\frac{\mathrm{P}\left(\theta_j \le \varepsilon\right)}{\mathrm{P}\left(\theta_j > \varepsilon\right)}\right)\sum_{i=1}^{n}\mathrm{P}\left(\theta_i X_i > x\right).
\end{aligned}$$

Since each θ_j is strictly positive, letting $\varepsilon \downarrow 0$ yields (5.3).

Finally, consider the general case where the random weights may take value 0 with a positive probability. Let I and I^c be as before and write

$$\Omega_I^0(\theta) = \{\omega : \theta_i > 0 \text{ for } i \in I \text{ and } \theta_j = 0 \text{ for } j \in I^c\}.$$

Then, by what we have just proven for (5.3),

$$\begin{aligned}
\mathrm{P}\left(\sum_{i=1}^{n}\theta_i X_i^+ > x\right) &= \sum_{\phi\ne I\subset\{1,\ldots,n\}}\mathrm{P}\left(\sum_{i\in I}\theta_i X_i^+ > x, \Omega_I^0(\theta)\right)\\
&\lesssim \sum_{\phi\ne I\subset\{1,\ldots,n\}}\sum_{i\in I}\mathrm{P}\left(\theta_i X_i > x, \Omega_I^0(\theta)\right)\\
&= \sum_{i=1}^{n}\mathrm{P}\left(\theta_i X_i > x\right). \qquad (5.7)
\end{aligned}$$

This ends the proof of Theorem 3.1. ■

5.2 Proof of Theorem 3.2

The following is an extension of Lemma 3.1 of Tang (2006):

Lemma 5.3 *In addition to the conditions of Lemma 5.1, assume that $M_*(F) > 0$. Then it holds uniformly for all $(c_1, \ldots, c_n) \in (0, 1]^n$ that*

$$\mathrm{P}\left(\sum_{i=1}^{n}c_i X_i > x\right) \lesssim \sum_{i=1}^{n}\mathrm{P}\left(c_i X_i > x\right).$$

Proof. If we rewrite $\mathrm{P}\left(\sum_{i=1}^{n}c_i X_i > x\right) = \mathrm{P}\left(\sum_{i=1}^{n}\widetilde{c}_i X_i > \widetilde{x}\right)$ with $\widetilde{c}_i = c_i/c_{(n)}$ and $\widetilde{x} = x/c_{(n)}$, then each $\widetilde{c}_i$ lies in $(0, 1]$, at least one of them equals 1, and $\widetilde{x} \to \infty$. Thus, without loss of generality, we may assume $c_1 = 1$.

For every $0 < \varepsilon < 1$ and $I \subset \{2, \ldots, n\}$, write $I^c = \{2, \ldots, n\} \backslash I$ and

$$A_I = \{(c_2, \ldots, c_n) : 0 < c_i \leq \varepsilon \text{ for } i \in I \text{ and } \varepsilon < c_j \leq 1 \text{ for } j \in I^c\}.$$

By Lemma 5.1 and inequality (2.4), uniformly over each A_I it holds for arbitrarily fixed $0 < \gamma < M_*(F)$ and some $C > 0$ that

$$\begin{aligned}
\frac{\mathrm{P}\left(\sum_{i=1}^n c_i X_i > x\right)}{\sum_{i=1}^n \mathrm{P}\left(c_i X_i > x\right)} &\lesssim \frac{\mathrm{P}\left(X_1^+ + \sum_{i\in I} \varepsilon X_i^+ + \sum_{j\in I^c} c_j X_j^+ > x\right)}{\mathrm{P}\left(X_1 > x\right) + \sum_{j\in I^c} \mathrm{P}\left(c_j X_j > x\right)} \\
&\sim \frac{\mathrm{P}\left(X_1 > x\right) + \sum_{i\in I} \mathrm{P}\left(\varepsilon X_i > x\right) + \sum_{j\in I^c} \mathrm{P}\left(c_j X_j > x\right)}{\mathrm{P}\left(X_1 > x\right) + \sum_{j\in I^c} \mathrm{P}\left(c_j X_j > x\right)} \\
&\leq 1 + \sum_{i\in I} \frac{\mathrm{P}\left(\varepsilon X_i > x\right)}{\mathrm{P}\left(X_1 > x\right)} \\
&\lesssim 1 + C\varepsilon^\gamma n.
\end{aligned}$$

Since $\{A_I : I \subset \{2, \ldots, n\}\}$ forms a finite partition of $(0, 1]^{n-1}$ and ε can be arbitrarily close to 0, the result follows. ■

The following lemma complements Lemma 5.3 with the opposite inequality:

Lemma 5.4 *Let $X_1, \ldots, X_n$ be n nonnegative independent random variables, each with an ultimate right tail. Then it holds uniformly for all $(c_1, \ldots, c_n) \in (0, 1]^n$ that*

$$\mathrm{P}\left(\sum_{i=1}^n c_i X_i > x\right) \gtrsim \sum_{i=1}^n \mathrm{P}\left(c_i X_i > x\right).$$

Proof. Note that $\mathrm{P}\left(\sum_{i=1}^n c_i X_i > x\right) \geq \mathrm{P}\left(\bigvee_{i=1}^n c_i X_i > x\right)$. A simple application of Bonferroni's inequality completes the proof. ■

A combination of Lemmas 5.3 and 5.4 extends Lemma 5.1 as follows:

Lemma 5.5 *Let $X_1, \ldots, X_n$ be n nonnegative independent random variables, each distributed by F_i satisfying $F_i \in \mathcal{L}$ and $\overline{F_i}(x) \asymp \overline{F}(x)$ for some $F \in \mathcal{A}$ and all $i = 1, \ldots, n$. Then relation (5.4) holds uniformly for all $(c_1, \ldots, c_n) \in (0, 1]^n$.*

The following lemma plays a crucial role in proving relation (5.2) for Theorem 3.2:

Lemma 5.6 *Let X, $\{Y_1, \ldots, Y_n\}$ and $\{\theta_0, \theta_1, \ldots, \theta_n\}$ be three independent groups of nonnegative random variables. If X is distributed by $F \in \mathcal{L}$ with $M_*(F) > 0$, and $\mathrm{P}\left(\theta_i > ux\right) = o(1)\mathrm{P}\left(\theta_i X > x\right)$ for all $u > 0$ and $i = 0, 1, \ldots, n$, then*

$$\mathrm{P}\left(\theta_0 X - \sum_{i=1}^n \theta_i Y_i > x\right) = \mathrm{P}\left(\theta_0 X > x\right) - o(1) \sum_{i=1}^n \mathrm{P}\left(\theta_i X > x\right).$$

Proof. Since the inequality $\mathrm{P}\left(\theta_0 X - \sum_{i=1}^n \theta_i Y_i > x\right) \leq \mathrm{P}\left(\theta_0 X > x\right)$ is trivial, we only need to prove the opposite inequality. As remarked after Theorem 3.2, there is some positive auxiliary function $a(\cdot)$, with $a(x) \uparrow \infty$ and $a(x) = o(x)$, such that $\mathrm{P}(\theta_i > a(x)) = o(1)\mathrm{P}(\theta_i X > x)$ holds for all $i = 0, 1, \ldots, n$. Let $\theta_{(n)} = \bigvee_{i=1}^n \theta_i$. For arbitrarily fixed $0 < \varepsilon < 1$, we derive

$$\begin{aligned}
\mathrm{P}\left(\theta_0 X - \sum_{i=1}^n \theta_i Y_i > x\right) &\geq \mathrm{P}\left(\theta_0 X - \sum_{i=1}^n \theta_i Y_i > x, \varepsilon\theta_{(n)} < \theta_0 \leq a(x)\right) \\
&\geq \mathrm{P}\left(\theta_0 X - \frac{\theta_0}{\varepsilon}\sum_{i=1}^n Y_i > x, \varepsilon\theta_{(n)} < \theta_0 \leq a(x)\right) \\
&\sim \mathrm{P}\left(\theta_0 X > x, \varepsilon\theta_{(n)} < \theta_0 \leq a(x)\right) \\
&\geq \mathrm{P}\left(\theta_0 X > x\right) - \mathrm{P}\left(\theta_0 X > x, \varepsilon\theta_{(n)} \geq \theta_0\right) - \mathrm{P}\left(\theta_0 > a(x)\right), \quad (5.8)
\end{aligned}$$

where in the third step we conditioned on θ_0 and applied the dominated convergence theorem and $F \in \mathcal{L}$. It holds for every $0 < \gamma < M_*(F)$, some $C > 0$ and all large x that

$$\begin{aligned}
\mathrm{P}\left(\theta_0 X > x, \varepsilon\theta_{(n)} \geq \theta_0\right) &\leq \mathrm{P}\left(\varepsilon\theta_{(n)} X > x\right) \\
&\leq \mathrm{P}\left(\varepsilon\theta_{(n)} X > x, \theta_{(n)} \leq a(x)\right) + \mathrm{P}\left(\theta_{(n)} > a(x)\right) \\
&\leq C\varepsilon^\gamma \mathrm{P}\left(\theta_{(n)} X > x\right) + \sum_{i=1}^n \mathrm{P}\left(\theta_i > a(x)\right) \\
&\lesssim C\varepsilon^\gamma \sum_{i=1}^n \mathrm{P}\left(\theta_i X > x\right),
\end{aligned}$$

where the third step is due to relation (2.4). Substituting this into (5.8), by the arbitrariness of ε we conclude the proof. ■

Proof of Theorem 3.2. Temporarily assume that the random weights are positive.

First we prove (5.2). For an arbitrary subset $I \subset \{1, \ldots, n\}$, write $I^c = \{1, \ldots, n\} \backslash I$ and

$$\Omega_I(X) = \left\{\omega : X_i \geq 0 \text{ for } i \in I,\ X_j < 0 \text{ for } j \in I^c\right\}.$$

Recall the equivalence between (3.1) and (3.2). With the positive auxiliary function $a(\cdot)$ specified in relation (3.2) for all $i = 1, \ldots, n$, it holds that

$$\mathrm{P}\left(S_n^\theta > x\right) \geq \sum_{\varnothing \neq I \subset \{1,\ldots,n\}} \mathrm{P}\left(\sum_{i \in I} \frac{\theta_i}{\theta_{(n)}} X_i + \sum_{j \in I^c} \frac{\theta_j}{\theta_{(n)}} X_j > \frac{x}{\theta_{(n)}}, \theta_{(n)} \leq a(x), \Omega_I(X)\right).$$

By conditioning on $\theta_1, \ldots, \theta_n$ and X_j for $j \in I^c$ and applying Lemma 5.4, we obtain

$$\begin{aligned}
\mathrm{P}\left(S_n^\theta > x\right) &\gtrsim \sum_{\varnothing \neq I \subset \{1,\ldots,n\}} \sum_{i \in I} \mathrm{P}\left(\frac{\theta_i}{\theta_{(n)}} X_i + \sum_{j \in I^c} \frac{\theta_j}{\theta_{(n)}} X_j > \frac{x}{\theta_{(n)}}, \theta_{(n)} \leq a(x), \Omega_I(X)\right) \\
&\geq \sum_{\varnothing \neq I \subset \{1,\ldots,n\}} \sum_{i \in I} \mathrm{P}\left(\theta_i X_i + \sum_{j \in I^c} \theta_j X_j > x, \Omega_I(X)\right) - \mathrm{P}\left(\theta_{(n)} > a(x)\right).
\end{aligned}$$

Note that $\mathrm{P}(\theta_j X_i > x) \asymp \mathrm{P}(\theta_i X_i > x)$ holds for all $1 \leq i, j \leq n$. Applying Lemma 5.6 and interchanging the order of the summations yield

$$\begin{aligned}
&\mathrm{P}\left(S_n^\theta > x\right) \\
&\gtrsim \sum_{\varnothing \neq I \subset \{1,\ldots,n\}} \sum_{i \in I} \left(\mathrm{P}\left(\theta_i X_i > x, \Omega_I(X)\right) - o(1) \sum_{j \in I^c} \mathrm{P}\left(\theta_j X_i > x\right)\right) - \sum_{i=1}^n \mathrm{P}\left(\theta_i > a(x)\right) \\
&= \sum_{i=1}^n \mathrm{P}\left(\theta_i X_i > x\right) - o(1) \sum_{\varnothing \neq I \subset \{1,\ldots,n\}} \sum_{i \in I, j \in I^c} \mathrm{P}\left(\theta_j X_i > x\right) - o(1) \sum_{i=1}^n \mathrm{P}\left(\theta_i X_i > x\right) \\
&\sim \sum_{i=1}^n \mathrm{P}\left(\theta_i X_i > x\right).
\end{aligned}$$

Next we prove (5.3). Clearly,

$$\begin{aligned}
\mathrm{P}\left(\sum_{i=1}^n \theta_i X_i^+ > x\right) &\leq \mathrm{P}\left(\sum_{i=1}^n \frac{\theta_i}{\theta_{(n)}} X_i^+ > \frac{x}{\theta_{(n)}}, \theta_{(n)} \leq a(x)\right) + \mathrm{P}\left(\theta_{(n)} > a(x)\right) \\
&\lesssim \sum_{i=1}^n \mathrm{P}\left(\frac{\theta_i}{\theta_{(n)}} X_i > \frac{x}{\theta_{(n)}}, \theta_{(n)} \leq a(x)\right) + \sum_{i=1}^n \mathrm{P}\left(\theta_i > a(x)\right) \\
&\sim \sum_{i=1}^n \mathrm{P}\left(\theta_i X_i > x\right),
\end{aligned}$$

where in the second step we applied Lemma 5.5.

So far we have proven Theorem 3.2 for the case with positive random weights. The extension to the case with nonnegative random weights can be done by copying (5.7). ■

5.3 Proof of Theorem 3.3

The following lemma is a slight extension of Lemma 6.2 of Tang and Yuan (2012) and is at the core of the proof of Theorem 3.3:

Lemma 5.7 *Let X be a random variable with a dominatedly-varying right tail and upper Matuszewska index M^*, let θ be a nonnegative random variable with $\mathrm{E}\left[\theta^\beta\right] < \infty$ for some*

$\beta > M^*$, let $\{\Delta_t, t \in \mathcal{T}\}$ be a set of random events satisfying $\lim_{t\to t_0} \mathrm{P}(\Delta_t) = 0$ for some t_0 in the closure of the index set $\mathcal{T}$, and let $\{\theta, \{\Delta_t, t \in \mathcal{T}\}\}$ be independent of X. Then

$$\lim_{t\to t_0} \limsup_{x\to\infty} \frac{\mathrm{P}(\theta X > x, \Delta_t)}{\mathrm{P}(\theta X > x)} = \lim_{t\to t_0} \limsup_{x\to\infty} \frac{\mathrm{P}(\theta X > x, \Delta_t)}{\mathrm{P}(X > x)} = 0.$$

Proof. Since $\mathrm{P}(\theta X > x) \asymp \mathrm{P}(X > x)$ by Theorem 3.3(iv) of Cline and Samorodnitsky (1994), we only need to prove the second relation. Choose some c such that $M^* < c\beta < \beta$ and do the split

$$\mathrm{P}(\theta X > x, \Delta_t) = \mathrm{P}(\theta X > x, \Delta_t, \theta \le x^c) + \mathrm{P}(\theta X > x, \Delta_t, \theta > x^c).$$

By inequality (2.3), there is some constant $C > 0$ such that, for all large x,

$$\mathrm{P}(\theta X > x, \Delta_t, \theta \le x^c) \le C\mathrm{P}(X > x)\,\mathrm{E}\left[(\theta \vee 1)^\beta \mathbf{1}_{\Delta_t}\right].$$

Moreover, by Markov's inequality and the second relation in (2.5),

$$\mathrm{P}(\theta X > x, \Delta_t, \theta > x^c) \le \mathrm{P}(\theta > x^c) \le x^{-c\beta}\mathrm{E}\left[\theta^\beta\right] = o(1)\mathrm{P}(X > x).$$

By these upper bounds we conclude the proof. ■

Proof of Theorem 3.3. We first prove (5.2). Since each $\theta_i X_i$ is long tailed by Lemma 5.2, we can choose some positive function $l(\cdot)$, with $l(x) \uparrow \infty$ and $l(x) \le x/2$, such that the relation $\mathrm{P}(\theta_i X_i > x + y) \sim \mathrm{P}(\theta_i X_i > x)$ holds uniformly for $-l(x) \le y \le l(x)$ and $i = 1, \ldots, n$. By Bonferroni's inequality, we have

$$\begin{aligned}
&\mathrm{P}\left(S_n^\theta > x\right) \\
&\ge \mathrm{P}\left(S_n^\theta > x, \bigvee_{i=1}^n \theta_i X_i > x + l(x)\right) \\
&\ge \sum_{i=1}^n \mathrm{P}\left(S_n^\theta > x, \theta_i X_i > x + l(x)\right) - \sum_{1\le j<k\le n} \mathrm{P}(\theta_j X_j > x + l(x), \theta_k X_k > x + l(x)) \\
&= (1 + o(1)) \sum_{i=1}^n \mathrm{P}(\theta_i X_i > x + l(x)) - \sum_{i=1}^n \mathrm{P}\left(S_n^\theta \le x, \theta_i X_i > x + l(x)\right) \\
&\gtrsim (1 + o(1)) \sum_{i=1}^n \mathrm{P}(\theta_i X_i > x) - \sum_{i=1}^n \mathrm{P}\left(\theta_i X_i > x + l(x), \sum_{j=1, j\ne i}^n \theta_j X_j < -l(x)\right) \\
&\sim \sum_{i=1}^n \mathrm{P}(\theta_i X_i > x),
\end{aligned}$$

where in the third and last steps we applied Lemma 5.7.

To show relation (5.3), we observe that

$$
\begin{aligned}
&\mathrm{P}\left(\sum_{i=1}^{n}\theta_i X_i^{+} > x\right) \\
&\leq \mathrm{P}\left(\bigvee_{i=1}^{n}\theta_i X_i > x - l(x)\right) + \mathrm{P}\left(\sum_{i=1}^{n}\theta_i X_i^{+} > x, \bigvee_{j=1}^{n}\theta_j X_j \leq x - l(x), \bigvee_{k=1}^{n}\theta_k X_k > \frac{x}{n}\right) \\
&\leq \sum_{i=1}^{n}\mathrm{P}\left(\theta_i X_i > x - l(x)\right) + \sum_{k=1}^{n}\mathrm{P}\left(\theta_k X_k > \frac{x}{n}, \sum_{i=1, i\neq k}^{n}\theta_i X_i^{+} > l(x)\right) \\
&\sim \sum_{i=1}^{n}\mathrm{P}\left(\theta_i X_i > x\right),
\end{aligned}
$$

where in the last step we applied Lemma 5.7 again. The proof is complete. ■

6 Proofs of Theorems 4.1–4.2 and Corollary 4.1

In the proofs of Theorems 4.1–4.2, we define a new probability measure Q on $(\Omega, \mathcal{F})$ by

$$\frac{\mathrm{dQ}}{\mathrm{dP}} = \frac{\theta_1 X_1^{+}}{\mathrm{E}\left[\theta_1\right]\mathrm{E}\left[X_1^{+}\right]}, \tag{6.1}$$

where, and throughout this section, the symbol E without a superscript means the expectation still under P. Under Q, it is easy to verify the following facts:

- the two sequences $\{X_1, \ldots, X_n\}$ and $\{\theta_1, \ldots, \theta_n\}$ are still independent;
- the primary random variables X_1, …, X_n are still independent with the distribution functions of X_2, …, X_n remaining intact;
- the tail probabilities of X_1 and $\theta_1 X_1$ are given by

  $$\mathrm{Q}\left(X_1 > x\right) = \frac{x\overline{F_1}(x) + \int_x^\infty \overline{F_1}(y)\mathrm{d}y}{\mathrm{E}\left[X_1\right]}, \qquad x > 0,$$

  and

  $$\mathrm{Q}\left(\theta_1 X_1 > x\right) = \frac{x\mathrm{P}(\theta_1 X_1 > y) + \int_x^\infty \mathrm{P}(\theta_1 X_1 > y)\mathrm{d}y}{\mathrm{E}\left[\theta_1\right]\mathrm{E}\left[X_1\right]}, \qquad x > 0,$$

  respectively, implying that the distribution functions of X_1 and $\theta_1 X_1$ are both long tailed.

In the proofs below, we shall often apply these facts tacitly.

6.1 Proof of Theorem 4.1

Clearly, it suffices to prove that

$$\mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{\left(S_n^\theta > x\right)}\right] \gtrsim \mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right] \tag{6.2}$$

and that

$$\mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(\sum_{i=1}^n \theta_i X_i^+ > x\right)}\right] \sim \mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right]. \tag{6.3}$$

To prove relation (6.2), note that

$$\begin{aligned}
&\mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{\left(S_n^\theta > x\right)}\right] \\
&= \mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(S_n^\theta > x\right)}\right] - \mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(S_n^\theta > x\right)}\right] \\
&\geq \mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(\theta_1 X_1 - \sum_{i=2}^n \theta_i X_i^- > x\right)}\right] - \mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(\sum_{i=2}^n \theta_i X_i > x\right)}\right] \\
&= \mathrm{E}\left[\theta_1\right] \mathrm{E}\left[X_1^+\right] \mathrm{Q}\left(\theta_1 X_1 - \sum_{i=2}^n \theta_i X_i^- > x\right) - \mathrm{E}\left[X_1^-\right] \mathrm{E}\left[\theta_1 \mathbf{1}_{\left(\sum_{i=2}^n \theta_i X_i > x\right)}\right],
\end{aligned}$$

while to prove relation (6.3), note that, for $x > 0$,

$$\mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(\sum_{i=1}^n \theta_i X_i^+ > x\right)}\right] = \mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right] + \mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(\theta_1 X_1^+ \leq x, \sum_{i=1}^n \theta_i X_i^+ > x\right)}\right].$$

Thus, it suffices to prove the following:

$$\mathrm{Q}\left(\theta_1 X_1 - \sum_{i=2}^n \theta_i X_i^- > x\right) \gtrsim \mathrm{Q}\left(\theta_1 X_1 > x\right); \tag{6.4}$$

$$\mathrm{E}\left[\theta_1 \mathbf{1}_{\left(\sum_{i=2}^n \theta_i X_i > x\right)}\right] = o(1) \mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right]; \tag{6.5}$$

$$\mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(\theta_1 X_1^+ \leq x, \sum_{i=1}^n \theta_i X_i^+ > x\right)}\right] = o(1) \mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right]. \tag{6.6}$$

Assuming that relation (4.2) is satisfied, we shall prove relations (6.4)–(6.6) under the conditions of Theorem 3.1, Theorem 3.2, or Theorem 3.3.

(a) Assume that the conditions of Theorem 3.1 hold, and assume without loss of generality that the random weights $\theta_1, \ldots, \theta_n$ are bounded above by 1. Relations (6.4) and (6.5) hold straightforwardly since

$$\mathrm{Q}\left(\theta_1 X_1 - \sum_{i=2}^n \theta_i X_i^- > x\right) \geq \mathrm{Q}\left(\theta_1 X_1 - \sum_{i=2}^n X_i^- > x\right) \sim \mathrm{Q}\left(\theta_1 X_1 > x\right),$$

and

$$\mathrm{E}\left[\theta_1 \mathbf{1}_{\left(\sum_{i=2}^n \theta_i X_i > x\right)}\right] \leq \mathrm{P}\left(\sum_{i=2}^n \theta_i X_i > x\right) = o(1) \mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right],$$

where in the last step we applied Theorem 3.1, relation (4.2), and the fact that $x\mathrm{P}\left(\theta_1 X_1 > x\right) \leq \mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right]$. To prove relation (6.6), note that, for $x > 0$,

$$\begin{aligned}
&\mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(\theta_1 X_1^+ \leq x, \sum_{i=1}^n \theta_i X_i^+ > x\right)}\right] \\
&\leq \sqrt{x}\mathrm{P}\left(\sum_{i=1}^n \theta_i X_i^+ > x\right) + x\mathrm{P}\left(\sqrt{x} < \theta_1 X_1 \leq x, \sum_{i=1}^n \theta_i X_i^+ > x\right). \qquad (6.7)
\end{aligned}$$

By Theorem 3.1 and relation (4.2), we have

$$\mathrm{P}\left(\sum_{i=1}^n \theta_i X_i^+ > x\right) \sim \sum_{i=1}^n \mathrm{P}\left(\theta_i X_i > x\right) = O(1)\mathrm{P}\left(\theta_1 X_1 > x\right). \qquad (6.8)$$

Moreover, it holds that

$$\begin{aligned}
&\mathrm{P}\left(\sqrt{x} < \theta_1 X_1 \leq x, \sum_{i=1}^n \theta_i X_i^+ > x\right) \\
&\leq \mathrm{P}\left(\sum_{i=1}^n \theta_i X_i^+ > x\right) - \mathrm{P}\left(\theta_1 X_1 > x\right) - \mathrm{P}\left(X_1 \leq \sqrt{x}\right)\mathrm{P}\left(\sum_{i=2}^n \theta_i X_i^+ > x\right) \\
&= o(1)\mathrm{P}\left(\theta_1 X_1 > x\right), \qquad (6.9)
\end{aligned}$$

where we applied Theorem 3.1 again in the last step. A combination of relations (6.7)–(6.9) yields relation (6.6).

(b) Assume that the conditions of Theorem 3.2 hold. First, to prove relation (6.4), observe that, for every $u > 0$,

$$\begin{aligned}
\mathrm{Q}\left(\theta_1 > ux\right) &= \int_0^\infty \mathrm{P}\left(\theta_1 > (ux) \vee s\right) \mathrm{d}s \\
&= u \int_0^\infty \mathrm{P}\left(\theta_1 > u\left(x \vee s\right)\right) \mathrm{d}s \\
&= o(1) \int_0^\infty \mathrm{P}\left(\theta_1 X_1 > \left(x \vee s\right)\right) \mathrm{d}s \\
&= o(1)\mathrm{Q}\left(\theta_1 X_1 > x\right).
\end{aligned}$$

Consequently, there is some positive auxiliary function $\tilde{a}(\cdot)$, with $\tilde{a}(x) \uparrow \infty$ and $\tilde{a}(x) = o(x)$, such that

$$\mathrm{Q}\left(\theta_1 > \tilde{a}(x)\right) = o(1)\mathrm{Q}\left(\theta_1 X_1 > x\right). \qquad (6.10)$$

Similarly to the derivation of relation (5.8), we have, for $\theta_{(n)} = \bigvee_{i=2}^n \theta_i$ and arbitrarily fixed $0 < \varepsilon < 1$,

$$\begin{aligned}
&\mathrm{Q}\left(\theta_1 X_1 - \sum_{i=2}^n \theta_i X_i^- > x\right) \\
&\gtrsim \mathrm{Q}\left(\theta_1 X_1 > x\right) - \mathrm{Q}\left(\theta_1 X_1 > x, \varepsilon\theta_{(n)} \geq \theta_1\right) - \mathrm{Q}\left(\theta_1 > \tilde{a}(x)\right). \qquad (6.11)
\end{aligned}$$

By relation (4.2), there is some $C > 0$ such that, for $x > 0$,

$$
\begin{aligned}
\mathrm{Q}\left(\theta_1 X_1 > x, \varepsilon\theta_{(n)} \geq \theta_1\right) &= \frac{1}{\mathrm{E}\left[\theta_1 X_1^+\right]}\mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{\left(\theta_1 X_1 > x,\ \varepsilon\theta_{(n)} \geq \theta_1\right)}\right] \\
&\leq \frac{1}{\mathrm{E}\left[\theta_1 X_1^+\right]}\mathrm{E}\left[\varepsilon\theta_{(n)} X_1 \mathbf{1}_{\left(\theta_{(n)} X_1 > x\right)}\right] \\
&= \frac{\varepsilon}{\mathrm{E}\left[\theta_1 X_1^+\right]}\int_0^\infty \mathrm{P}\left(\theta_{(n)} X_1 > (x \vee s)\right) \mathrm{d}s \\
&\leq \frac{C\varepsilon}{\mathrm{E}\left[\theta_1 X_1^+\right]}\int_0^\infty \mathrm{P}\left(\theta_1 X_1 > (x \vee s)\right) \mathrm{d}s \\
&= C\varepsilon \mathrm{Q}\left(\theta_1 X_1 > x\right).
\end{aligned}
$$

Substituting this and (6.10) into (6.11) and noticing the arbitrariness of ε, we obtain (6.4).

Next, to prove relation (6.5), we derive

$$
\begin{aligned}
\mathrm{E}\left[\theta_1 \mathbf{1}_{\left(\sum_{i=2}^n \theta_i X_i > x\right)}\right] &= \mathrm{E}\left[\theta_1 \mathbf{1}_{\left(\theta_1 \leq \tilde{a}(x), \sum_{i=2}^n \theta_i X_i > x\right)}\right] + \mathrm{E}\left[\theta_1 \mathbf{1}_{\left(\theta_1 > \tilde{a}(x), \sum_{i=2}^n \theta_i X_i > x\right)}\right] \\
&\leq \tilde{a}(x)\mathrm{P}\left(\sum_{i=2}^n \theta_i X_i > x\right) + \mathrm{E}\left[\theta_1 \mathbf{1}_{\left(\theta_1 > \tilde{a}(x)\right)}\right] \\
&\sim \tilde{a}(x)\sum_{i=2}^n \mathrm{P}\left(\theta_i X_i > x\right) + \mathrm{E}\left[\theta_1\right]\mathrm{Q}\left(\theta_1 > \tilde{a}(x)\right) \\
&= o(1)\mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{\left(\theta_1 X_1 > x\right)}\right],
\end{aligned}
$$

where we applied Theorem 3.2 in the third step and relation (6.10) in the last step.

Finally, we prove relation (6.6). For every $0 < \varepsilon < 1$, it holds for $x > 0$ that

$$
\begin{aligned}
&\mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(\theta_1 X_1 \leq x, \sum_{i=1}^n \theta_i X_i^+ > x\right)}\right] \\
&\leq \varepsilon x \mathrm{P}\left(\sum_{i=1}^n \theta_i X_i^+ > x\right) + x\mathrm{P}\left(\varepsilon x < \theta_1 X_1 \leq x, \sum_{i=1}^n \theta_i X_i^+ > x\right).
\end{aligned}
$$

By Theorem 3.2 and relation (4.2), we have

$$
\mathrm{P}\left(\sum_{i=1}^n \theta_i X_i^+ > x\right) \sim \sum_{i=1}^n \mathrm{P}\left(\theta_i X_i > x\right) = O(1)\mathrm{P}\left(\theta_1 X_1 > x\right).
$$

With the auxiliary function $a(\cdot)$ given in (3.2), we have

$$
\begin{aligned}
&\mathrm{P}\left(\varepsilon x < \theta_1 X_1 \leq x, \sum_{i=1}^n \theta_i X_i^+ > x\right) \\
&\leq \mathrm{P}\left(\sum_{i=1}^n \theta_i X_i^+ > x\right) - \mathrm{P}\left(\theta_1 X_1 > x\right) - \mathrm{P}\left(X_1 \leq \varepsilon\frac{x}{a(x)}\right)\mathrm{P}\left(\theta_1 \leq a(x), \sum_{i=2}^n \theta_i X_i^+ > x\right) \\
&= o(1)\mathrm{P}\left(\theta_1 X_1 > x\right),
\end{aligned}
$$

where in the last step we applied Theorem 3.2 twice. By the arbitrariness of ε, we conclude that

$$\mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(\theta_1 X_1 \le x, \sum_{i=1}^n \theta_i X_i^+ > x\right)}\right] = o(x)\mathrm{P}\left(\theta_1 X_1 > x\right) = o(1)\mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right].$$

(c) Assume that the conditions of Theorem 3.3 hold. First, to prove relation (6.4), observe that there is some positive function $l(\cdot)$, with $l(x) \le x/2$ and $l(x) \uparrow \infty$, such that $\mathrm{Q}\left(\theta_1 X_1 > x + l(x)\right) \sim \mathrm{Q}\left(\theta_1 X_1 > x\right)$. Also observe that the distribution function of X_1 under Q has an upper Matuszewska index not greater than $M^*(F_1) - 1 \in [0, \infty)$ and that $\mathrm{E}^{\mathrm{Q}}\left[\theta_1^{\beta_1 - 1}\right] = \mathrm{E}\left[\theta_1^{\beta_1}\right] / \mathrm{E}\left[\theta_1\right] < \infty$. We have

$$\begin{aligned}
&\mathrm{Q}\left(\theta_1 X_1 - \sum_{i=2}^n \theta_i X_i^- > x\right) \\
&\ge \mathrm{Q}\left(\theta_1 X_1 > x + l(x), \sum_{i=2}^n \theta_i X_i^- \le l(x)\right) \\
&= \mathrm{Q}\left(\theta_1 X_1 > x + l(x)\right) - \mathrm{Q}\left(\theta_1 X_1 > x + l(x), \sum_{i=2}^n \theta_i X_i^- > l(x)\right) \\
&\sim \mathrm{Q}\left(\theta_1 X_1 > x\right),
\end{aligned}$$

where in the last step we applied Lemma 5.7.

Next, to prove relation (6.5), choose some c such that $M^*(F_1) < c\beta_1 < \beta_1$. For every $\varepsilon > 0$, it holds for $x > 0$ that

$$\begin{aligned}
\mathrm{E}\left[\theta_1 \mathbf{1}_{\left(\sum_{i=2}^n \theta_i X_i > x\right)}\right] &= \mathrm{E}\left[\theta_1 \mathbf{1}_{\left(\theta_1 \le x^c, \sum_{i=2}^n \theta_i X_i > x\right)}\right] + \mathrm{E}\left[\theta_1 \mathbf{1}_{\left(\theta_1 > x^c, \sum_{i=2}^n \theta_i X_i > x\right)}\right] \\
&\le x^c \mathrm{P}\left(\sum_{i=2}^n \theta_i X_i > x\right) + \mathrm{E}\left[\theta_1 \mathbf{1}_{(\theta_1 > x^c)}\right] \\
&\le o(x) \sum_{i=2}^n \mathrm{P}\left(\theta_i X_i > x\right) + \left(\mathrm{E}\left[\theta_1^{\beta_1}\right]\right)^{1/\beta_1} \left(\mathrm{P}\left(\theta_1 > x^c\right)\right)^{(\beta_1 - 1)/\beta_1} \\
&\le o(x)\mathrm{P}\left(\theta_1 X_1 > x\right) + \left(\mathrm{E}\left[\theta_1^{\beta_1}\right]\right)^{1/\beta_1 + 1} x^{-c(\beta_1 - 1)},
\end{aligned}$$

where we applied Theorem 3.3 and Hölder's inequality in the third step and Markov's inequality in the last step. Since $c(\beta_1 - 1) + 1 > M^*(F_1)$ and $\mathrm{P}\left(\theta_1 X_1 > x\right) \asymp \mathrm{P}\left(X_1 > x\right)$, we have $x^{-c(\beta_1 - 1)} = o(x)\mathrm{P}\left(\theta_1 X_1 > x\right)$. Therefore, relation (6.5) holds.

Finally, to prove relation (6.6), let c_1 and c_2 be arbitrarily fixed constants satisfying $0 < c_2 < c_1 < 1$ and $M^*(F_1) < c_2 \beta_1$. We have

$$\begin{aligned}
&\mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(\theta_1 X_1 \le x, \sum_{i=1}^n \theta_i X_i^+ > x\right)}\right] \\
&\le x^{c_1} \mathrm{P}\left(\sum_{i=1}^n \theta_i X_i^+ > x\right) + x\mathrm{P}\left(x^{c_1} < \theta_1 X_1 \le x, \sum_{i=1}^n \theta_i X_i^+ > x\right).
\end{aligned}$$

By Theorem 3.3 and relation (4.2), we have

$$\mathrm{P}\left(\sum_{i=1}^{n}\theta_i X_i^+ > x\right) \sim \sum_{i=1}^{n}\mathrm{P}\left(\theta_i X_i > x\right) = O(1)\mathrm{P}\left(\theta_1 X_1 > x\right).$$

Furthermore, it holds that

$$\begin{aligned}
&\mathrm{P}\left(x^{c_1} < \theta_1 X_1 \le x, \sum_{i=1}^{n}\theta_i X_i^+ > x\right) \\
&\le \mathrm{P}\left(\sum_{i=1}^{n}\theta_i X_i^+ > x\right) - \mathrm{P}\left(\theta_1 X_1 > x\right) - \mathrm{P}\left(X_1 \le x^{c_1-c_2}\right)\mathrm{P}\left(\theta_1 \le x^{c_2}, \sum_{i=2}^{n}\theta_i X_i^+ > x\right) \\
&= o(1)\mathrm{P}\left(\theta_1 X_1 > x\right),
\end{aligned}$$

where in the last step we applied Theorem 3.3 again and the fact that $\mathrm{P}\left(\theta_1 > x^{c_2}\right) = o(1)\mathrm{P}(\theta_1 X_1 > x)$. It follows that

$$\mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(\theta_1 X_1 \le x, \sum_{i=1}^{n}\theta_i X_i^+ > x\right)}\right] = o(x)\mathrm{P}\left(\theta_1 X_1 > x\right) = o(1)\mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right].$$

The proof is complete.

6.2 Proof of Theorem 4.2

It is easy to verify that, under Q defined by (6.1), the distribution function of X_1 still belongs to the class $\mathcal{L}\cap\mathcal{D}$ with an upper Matuszewska index not greater than $M^*(F_1)-1 \in [0,\infty)$. Moreover, it holds for every $i = 2,\ldots,n$ that

$$\mathrm{E}^{\mathrm{Q}}\left[\theta_1^{\beta_1-1}\right] = \frac{\mathrm{E}\left[\theta_1^{\beta_1}\right]}{\mathrm{E}\left[\theta_1\right]} < \infty \qquad \text{and} \qquad \mathrm{E}^{\mathrm{Q}}\left[\theta_i^{\beta_i}\right] = \frac{\mathrm{E}\left[\theta_1\theta_i^{\beta_i}\right]}{\mathrm{E}\left[\theta_1\right]} < \infty. \tag{6.12}$$

Therefore, an application of Theorem 3.3 immediately gives

$$\begin{aligned}
\mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(S_n^\theta > x\right)}\right] &= \mathrm{E}\left[\theta_1\right]\mathrm{E}\left[X_1^+\right]\mathrm{Q}\left(S_n^\theta > x\right) \\
&\sim \mathrm{E}\left[\theta_1\right]\mathrm{E}\left[X_1^+\right]\sum_{i=1}^{n}\mathrm{Q}\left(\theta_i X_i > x\right) \\
&= \sum_{i=1}^{n}\mathrm{E}\left[\theta_1 X_1^+ \mathbf{1}_{(\theta_i X_i > x)}\right].
\end{aligned}$$

When X_1 has a nontrivial negative part X_1^-, define another probability measure Q^- on $(\Omega,\mathcal{F})$ by

$$\frac{\mathrm{dQ}^-}{\mathrm{dP}} = \frac{\theta_1 X_1^-}{\mathrm{E}\left[\theta_1\right]\mathrm{E}\left[X_1^-\right]}.$$

Note that under Q^- all distributional results of $\{X_2, \dots, X_n; \theta_1, \dots, \theta_n\}$ remain the same as under Q. Thus, by the moment conditions in (6.12) and Theorem 3.3, we have

$$\mathrm{Q}^-\left(\sum_{i=2}^n \theta_i X_i > x\right) \sim \sum_{i=2}^n \mathrm{Q}^-\left(\theta_i X_i > x\right). \tag{6.13}$$

On the one hand, it follows from relation (6.13) that

$$\begin{aligned}
\mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(S_n^\theta > x\right)}\right] &\le \mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(\sum_{i=2}^n \theta_i X_i > x\right)}\right] \\
&= \mathrm{E}\left[\theta_1\right] \mathrm{E}\left[X_1^-\right] \mathrm{Q}^-\left(\sum_{i=2}^n \theta_i X_i > x\right) \\
&\sim \mathrm{E}\left[\theta_1\right] \mathrm{E}\left[X_1^-\right] \sum_{i=2}^n \mathrm{Q}^-\left(\theta_i X_i > x\right).
\end{aligned} \tag{6.14}$$

On the other hand, to derive an asymptotic lower bound for $\mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(S_n^\theta > x\right)}\right]$, observe that by Lemma 5.2 the distribution function of each product $\theta_i X_i$ under Q^- belongs to $\mathcal{L} \cap \mathcal{D}$. Hence, there is some positive function $l(\cdot)$, with $l(x) \uparrow \infty$ and $l(x) \le x/2$, such that

$$\mathrm{Q}^-\left(\theta_i X_i > x + l(x)\right) \sim \mathrm{Q}^-\left(\theta_i X_i > x\right), \qquad i = 2, \dots, n. \tag{6.15}$$

Moreover,

$$\begin{aligned}
\mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(S_n^\theta > x\right)}\right] &\ge \mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(\sum_{i=2}^n \theta_i X_i > x + l(x), \theta_1 X_1^- \le l(x)\right)}\right] \\
&= \mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(\sum_{i=2}^n \theta_i X_i > x + l(x)\right)}\right] - \mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(\sum_{i=2}^n \theta_i X_i > x + l(x), \theta_1 X_1^- > l(x)\right)}\right].
\end{aligned}$$

By relations (6.13) and (6.15),

$$\begin{aligned}
\mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(\sum_{i=2}^n \theta_i X_i > x + l(x)\right)}\right] &= \mathrm{E}\left[\theta_1\right] \mathrm{E}\left[X_1^-\right] \mathrm{Q}^-\left(\sum_{i=2}^n \theta_i X_i > x + l(x)\right) \\
&\sim \mathrm{E}\left[\theta_1\right] \mathrm{E}\left[X_1^-\right] \sum_{i=2}^n \mathrm{Q}^-\left(\theta_i X_i > x\right).
\end{aligned}$$

Also, note that

$$\begin{aligned}
&\mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(\sum_{i=2}^n \theta_i X_i > x + l(x), \theta_1 X_1^- > l(x)\right)}\right] \\
&\le \sum_{i=2}^n \mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(\theta_i X_i > \frac{x}{n-1}, \theta_1 X_1^- > l(x)\right)}\right] \\
&= \sum_{i=2}^n \mathrm{E}\left[\theta_1\right] \mathrm{E}\left[X_1^-\right] \mathrm{Q}^-\left(\theta_i X_i > \frac{x}{n-1}, \theta_1 X_1^- > l(x)\right) \\
&= o(1) \sum_{i=2}^n \mathrm{Q}^-\left(\theta_i X_i > x\right),
\end{aligned}$$

where the last step is due to Lemma 5.7. We then have

$$\mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(S_n^\theta > x\right)}\right] \gtrsim \mathrm{E}\left[\theta_1\right] \mathrm{E}\left[X_1^-\right] \sum_{i=2}^n \mathrm{Q}^-\left(\theta_i X_i > x\right),$$

which together with relation (6.14) gives

$$\mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(S_n^\theta > x\right)}\right] \sim \sum_{i=2}^n \mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{(\theta_i X_i > x)}\right] = \sum_{i=1}^n \mathrm{E}\left[\theta_1 X_1^- \mathbf{1}_{(\theta_i X_i > x)}\right].$$

This completes the proof.

6.3 Proof of Corollary 4.1

(a) For each $i = 1, \ldots, n$, by conditioning on θ_i we obtain $\mathrm{P}\left(\theta_i X_i > x\right) \sim c_i \mathrm{P}\left(\theta_i X > x\right)$. For every $\varepsilon > 0$, there is some small $0 < \delta < \hat{t}$ such that

$$\mathrm{P}\left(\theta_i > t\right) \leq \left(d_i + \varepsilon\right) \mathrm{P}\left(\theta > t\right)$$

holds for all $\hat{t} - \delta < t \leq \hat{t}$. Since X is rapidly-varying tailed, we have

$$\frac{\mathrm{P}\left(\theta_i X_i > x\right)}{\mathrm{P}\left(\theta X > x\right)} \sim c_i \frac{\mathrm{P}\left(\theta_i X > x\right)}{\mathrm{P}\left(\theta X > x\right)} \sim c_i \frac{\mathrm{P}\left(\theta_i X > x, \theta_i > \hat{t} - \delta\right)}{\mathrm{P}\left(\theta X > x, \theta > \hat{t} - \delta\right)} \leq c_i (d_i + \varepsilon),$$

where the last step is obtained by conditioning on X in both the numerator and the denominator. The opposite asymptotic inequality can be established similarly. Thus, by the arbitrariness of ε we obtain

$$\lim_{x \to \infty} \frac{\mathrm{P}\left(\theta_i X_i > x\right)}{\mathrm{P}\left(\theta X > x\right)} = c_i d_i, \qquad i = 1, \ldots, n. \tag{6.16}$$

It follows from Theorem 3.1 and relation (6.16) that

$$\mathrm{P}\left(S_n^\theta > x\right) \sim \mathrm{P}\left(\theta X > x\right) \sum_{i=1}^n c_i d_i \tag{6.17}$$

and, hence, that

$$x = \mathrm{VaR}_q\left[S_n^\theta\right] \sim \mathrm{VaR}_{\tilde{q}}\left[\theta X\right], \qquad q \uparrow 1. \tag{6.18}$$

Moreover, since the product θX is rapidly-varying tailed, by (2.4), for arbitrarily fixed $\gamma > 1$, there is some $C > 0$ such that

$$\frac{\mathrm{P}\left(\theta X > xy\right)}{\mathrm{P}\left(\theta X > x\right)} \leq C y^{-\gamma}$$

holds for all large x and all $y \geq 1$. Therefore, by the dominated convergence theorem and relation (6.16),

$$\begin{aligned}
\mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right] &= x\mathrm{P}\left(\theta_1 X_1 > x\right)\left(1 + \int_1^\infty \frac{\mathrm{P}\left(\theta_1 X_1 > xy\right)}{\mathrm{P}\left(\theta_1 X_1 > x\right)}\mathrm{d}y\right) \\
&\sim c_1 d_1 x\mathrm{P}\left(\theta X > x\right)\left(1 + \int_1^\infty \frac{\mathrm{P}\left(\theta X > xy\right)}{\mathrm{P}\left(\theta X > x\right)}\mathrm{d}y\right) \\
&\sim c_1 d_1 x\mathrm{P}\left(\theta X > x\right). \qquad (6.19)
\end{aligned}$$

Starting from relation (4.1) and applying Theorem 4.1(a) and relations (6.17)–(6.19), we obtain

$$\mathrm{AC}_1 \sim \frac{\mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right]}{\mathrm{P}\left(\theta X > x\right)\sum_{i=1}^n c_i d_i} \sim \frac{c_1 d_1}{\sum_{i=1}^n c_i d_i}\mathrm{VaR}_{\tilde{q}}\left[\theta X\right], \qquad q \uparrow 1.$$

(b) The proof is similar to that of (a) and, hence, is omitted.

(c) By Theorem 3.3 and Breiman's theorem (see Breiman (1965) and Cline and Samorodnitsky (1994)), we have

$$\mathrm{P}\left(S_n^\theta > x\right) \sim \overline{F}(x)\sum_{i=1}^n c_i \mathrm{E}\left[\theta_i^\alpha\right]. \qquad (6.20)$$

By Proposition 0.8(v) of Resnick (1987), $F^{\leftarrow}(1 - 1/x)$ is regularly varying with index $1/\alpha$. Therefore, as $q \uparrow 1$,

$$x = \mathrm{VaR}_q\left[S_n^\theta\right] \sim F^{\leftarrow}\left(1 - \frac{1-q}{\sum_{i=1}^n c_i \mathrm{E}\left[\theta_i^\alpha\right]}\right) \sim \left(\sum_{i=1}^n c_i \mathrm{E}\left[\theta_i^\alpha\right]\right)^{1/\alpha}\mathrm{VaR}_q[X]. \qquad (6.21)$$

Starting from relation (4.1) and applying Theorem 4.1(c), Breiman's theorem, the dominated convergence theorem and relations (6.20)–(6.21), we obtain

$$\begin{aligned}
\mathrm{AC}_1 &\sim \frac{\mathrm{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right]}{\overline{F}(x)\sum_{i=1}^n c_i \mathrm{E}\left[\theta_i^\alpha\right]} \\
&= \frac{1}{\sum_{i=1}^n c_i \mathrm{E}\left[\theta_i^\alpha\right]}\frac{x\mathrm{P}\left(\theta_1 X_1 > x\right) + \int_x^\infty \mathrm{P}\left(\theta_1 X_1 > y\right)\mathrm{d}y}{\overline{F}(x)} \\
&\sim \frac{c_1 \mathrm{E}\left[\theta_1^\alpha\right]}{\sum_{i=1}^n c_i \mathrm{E}\left[\theta_i^\alpha\right]} x\left(1 + \int_1^\infty y^{-\alpha}\mathrm{d}y\right) \\
&\sim \frac{\alpha}{\alpha - 1}\frac{c_1 \mathrm{E}\left[\theta_1^\alpha\right]}{\left(\sum_{i=1}^n c_i \mathrm{E}\left[\theta_i^\alpha\right]\right)^{1-1/\alpha}}\mathrm{VaR}_q\left[X\right], \qquad q \uparrow 1.
\end{aligned}$$

This concludes the proof.

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