# Interplay of Insurance and Financial Risks in a Discrete-time Model with Strongly Regular Variation

Jinzhu Li[a],* and Qihe Tang[b]

[a] School of Mathematical Science and LPMC, Nankai University

Tianjin 300071, P.R. China

Email: lijinzhu@nankai.edu.cn

[b] Department of Statistics and Actuarial Science, The University of Iowa

241 Schaeffer Hall, Iowa City, IA 52242, USA

Email: qihe-tang@uiowa.edu

April 12, 2014

**Abstract**

Consider an insurance company exposed to a stochastic economic environment that contains two kinds of risk. The first kind is the insurance risk caused by traditional insurance claims, and the second kind is the financial risk resulting from investments. Its wealth process is described in a standard discrete-time model in which, during each period, the insurance risk is quantified as a real-valued random variable $X$ equal to the total amount of claims less premiums, and the financial risk as a positive random variable $Y$ equal to the reciprocal of the stochastic accumulation factor. This risk model builds an efficient platform for investigating the interplay of the two kinds of risk. We focus on the ruin probability and the tail probability of the aggregate risk amount. Assuming that every convex combination of the distributions of $X$ and $Y$ is of strongly regular variation, we derive some precise asymptotic formulas for these probabilities with both finite and infinite time horizons, all in the form of linear combinations of the tail probabilities of $X$ and $Y$. Our treatment is unified in the sense that no dominating relationship between $X$ and $Y$ is required.

*Keywords:* asymptotics; convolution equivalence; financial risk; insurance risk; ruin probabilities; (strongly) regular variation; tail probabilities

*Mathematics Subject Classification*: Primary 62P05; Secondary 62E20, 91B30

# 1 Introduction

As summarized by Norberg (1999), an insurance company which invests its wealth in a financial market is exposed to two kinds of risk. The first kind, called insurance risk, is the

*Corresponding author.

traditional liability risk caused by insurance claims, and the second kind, called financial risk, is the asset risk related to risky investments. The interplay of the two risks unavoidably leads to a complicated stochastic structure for the wealth process of the insurance company. Paulsen (1993) proposed a general continuous-time risk model in which the cash flow of premiums less claims is described as a semimartingale and the log price of the investment portfolio as another semimartingale. Since then the study of ruin in the presence of risky investments has experienced a vital development in modern risk theory; some recent works include Paulsen (2008), Klüppelberg and Kostadinova (2008), Heyde and Wang (2009), Hult and Lindskog (2011), Bankovsky et al. (2011), and Hao and Tang (2012). During this research, much attention has been paid to an important special case of Paulsen's set-up, the so-called bivariate Lévy-driven risk model, in which the two semimartingales are independent Lévy processes fulfilling certain conditions so that insurance claims dominate financial uncertainties.

A well-known folklore says that risky investments may impair the insurer's solvency just as severely as do large claims; see Norberg (1999), Kalashnikov and Norberg (2002), Frolova et al. (2002), and Pergamenshchikov and Zeitouny (2006).

In this paper, we describe the insurance business in a discrete-time risk model in which the two risks are quantified as concrete random variables. This discrete-time risk model builds an efficient platform for investigating the interplay of the two risks. The ruin probabilities of this model have been investigated by Nyrhinen (1999, 2001), Tang and Tsitsiashvili (2003, 2004), Collamore (2009), and Chen (2011), among many others.

Concretely, for each $n \in \mathbb{N} = \{1, 2, \ldots\}$, denote by $X_n$ the insurer's net loss (the total amount of claims less premiums) within period $n$ and by $Y_n$ the stochastic discount factor (the reciprocal of the stochastic accumulation factor) over the same time period. Then the random variables $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ represent the corresponding insurance risks and financial risks, respectively. In this framework, we consider the stochastic present values of aggregate net losses specified as

$$S_0 = 0, \qquad S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j, \qquad n \in \mathbb{N}, \tag{1.1}$$

and consider their maxima

$$M_n = \max_{0 \le k \le n} S_k, \qquad n \in \mathbb{N}. \tag{1.2}$$

If $(X_1, Y_1)$, $(X_2, Y_2)$, $\ldots$ form a sequence of independent and identically distributed (i.i.d.) random pairs fulfilling $-\infty \le \mathrm{E}\ln Y_1 < 0$ and $\mathrm{E}\ln(|X_1| \vee 1) < \infty$, then, by Lemma 1.7 of Vervaat (1979), $S_n$ converges almost surely (a.s.) as $n \to \infty$. In this case, denote by

$S_\infty$ the a.s. limit. Clearly, $M_n$ is non-decreasing in $n$ and

$$0 \leq M_n \leq \sum_{i=1}^{n} (X_i \vee 0) \prod_{j=1}^{i} Y_j.$$

Thus, if $-\infty \leq \mathrm{E} \ln Y_1 < 0$ and $\mathrm{E} \ln (X_1 \vee 1) < \infty$, then $M_n$ also converges a.s. to a limit, denoted by $M_\infty$, as $n \to \infty$.

We conduct risk analysis of the insurance business through studying the tail probabilities of $S_n$ and $M_n$ for $n \in \mathbb{N} \cup \{\infty\}$. The study of tail probabilities is of fundamental interest in insurance, finance, and, in particular, quantitative risk management. Moreover, the tail probability of $M_n$ with $n \in \mathbb{N} \cup \{\infty\}$ is immediately interpreted as the finite-time or infinite-time ruin probability.

In most places of the paper, we restrict ourselves to the standard framework in which $X_1$, $X_2$, ... form a sequence of i.i.d. random variables with generic random variable $X$ and common distribution $F = 1 - \overline{F}$ on $\mathbb{R} = (-\infty, \infty)$, $Y_1$, $Y_2$, ... form another sequence of i.i.d. random variables with generic random variable $Y$ and common distribution $G$ on $(0, \infty)$, and the two sequences are mutually independent.

Under the assumption that the insurance risk $X$ has a regularly-varying tail dominating that of the financial risk $Y$, Tang and Tsitsiashvili (2003, 2004) obtained some precise asymptotic formulas for the finite-time and infinite-time ruin probabilities. The dominating relationship between $X$ and $Y$ holds true if we consider the classical Black–Scholes market in which the log price of the investment portfolio is modelled as a Brownian motion with drift and, hence, $Y$ has a lognormal tail, lighter than every regularly-varying tail. However, empirical data often reveal that the lognormal model significantly underestimates the financial risk. It shows particularly poor performance in reflecting financial catastrophes such as the recent Great Recession since 2008. This intensifies the need to investigate the opposite case where the financial risk $Y$ has a regularly-varying tail dominating that of the insurance risk $X$. In this case, the stochastic quantities in (1.1) and (1.2) become much harder to tackle with the difficulty in studying the tail probability of the product of many independent regularly-varying random variables. Tang and Tsitsiashvili (2003) gave two examples for this opposite case illustrating that, as anticipated, the finite-time ruin probability is mainly determined by the financial risk. Chen and Xie (2005) also studied the finite-time ruin probability of this model and they obtained some related results applicable to the case with the same heavy-tailed insurance and financial risks.

In this paper, under certain technical conditions, we give a unified treatment in the sense that no dominating relationship between the two risks is required. That is to say, the obtained formulas hold uniformly for the cases in which the insurance risk $X$ is more heavy-tailed than, less heavy-tailed than, and equally heavy-tailed as the financial risk $Y$. In our main result, under the assumption that every convex combination of $F$ and $G$ is of strongly regular variation (see Definition 2.1 below), we derive some precise asymptotic formulas for the tail probabilities of $S_n$ and $M_n$ for $n \in \mathbb{N} \cup \{\infty\}$. All the obtained formulas appear to be linear combinations of $\overline{F}$ and $\overline{G}$. Hence, if one of $\overline{F}$ and $\overline{G}$ dominates the other, then this term remains in the formulas but the other term is negligible; otherwise, both terms should simultaneously present. These formulas are in line with the folklore quoted before, confirming that whichever one of the insurance and financial risks with a heavier tail plays a dominating role in leading to the insurer's insolvency.

In the rest of this paper, Section 2 displays our results and some related discussions after introducing the assumptions, Section 3 prepares some necessary lemmas, and Section 4 proves the results.

# 2 Preliminaries and results

Throughout this paper, all limit relationships hold for $x \to \infty$ unless otherwise stated. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \lesssim b(x)$ or $b(x) \gtrsim a(x)$ if $\limsup a(x)/b(x) \le 1$, write $a(x) \sim b(x)$ if both $a(x) \lesssim b(x)$ and $a(x) \gtrsim b(x)$, and write $a(x) \asymp b(x)$ if both $a(x) = O(b(x))$ and $b(x) = O(a(x))$. For a real number $x$, we write $x_+ = x \vee 0$ and $x_- = -(x \wedge 0)$.

## 2.1 Assumptions

We restrict our discussions within the scope of regular variation. A distribution $U$ on $\mathbb{R}$ is said to be of regular variation if $\overline{U}(x) > 0$ for all $x$ and the relation

$$\lim_{x\to\infty} \frac{\overline{U}(xy)}{\overline{U}(x)} = y^{-\alpha}, \qquad y > 0,$$

holds for some $0 \le \alpha < \infty$. In this case we write $U \in \mathcal{R}_{-\alpha}$. However, such a condition is too general to enable us to derive explicit asymptotic formulas for the tail probabilities of the quantities defined in (1.1) and (1.2). To overcome this difficulty, our idea is to employ some existing results and techniques related to the well-developed concept of convolution equivalence.

A distribution $V$ on $[0, \infty)$ is said to be convolution equivalent if $\overline{V}(x) > 0$ for all $x$ and the relations

$$\lim_{x\to\infty} \frac{\overline{V}(x-y)}{\overline{V}(x)} = \mathrm{e}^{\alpha y}, \qquad y \in \mathbb{R}, \tag{2.1}$$

and

$$\lim_{x\to\infty} \frac{\overline{V^{2*}}(x)}{\overline{V}(x)} = 2c < \infty \tag{2.2}$$

hold for some $\alpha \geq 0$, where $V^{2*}$ stands for the 2-fold convolution of $V$. More generally, a distribution $V$ on $\mathbb{R}$ is still said to be convolution equivalent if $V(x)\mathbf{1}_{(x\geq 0)}$ is. In this case we write $V \in \mathcal{S}(\alpha)$. Relation (2.1) itself defines a larger class denoted by $\mathcal{L}(\alpha)$. It is known that the constant $c$ in relation (2.2) is equal to

$$\hat{V}(\alpha) = \int_{-\infty}^{\infty} \mathrm{e}^{\alpha x} V(\mathrm{d}x) < \infty;$$

see Cline (1987) and Pakes (2004). We shall use the notation $\hat{V}(\cdot)$ as above for the moment generating function of a distribution $V$ throughout the paper. The class $\mathcal{S}(0)$ coincides with the well-known subexponential class. Examples and criteria for membership of the class $\mathcal{S}(\alpha)$ for $\alpha > 0$ can be found in Embrechts (1983) and Cline (1986). Note that the gamma distribution belongs to the class $\mathcal{L}(\alpha)$ for some $\alpha > 0$ but does not belong to the class $\mathcal{S}(\alpha)$. Hence, the inclusion $\mathcal{S}(\alpha) \subset \mathcal{L}(\alpha)$ is proper. Recent works in risk theory using convolution equivalence include Klüppelberg et al. (2004), Doney and Kyprianou (2006), Tang and Wei (2010), Griffin and Maller (2012), Griffin et al. (2012), and Griffin (2013).

For a distribution $U$ on $\mathbb{R}$, define

$$V(x) = 1 - \frac{\overline{U}(\mathrm{e}^x)}{\overline{U}(0)}, \qquad x \in \mathbb{R}, \tag{2.3}$$

which is still a proper distribution on $\mathbb{R}$. Actually, if $\xi$ is a real-valued random variable distributed as $U$, then $V$ denotes the conditional distribution of $\ln \xi$ on $\xi > 0$. For every $\alpha \geq 0$, it is clear that $U \in \mathcal{R}_{-\alpha}$ if and only if $V \in \mathcal{L}(\alpha)$. We now introduce a proper subclass of the class $\mathcal{R}_{-\alpha}$.

**Definition 2.1** *A distribution $U$ on $\mathbb{R}$ is said to be of strongly regular variation if $V$ defined by (2.3) belongs to the class $\mathcal{S}(\alpha)$ for some $\alpha \geq 0$. In this case we write $U \in \mathcal{R}^*_{-\alpha}$.*

Examples and criteria for membership of the class $\mathcal{R}^*_{-\alpha}$ can be given completely in parallel with those in Embrechts (1983) and Cline (1986). This distribution class turns out to be crucial for our purpose. Clearly, if $\xi$ follows $U \in \mathcal{R}^*_{-\alpha}$ for some $\alpha \geq 0$ then

$$\mathrm{E}\xi_+^\alpha = \overline{U}(0)\mathrm{E}\left(\mathrm{e}^{\alpha \ln \xi} \middle| \, \xi > 0\right) < \infty$$

since the conditional distribution of $\ln \xi$ on $\xi > 0$ belongs to the class $\mathcal{S}(\alpha)$.

Our standing assumption is as follows:

**Assumption 2.1** *Every convex combination of $F$ and $G$, namely $pF + (1-p)G$ for $0 < p < 1$, belongs to the class $\mathcal{R}^*_{-\alpha}$.*

Some interesting special cases of Assumption 2.1 include:

(a) $F \in \mathcal{R}^*_{-\alpha}$ and $\overline{G}(x) = o(\overline{F}(x))$; or, symmetrically, $G \in \mathcal{R}^*_{-\alpha}$ and $\overline{F}(x) = o(\overline{G}(x))$.

(b) $F \in \mathcal{R}^*_{-\alpha}$, $G \in \mathcal{R}_{-\alpha}$, and $\overline{G}(x) = O(\overline{F}(x))$; or, symmetrically, $G \in \mathcal{R}^*_{-\alpha}$, $F \in \mathcal{R}_{-\alpha}$, and $\overline{F}(x) = O(\overline{G}(x))$.

(c) $F \in \mathcal{R}^*_{-\alpha}$, $G \in \mathcal{R}^*_{-\alpha}$, and the function $b(x) = \overline{F}(\mathrm{e}^x)/\overline{G}(\mathrm{e}^x)$ is $O$-regularly varying (that is to say, $b(xy) \asymp b(x)$ for every $y > 0$).

For (a) and (b), recall a fact that, if $V_1 \in \mathcal{L}(\alpha)$, $V_2 \in \mathcal{L}(\alpha)$, and $\overline{V_1}(x) \asymp \overline{V_2}(x)$, then $V_1 \in \mathcal{S}(\alpha)$ and $V_2 \in \mathcal{S}(\alpha)$ are equivalent; see Theorem 2.1(a) of Klüppelberg (1988) and the sentences before it. This fact can be restated as that, if $U_1 \in \mathcal{R}_{-\alpha}$, $U_2 \in \mathcal{R}_{-\alpha}$, and $\overline{U_1}(x) \asymp \overline{U_2}(x)$, then $U_1 \in \mathcal{R}^*_{-\alpha}$ and $U_2 \in \mathcal{R}^*_{-\alpha}$ are equivalent. By this fact the verifications of (a) and (b) are straightforward. For (c), by Theorem 2.0.8 of Bingham et al. (1987), the relation $b(xy) \asymp b(x)$ holds uniformly on every compact $y$-set of $(0, \infty)$. Then the verification can be done by using Theorems 3.4 and 3.5 of Cline (1987).

## 2.2 The main result

In this subsection, we assume that $\{X, X_1, X_2, \ldots\}$ and $\{Y, Y_1, Y_2, \ldots\}$ are two independent sequences of i.i.d. random variables with $X$ distributed as $F$ on $\mathbb{R}$ and $Y$ as $G$ on $(0, \infty)$. Under Assumption 2.1, by Lemma 3.5 below (with $n = 2$), we have

$$\Pr(XY > x) = \Pr(X_+ Y > x) \sim \mathrm{E}Y^\alpha \overline{F}(x) + \mathrm{E}X_+^\alpha \overline{G}(x).$$

Note that both $\mathrm{E}Y^\alpha$ and $\mathrm{E}X_+^\alpha$ are finite under Assumption 2.1. The moments of $Y$ will appear frequently in the paper, so we introduce a shorthand $\mu_\alpha = \mathrm{E}Y^\alpha$ for $\alpha \geq 0$ to help with the presentation. Starting with this asymptotic formula and proceeding with induction, we shall show in our main result that the relations

$$\Pr(M_n > x) \sim A_n \overline{F}(x) + B_n \overline{G}(x) \tag{2.4}$$

and

$$\Pr(S_n > x) \sim A_n \overline{F}(x) + C_n \overline{G}(x) \tag{2.5}$$

hold for every $n \in \mathbb{N}$, where the coefficients $A_n$, $B_n$, and $C_n$ are given by

$$A_n = \sum_{i=1}^n \mu_\alpha^i, \quad B_n = \sum_{i=1}^n \mu_\alpha^{i-2} \mathrm{E}M_{n-i+1}^\alpha, \quad C_n = \sum_{i=1}^n \mu_\alpha^{i-2} \mathrm{E}S_{n-i+1,+}^\alpha.$$

Furthermore, we shall seek to extend relations (2.4) and (2.5) to $n = \infty$. For this purpose, it is natural to assume $\mu_\alpha < 1$ (which excludes the case $\alpha = 0$) to guarantee the finiteness of the constants $A_\infty$, $B_\infty$, and $C_\infty$. Note in passing that $\mu_\alpha < 1$ implies

$-\infty \leq \mathrm{E}\ln Y < 0$, which is an aforementioned requirement for $S_\infty$ and $M_\infty$ to be a.s. finite. Straightforwardly,

$$A_\infty = \frac{\mu_\alpha}{1-\mu_\alpha} < \infty.$$

It is easy to see that

$$\mathrm{E}M_\infty^\alpha \leq \mathrm{E}\left(\sum_{i=1}^{\infty} X_{i,+}\prod_{j=1}^{i} Y_j\right)^\alpha < \infty. \tag{2.6}$$

Actually, when $0 < \alpha \leq 1$ we use the elementary inequality $\left(\sum_{i=1}^\infty x_i\right)^\alpha \leq \sum_{i=1}^\infty x_i^\alpha$ for any nonnegative sequence $\{x_1, x_2, \ldots\}$, and when $\alpha > 1$ we use Minkowski's inequality. In order for $S_\infty$ to be a.s. finite, we need another technical condition $\mathrm{E}\ln(X_- \vee 1) < \infty$. The finiteness of $\mathrm{E}S_{\infty,+}^\alpha$ can be verified similarly to (2.6). Applying the dominated convergence theorem to the expressions for $B_n$ and $C_n$, we obtain

$$B_\infty = \frac{\mathrm{E}M_\infty^\alpha}{\mu_\alpha(1-\mu_\alpha)} < \infty, \qquad C_\infty = \frac{\mathrm{E}S_{\infty,+}^\alpha}{\mu_\alpha(1-\mu_\alpha)} < \infty. \tag{2.7}$$

Now we are ready to state our main result, whose proof is postponed to Subsections 4.1–4.3.

**Theorem 2.1** *Let $\{X, X_1, X_2, \ldots\}$ and $\{Y, Y_1, Y_2, \ldots\}$ be two independent sequences of i.i.d. random variables with $X$ distributed as $F$ on $\mathbb{R}$ and $Y$ as $G$ on $(0,\infty)$. Under Assumption 2.1, we have the following:*

*(a) Relations (2.4) and (2.5) hold for every $n \in \mathbb{N}$;*

*(b) If $\mu_\alpha < 1$ then relation (2.4) holds for $n = \infty$;*

*(c) If $\mu_\alpha < 1$ and $\mathrm{E}\ln(X_- \vee 1) < \infty$ then relation (2.5) holds for $n = \infty$.*

As we pointed out before, Theorem 2.1 does not require a dominating relationship between $\overline{F}$ and $\overline{G}$. Even in assertions (b) and (c) where $\mu_\alpha < 1$ is assumed, there is not necessarily a dominating relationship between $\overline{F}$ and $\overline{G}$, though the conditions on $\overline{F}$ and $\overline{G}$ become not exactly symmetric any more. Additionally, Theorems 5.2(3) and 6.1 of Tang and Tsitsiashvili (2003) are two special cases of our Theorem 2.1(a) with $\overline{G}(x) = o(\overline{F}(x))$ and $\overline{F}(x) = o(\overline{G}(x))$, respectively.

Since the famous work of Kesten (1973), the tail probabilities of $S_\infty$ and $M_\infty$ have been extensively investigated, mainly in the framework of random difference equations and most under so-called Cramér's condition that $\mu_\alpha = 1$ holds for some $\alpha > 0$. Traditional random difference equations appearing in the literature are often different from ones such as (4.1) and (4.3) below associated to our model. Nevertheless, under our standard assumptions on $\{X_1, X_2, \ldots\}$ and $\{Y_1, Y_2, \ldots\}$, these subtle differences are not essential and the existing results can easily be transformed to our framework. We omit such details here. Corresponding to our model, Kesten's (1973) work shows an asymptotic formula of the form $Cx^{-\alpha}$ assuming, among others, that $Y$ fulfills Cramér's condition and $X$ fulfills a certain integrability condition involving $Y$. Kesten's constant $C$, though positive, is generally unknown. See Enriquez et al. (2009) for a probabilistic representation for this constant. Goldie (1991) studied the same problem but in a broader scope and he simplified Kesten's argument. Note that Cramér's condition is essentially used in these works. Among few works on this topic beyond Cramér's condition we mention Grey (1994) and Goldie and Grübel (1996). For the case where $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, $\mu_{\alpha+\varepsilon} < \infty$ for some $\varepsilon > 0$, and $\mu_\alpha < 1$, indicating that the insurance risk dominates the financial risk, Grey's (1994) work shows a precise asymptotic formula similar to ours. Goldie and Grübel (1996) interpreted the study in terms of perpetuities in insurance and finance and they derived some rough asymptotic formulas. Corresponding to our model, their results show that $S_\infty$ exhibits a light tail if $X$ is light tailed and $\Pr(Y \leq 1) = 1$, while $S_\infty$ must exhibit a heavy tail once $\Pr(Y > 1) > 0$, regardless of the tail behavior of $X$, all being consistent with the consensus on this topic that risky investments are dangerous. We also refer the reader to Hult and Samorodnitsky (2008), Collamore (2009), Blanchet and Sigman (2011), and Hitczenko and Wesołowski (2011) for recent interesting developments on the topic.

In contrast to these existing results, we do not require Cramér's condition or a dominating relationship between $\overline{F}$ and $\overline{G}$ in Theorem 2.1(b, c). The coefficients $B_\infty$ and $C_\infty$ appearing in our formulas, though still generally unknown, assume transparent structures as given in (2.7), which enable one to easily conduct numerical estimates.

The condition $\mu_\alpha < 1$ in Theorem 2.1(b, c) is made mainly to ensure the finiteness of $B_\infty$ and $C_\infty$. However, it excludes some apparently simpler cases such as $G \in \mathcal{R}_0^*$ and classical random walks (corresponding to $\Pr(Y = 1) = 1$). The tail behavior of the maximum of a random walk with negative drift, especially with heavy-tailed increments, has been systematically investigated by many people; see, e.g. Feller (1971), Veraverbeke (1977), Korshunov (1997), Borovkov (2003), Denisov et al. (2004), and Foss et al. (2011), among many others. The study of random walks hints that the tail probabilities of $S_\infty$ and $M_\infty$ behave essentially differently between the cases $\mu_\alpha < 1$ and $\mu_\alpha = 1$. Actually, if $\mu_\alpha = 1$, then all of $A_n$, $B_n$, and $C_n$ diverge to $\infty$ as $n \to \infty$, and Theorem 2.1 leads to

$$\lim_{x\to\infty} \frac{\Pr(S_\infty > x)}{\overline{F}(x) + \overline{G}(x)} = \lim_{x\to\infty} \frac{\Pr(M_\infty > x)}{\overline{F}(x) + \overline{G}(x)} = \infty.$$

This fails to give precise asymptotic formulas for $\Pr(S_\infty > x)$ and $\Pr(M_\infty > x)$, though still consistent with Kesten and Goldie's formula $Cx^{-\alpha}$ since $\overline{F}(x) + \overline{G}(x) = o(x^{-\alpha})$. For this case, intriguing questions include how to capture the precise asymptotics other than Kesten and Goldie's for $\Pr(S_\infty > x)$ and $\Pr(M_\infty > x)$ and how to connect the asymptotics for $\Pr(M_n > x)$ and $\Pr(S_n > x)$ as $x \wedge n \to \infty$ to Kesten and Goldie's formula $Cx^{-\alpha}$. The approach developed in the present paper seems not efficient to give a satisfactory answer to either of these questions.

Admittedly, the standard complete independence assumptions on the two sequences $\{X_1, X_2, \ldots\}$ and $\{Y_1, Y_2, \ldots\}$, though often appearing in the literature, are not of practical relevance. However, Theorem 2.1 offers new insights into the tail probabilities of the sums in (1.1) and their maxima in (1.2), revealing the interplay between the insurance and financial risks. Furthermore, extensions that incorporate various dependence structures into the model are expected and usually without much difficulty. We show in the next subsection a simple example for such extensions.

## 2.3 An extension

As done by Chen (2011), in this subsection we assume that $\{(X, Y), (X_1, Y_1), (X_2, Y_2), \ldots\}$ is a sequence of i.i.d. random pairs with $(X, Y)$ following a Farlie–Gumbel–Morgenstern (FGM) distribution

$$\pi(x, y) = F(x)G(y)\left(1 + \theta\overline{F}(x)\overline{G}(y)\right), \qquad \theta \in [-1, 1],\ x \in \mathbb{R},\ y > 0, \tag{2.8}$$

where $F$ on $\mathbb{R}$ and $G$ on $(0, \infty)$ are two marginal distributions. In view of the decomposition

$$\pi = (1 + \theta)FG - \theta F^2 G - \theta F G^2 + \theta F^2 G^2, \tag{2.9}$$

the FGM structure can easily be dissolved. Hereafter, for a random variable $\xi$ and its i.i.d. copies $\xi_1$ and $\xi_2$, denote by $\check{\xi}$ a random variable identically distributed as $\xi_1 \vee \xi_2$ and independent of all other sources of randomness. Under Assumption 2.1, by (2.9) and Lemma 3.5 below, we can conduct an induction procedure to obtain

$$\Pr(M_n > x) \sim A'_n \overline{F}(x) + B'_n \overline{G}(x) \tag{2.10}$$

and

$$\Pr(S_n > x) \sim A'_n \overline{F}(x) + C'_n \overline{G}(x) \tag{2.11}$$

for every $n \in \mathbb{N}$, where

$$A'_n = \left((1 - \theta)\,\mu_\alpha + \theta \mathrm{E}\check{Y}^\alpha\right) \sum_{i=1}^{n} \mu_\alpha^{i-1},$$

$$B'_n = \sum_{i=1}^{n} \mu_\alpha^{i-1} \left((1 - \theta)\,\mathrm{E}\left(M_{n-i} + X_{n-i+1}\right)_+^\alpha + \theta \mathrm{E}\left(M_{n-i} + \check{X}_{n-i+1}\right)_+^\alpha\right),$$

$$C'_n = \sum_{i=1}^{n} \mu_\alpha^{i-1} \left((1 - \theta)\,\mathrm{E}\left(S_{n-i} + X_{n-i+1}\right)_+^\alpha + \theta \mathrm{E}\left(S_{n-i} + \check{X}_{n-i+1}\right)_+^\alpha\right).$$

Additionally, under the conditions of Theorem 2.1(b, c), letting $n \to \infty$ leads to

$$\begin{aligned} A'_\infty &= \frac{1}{1-\mu_\alpha}\left((1-\theta)\,\mu_\alpha + \theta \mathrm{E}\check{Y}^\alpha\right), \\ B'_\infty &= \frac{1}{1-\mu_\alpha}\left((1-\theta)\,\mathrm{E}\left(M_\infty + X\right)_+^\alpha + \theta \mathrm{E}\left(M_\infty + \check{X}\right)_+^\alpha\right), \\ C'_\infty &= \frac{1}{1-\mu_\alpha}\left((1-\theta)\,\mathrm{E}\left(S_\infty + X\right)_+^\alpha + \theta \mathrm{E}\left(S_\infty + \check{X}\right)_+^\alpha\right), \end{aligned}$$

where $X$ and $\check{X}$ are independent of $M_\infty$ and $S_\infty$. It is easy to verify the finiteness of $B'_\infty$ and $C'_\infty$.

We summarize the analysis above into the following corollary and will show a sketch of its proof in Subsection 4.4.

**Corollary 2.1** *Let $\{(X,Y),(X_1,Y_1),(X_2,Y_2),\ldots\}$ be a sequence of i.i.d. random pairs with common FGM distribution (2.8). Under Assumption 2.1, we have the following:*

*(a) Relations (2.10) and (2.11) hold for every $n \in \mathbb{N}$;*

*(b) If $\mu_\alpha < 1$ then relation (2.10) holds for $n = \infty$;*

*(c) If $\mu_\alpha < 1$ and $\mathrm{E}\ln\left(X_- \vee 1\right) < \infty$ then relation (2.11) holds for $n = \infty$.*

As a sanity check, letting $\theta = 0$, the results in Corollary 2.1 coincide with those in Theorem 2.1.

# 3 Lemmas

In this section, we prepare a series of lemmas, some of which are interesting in their own right. We first recall some well-known properties of distributions of regular variation and convolution equivalence. If $U \in \mathcal{R}_{-\alpha}$ for some $0 \le \alpha < \infty$, then for every $\varepsilon > 0$ and every $b > 1$ there is some constant $x_0 > 0$ such that Potter's bounds

$$\frac{1}{b}\left(y^{-\alpha-\varepsilon} \wedge y^{-\alpha+\varepsilon}\right) \le \frac{\overline{U}(xy)}{\overline{U}(x)} \le b\left(y^{-\alpha-\varepsilon} \vee y^{-\alpha+\varepsilon}\right) \tag{3.1}$$

hold whenever $x \ge x_0$ and $xy \ge x_0$; see Theorem 1.5.6(iii) of Bingham et al. (1987). Since $U \in \mathcal{R}_{-\alpha}$ if and only if $V$ defined by (2.3) belongs to $\mathcal{L}(\alpha)$, Potter's bounds above can easily be restated in terms of a distribution $V \in \mathcal{L}(\alpha)$ as that, for every $\varepsilon > 0$ and every $b > 1$ there is some constant $x_0 > 0$ such that the inequalities

$$\frac{1}{b}\left(\mathrm{e}^{-(\alpha+\varepsilon)y} \wedge \mathrm{e}^{-(\alpha-\varepsilon)y}\right) \le \frac{\overline{V}(x+y)}{\overline{V}(x)} \le b\left(\mathrm{e}^{-(\alpha+\varepsilon)y} \vee \mathrm{e}^{-(\alpha-\varepsilon)y}\right) \tag{3.2}$$

hold whenever $x \geq x_0$ and $x + y \geq x_0$. By Lemma 5.2 of Pakes (2004), if $V \in \mathcal{S}(\alpha)$ then it holds for every $n \in \mathbb{N}$ that

$$\lim_{x\to\infty} \frac{\overline{V^{n*}}(x)}{\overline{V}(x)} = n \left(\hat{V}(\alpha)\right)^{n-1}. \tag{3.3}$$

The first lemma below describes an elementary property of convolution equivalence.

**Lemma 3.1** *Let $\eta_1, \ldots, \eta_n$ be $n \geq 2$ i.i.d. real-valued random variables with common distribution $V \in \mathcal{S}(\alpha)$ for some $\alpha \geq 0$. Then*

$$\lim_{c\to\infty} \lim_{x\to\infty} \frac{\Pr\left(\sum_{i=1}^{n} \eta_i > x, \eta_1 > c, \eta_2 > c\right)}{\overline{V}(x)} = 0.$$

**Proof.** For every $x \geq 0$ and $c \geq 0$, write

$$\begin{aligned}
&\Pr\left(\sum_{i=1}^{n} \eta_i > x, \eta_1 > c, \eta_2 > c\right) \\
&= \Pr\left(\sum_{i=1}^{n} \eta_i > x\right) - 2\Pr\left(\sum_{i=1}^{n} \eta_i > x, \eta_1 \leq c\right) + \Pr\left(\sum_{i=1}^{n} \eta_i > x, \eta_1 \leq c, \eta_2 \leq c\right) \\
&= I_1(x) - 2I_2(x,c) + I_3(x,c).
\end{aligned} \tag{3.4}$$

By relation (3.3), we have

$$\lim_{x\to\infty} \frac{I_1(x)}{\overline{V}(x)} = n\left(\hat{V}(\alpha)\right)^{n-1}$$

and

$$\begin{aligned}
\lim_{x\to\infty} \frac{I_2(x,c)}{\overline{V}(x)} &= \lim_{x\to\infty} \int_{-\infty}^{c} \frac{\Pr\left(\sum_{i=1}^{n-1} \eta_i > x - y\right)}{\overline{V}(x-y)} \frac{\overline{V}(x-y)}{\overline{V}(x)} V(\mathrm{d}y) \\
&= (n-1)\left(\hat{V}(\alpha)\right)^{n-2} \int_{-\infty}^{c} \mathrm{e}^{\alpha y} V(\mathrm{d}y),
\end{aligned}$$

where in the last step we used $V \in \mathcal{L}(\alpha)$ and the dominated convergence theorem. Similarly,

$$\begin{aligned}
\lim_{x\to\infty} \frac{I_3(x,c)}{\overline{V}(x)} &= \lim_{x\to\infty} \int_{-\infty}^{c} \int_{-\infty}^{c} \frac{\Pr\left(\sum_{i=1}^{n-2} \eta_i > x - y_1 - y_2\right)}{\overline{V}(x)} V(\mathrm{d}y_1)V(\mathrm{d}y_2) \\
&= (n-2)\left(\hat{V}(\alpha)\right)^{n-3} \left(\int_{-\infty}^{c} \mathrm{e}^{\alpha y} V(\mathrm{d}y)\right)^2.
\end{aligned}$$

Plugging these limits into (3.4) yields the desired result. ■

Hereafter, for $n \geq 2$ distributions $V_1, \ldots, V_n$, denote by $V_{\mathbf{p}} = \sum_{i=1}^{n} p_i V_i$ a convex combination of $V_1, \ldots, V_n$, where $\mathbf{p} \in \Delta = \{(p_1, \ldots, p_n) \in (0,1)^n : \sum_{i=1}^{n} p_i = 1\}$.

**Lemma 3.2** *Let $V_1$, ..., $V_n$ be $n \geq 2$ distributions and let $\alpha \geq 0$. The following are equivalent:*

*(a) $V_{\mathbf{p}} \in \mathcal{S}(\alpha)$ for every $\mathbf{p} \in \Delta$;*

*(b) $V_{\mathbf{p}} \in \mathcal{S}(\alpha)$ for some $\mathbf{p} \in \Delta$ and the relation*

$$\overline{V_i}(x-y) - \mathrm{e}^{\alpha y}\overline{V_i}(x) = o\left(\sum_{j=1}^{n} \overline{V_j}(x)\right) \tag{3.5}$$

*holds for every $y \in \mathbb{R}$ and every $i = 1, \ldots, n$.*

**Proof.** First prove that (b) implies (a). Denote by $\mathbf{p}^*$ this specific element in $\Delta$ such that $V_{\mathbf{p}^*} \in \mathcal{S}(\alpha)$. For every $\mathbf{p} \in \Delta$, it is easy to see that $\overline{V_{\mathbf{p}}}(x) \asymp \sum_{j=1}^{n} \overline{V_j}(x) \asymp \overline{V_{\mathbf{p}^*}}(x)$ and that $V_{\mathbf{p}} \in \mathcal{L}(\alpha)$ by (3.5). Thus, $V_{\mathbf{p}} \in \mathcal{S}(\alpha)$ follows from the closure of the class $\mathcal{S}(\alpha)$ under weak equivalence as mentioned in the last paragraph of Subsection 2.1.

For the other implication, we only need to use (a) to verify (3.5). For arbitrarily fixed $0 < \varepsilon < 1$ and every $i = 1, \ldots, n$, each of the sums $\overline{V_i}(x) + \varepsilon \sum_{j=1, j\neq i}^{n} \overline{V_j}(x)$ and $\sum_{j=1}^{n} \overline{V_j}(x)$ is proportional to a convolution-equivalent tail. Thus,

$$\begin{aligned}
\left|\overline{V_i}(x-y) - \mathrm{e}^{\alpha y}\overline{V_i}(x)\right| \leq & \left| \left(\overline{V_i}(x-y) - \mathrm{e}^{\alpha y}\overline{V_i}(x)\right) + \varepsilon \sum_{j=1, j\neq i}^{n} \left(\overline{V_j}(x-y) - \mathrm{e}^{\alpha y}\overline{V_j}(x)\right) \right| \\
& + \varepsilon \sum_{j=1, j\neq i}^{n} \left|\overline{V_j}(x-y) - \mathrm{e}^{\alpha y}\overline{V_j}(x)\right| \\
\leq & \left| \left(\overline{V_i}(x-y) + \varepsilon \sum_{j=1, j\neq i}^{n} \overline{V_j}(x-y)\right) - \mathrm{e}^{\alpha y} \left(\overline{V_i}(x) + \varepsilon \sum_{j=1, j\neq i}^{n} \overline{V_j}(x)\right) \right| \\
& + \varepsilon \sum_{j=1}^{n} \overline{V_j}(x-y) + \varepsilon \mathrm{e}^{\alpha y} \sum_{j=1}^{n} \overline{V_j}(x) \\
= & \, o(1) \left(\overline{V_i}(x) + \varepsilon \sum_{j=1, j\neq i}^{n} \overline{V_j}(x)\right) + 2\varepsilon \left(\mathrm{e}^{\alpha y} + o(1)\right) \sum_{j=1}^{n} \overline{V_j}(x).
\end{aligned}$$

By the arbitrariness of $\varepsilon$, relation (3.5) follows. ■

The following lemma shows the usefulness of convolution equivalence in dealing with the tail probability of the sum of independent random variables. Note that the lemma does not require any dominating relationship among the individual tails. Additionally, in view of Lemma 3.2, letting $\alpha = 0$ in Lemma 3.3 retrieves Theorem 1 of Li and Tang (2010).

**Lemma 3.3** *Let $V_1$, ..., $V_n$ be $n \geq 2$ distributions on $\mathbb{R}$ and let $\alpha \geq 0$. If $V_{\mathbf{p}} \in \mathcal{S}(\alpha)$ for every $\mathbf{p} \in \Delta$, then $V_1 * \cdots * V_n \in \mathcal{S}(\alpha)$ and*

$$\overline{V_1 * \cdots * V_n}(x) \sim \sum_{i=1}^{n} \left(\prod_{j=1, j\neq i}^{n} \hat{V}_j(\alpha)\right) \overline{V_i}(x). \tag{3.6}$$

**Proof.** Clearly, we only need to prove relation (3.6). Introduce $n$ independent random variables $\eta_1, \ldots, \eta_n$ with distributions $V_1, \ldots, V_n$, respectively. For every $x \geq 0$ and $0 \leq c \leq x/n$,

$$\overline{V_1 * \cdots * V_n}(x) = \Pr\left(\sum_{i=1}^{n} \eta_i > x, \bigcup_{j=1}^{n} (\eta_j > c)\right).$$

According to whether or not there is exactly only one $(\eta_j > c)$ occurring in the union, we split the probability on the right-hand side into two parts as

$$\overline{V_1 * \cdots * V_n}(x) = I_1(x,c) + I_2(x,c). \tag{3.7}$$

First we deal with $I_1(x,c)$. For a real vector $\mathbf{y} = (y_1, \ldots, y_{n-1})'$, write $\Sigma = \sum_{k=1}^{n-1} y_k$, and for each $j = 1, \ldots, n$, write

$$\left(\prod_{k=1,k\neq j}^{n} \mathrm{d}V_k\right)(\mathbf{y}) = V_1(\mathrm{d}y_1)\cdots V_{j-1}(\mathrm{d}y_{j-1})V_{j+1}(\mathrm{d}y_j)\cdots V_n(\mathrm{d}y_{n-1}).$$

We have

$$\begin{aligned}
I_1(x,c) &= \sum_{j=1}^{n} \Pr\left(\sum_{i=1}^{n} \eta_i > x, \eta_j > c, \bigcap_{k=1,k\neq j}^{n} (\eta_k \leq c)\right) \\
&= \sum_{j=1}^{n} \int_{-\infty}^{c} \cdots \int_{-\infty}^{c} \overline{V_j}\,(x-\Sigma) \left(\prod_{k=1,k\neq j}^{n} \mathrm{d}V_k\right)(\mathbf{y}) \\
&= \int_{-\infty}^{c} \cdots \int_{-\infty}^{c} \left(\sum_{j=1}^{n} \overline{V_j}\,(x-\Sigma)\right)\left(\sum_{h=1}^{n}\left(\prod_{k=1,k\neq h}^{n} \mathrm{d}V_k\right)(\mathbf{y})\right) \\
&\quad - \sum_{j=1}^{n} \sum_{h=1,h\neq j}^{n} \int_{-\infty}^{c} \cdots \int_{-\infty}^{c} \overline{V_j}\,(x-\Sigma) \left(\prod_{k=1,k\neq h}^{n} \mathrm{d}V_k\right)(\mathbf{y}).
\end{aligned}$$

Since $\sum_{j=1}^{n} \overline{V_j}\,(x)$ is proportional to a convolution-equivalent tail, by the dominated convergence theorem,

$$\begin{aligned}
I_1(x,c) &\sim \left(\sum_{j=1}^{n} \overline{V_j}\,(x)\right) \int_{-\infty}^{c} \cdots \int_{-\infty}^{c} \mathrm{e}^{\alpha\Sigma} \left(\sum_{h=1}^{n}\left(\prod_{k=1,k\neq h}^{n} \mathrm{d}V_k\right)(\mathbf{y})\right) \\
&\quad - \sum_{j=1}^{n} \sum_{h=1,h\neq j}^{n} \int_{-\infty}^{c} \cdots \int_{-\infty}^{c} \overline{V_j}\,(x-\Sigma) \left(\prod_{k=1,k\neq h}^{n} \mathrm{d}V_k\right)(\mathbf{y}) \\
&= \sum_{j=1}^{n} \overline{V_j}\,(x) \int_{-\infty}^{c} \cdots \int_{-\infty}^{c} \mathrm{e}^{\alpha\Sigma} \left(\prod_{k=1,k\neq j}^{n} \mathrm{d}V_k\right)(\mathbf{y}) \\
&\quad - \sum_{j=1}^{n} \sum_{h=1,h\neq j}^{n} \int_{-\infty}^{c} \cdots \int_{-\infty}^{c} \left(\overline{V_j}\,(x-\Sigma) - \mathrm{e}^{\alpha\Sigma}\overline{V_j}\,(x)\right) \left(\prod_{k=1,k\neq h}^{n} \mathrm{d}V_k\right)(\mathbf{y}).
\end{aligned}$$

Hence, it follows from (3.5) and the dominated convergence theorem that

$$\lim_{c\to\infty}\lim_{x\to\infty}\frac{I_1(x,c)}{\sum_{i=1}^n\left(\prod_{j=1,j\neq i}^n\hat{V}_j(\alpha)\right)\overline{V_i}(x)}=1. \tag{3.8}$$

Next we turn to $I_2(x,c)$. Write $\tilde{\eta}=\max\{\eta_1,\ldots,\eta_n\}$, which has a convolution-equivalent tail proportional to $\sum_{j=1}^n\overline{V_j}\,(x)$, and let $\tilde{\eta}_1$, ..., $\tilde{\eta}_n$ be i.i.d. copies of $\tilde{\eta}$. Clearly,

$$\begin{aligned}I_2(x,c)&=\Pr\left(\sum_{i=1}^n\eta_i>x,\bigcup_{1\leq j<k\leq n}(\eta_j>c,\eta_k>c)\right)\\&\leq\sum_{1\leq j<k\leq n}\Pr\left(\sum_{i=1}^n\tilde{\eta}_i>x,\tilde{\eta}_j>c,\tilde{\eta}_k>c\right).\end{aligned}$$

Thus, by Lemma 3.1,

$$\begin{aligned}&\lim_{c\to\infty}\limsup_{x\to\infty}\frac{I_2(x,c)}{\sum_{i=1}^n\left(\prod_{j=1,j\neq i}^n\hat{V}_j(\alpha)\right)\overline{V_i}(x)}\\&\leq\lim_{c\to\infty}\lim_{x\to\infty}\frac{I_2(x,c)}{\sum_{j=1}^n\overline{V_j}\,(x)}\limsup_{x\to\infty}\frac{\sum_{j=1}^n\overline{V_j}\,(x)}{\sum_{i=1}^n\left(\prod_{j=1,j\neq i}^n\hat{V}_j(\alpha)\right)\overline{V_i}(x)}\\&=0.\end{aligned}\tag{3.9}$$

Plugging (3.8) and (3.9) into (3.7) yields the desired result. ■

Due to the connection between convolution equivalence and strongly regular variation, we can restate Lemmas 3.2 and 3.3 in terms of strongly regular variation. Actually, the next lemma shows an equivalent condition for Assumption 2.1.

**Lemma 3.4** *Let $U_1$, ..., $U_n$ be $n\geq 2$ distributions and let $\alpha\geq 0$. The following are equivalent:*

*(a) $U_{\mathbf{p}}\in\mathcal{R}_{-\alpha}^*$ for every $\mathbf{p}\in\Delta$;*

*(b) $U_{\mathbf{p}}\in\mathcal{R}_{-\alpha}^*$ for some $\mathbf{p}\in\Delta$ and the relation*

$$\overline{U_i}(x/y)-y^\alpha\overline{U_i}(x)=o\left(\sum_{j=1}^n\overline{U_j}(x)\right)$$

*holds for every $y>0$ and every $i=1,\ldots,n$.*

The lemma below expands the tail probability of the product of independent, nonnegative, and strongly regular random variables, forming an analogue of the well-known Breiman's theorem in a different situation. For Breiman's theorem, see Breiman (1965) and Cline and Samorodnitsky (1994).

**Lemma 3.5** *Let $\xi_1, \ldots, \xi_n$ be $n \geq 2$ independent nonnegative random variables with distributions $U_1, \ldots, U_n$, respectively, and let $\alpha \geq 0$. If $U_{\mathbf{p}} \in \mathcal{R}^*_{-\alpha}$ for every $\mathbf{p} \in \Delta$, then the distribution of $\prod_{i=1}^n \xi_i$ belongs to the class $\mathcal{R}^*_{-\alpha}$ and*

$$\Pr\left(\prod_{i=1}^n \xi_i > x\right) \sim \sum_{i=1}^n \left(\prod_{j=1, j\neq i}^n \mathrm{E}\xi_j^\alpha\right) \overline{U_i}(x).$$

The next lemma shows Kesten's bound for convolution tails without the usual requirement $\hat{V}(\alpha) \geq 1$. It improves Lemma 5.3 of Pakes (2004) for the case $0 < \hat{V}(\alpha) < 1$.

**Lemma 3.6** *Let $V$ be a distribution on $\mathbb{R}$. If $V \in \mathcal{S}(\alpha)$ for some $\alpha \geq 0$, then for every $\varepsilon > 0$ there is some constant $K > 0$ such that the relation*

$$\overline{V^{n*}}(x) \leq K\left(\hat{V}(\alpha) + \varepsilon\right)^n \overline{V}(x)$$

*holds for all $n \in \mathbb{N}$ and all $x \geq 0$.*

**Proof.** When $\hat{V}(\alpha) \geq 1$, the assertion has been given in Lemma 5.3 of Pakes (2004). Hence, we only need to consider $\hat{V}(\alpha) < 1$ (for which $\alpha > 0$ must hold). Let $\{\eta, \eta_1, \eta_2, \ldots\}$ be a sequence of i.i.d. random variables with common distribution $V$, and set $c = -\alpha^{-1}\ln \hat{V}(\alpha) > 0$. Clearly,

$$\overline{V^{n*}}(x) = \Pr\left(\sum_{i=1}^n (\eta_i + c) > x + nc\right).$$

Note that the distribution of $\eta + c$ still belongs to the class $\mathcal{S}(\alpha)$ and $\mathrm{E}\mathrm{e}^{\alpha(\eta+c)} = 1$. Hence, for every $\delta > 0$, by Lemma 5.3 of Pakes (2004), there is some constant $K_1 > 0$ such that, for all $n \in \mathbb{N}$ and all $x \geq 0$,

$$\overline{V^{n*}}(x) \leq K_1(1+\delta)^n \Pr(\eta + c > x + nc) = K_1(1+\delta)^n \overline{V}(x + (n-1)c). \tag{3.10}$$

By (3.2), there are some constants $K_2 > 0$ and $x_0 > 0$ such that, for all $n \in \mathbb{N}$ and all $x \geq x_0$,

$$\overline{V}(x + (n-1)c) \leq K_2 \mathrm{e}^{-(\alpha-\delta)(n-1)c}\overline{V}(x). \tag{3.11}$$

Plugging (3.11) into (3.10) and noticing that $\mathrm{e}^{-\alpha c} = \hat{V}(\alpha)$, we have, for all $n \in \mathbb{N}$ and all $x \geq x_0$,

$$\overline{V^{n*}}(x) \leq K_1K_2\mathrm{e}^{(\alpha-\delta)c}\left((1+\delta)\mathrm{e}^{c\delta}\hat{V}(\alpha)\right)^n \overline{V}(x). \tag{3.12}$$

For $0 \leq x < x_0$, we choose an integer $n_0 \geq x_0/c$. Then, for $0 \leq x < x_0$ and $n > n_0$, using the same derivations as in (3.10)–(3.12), we obtain

$$\begin{aligned}\overline{V^{n*}}(x) &\leq K_1(1+\delta)^n \overline{V}(x + n_0c + (n - n_0 - 1)c) \\ &\leq K_1K_2\mathrm{e}^{(\alpha-\delta)(n_0+1)c}\left((1+\delta)\mathrm{e}^{c\delta}\hat{V}(\alpha)\right)^n \overline{V}(x + n_0c) \\ &\leq K_1K_2\mathrm{e}^{(\alpha-\delta)(n_0+1)c}\left((1+\delta)\mathrm{e}^{c\delta}\hat{V}(\alpha)\right)^n \overline{V}(x).\end{aligned} \tag{3.13}$$

At last, for $0 \leq x < x_0$ and $1 \leq n \leq n_0$, it is obvious that

$$\overline{V^{n*}}(x) \leq 1 \leq \frac{\left((1+\delta)\mathrm{e}^{c\delta}\hat{V}(\alpha)\right)^n}{\left((1+\delta)\mathrm{e}^{c\delta}\hat{V}(\alpha)\right)^{n_0} \wedge 1} \frac{\overline{V}(x)}{\overline{V}(x_0)}. \tag{3.14}$$

A combination of (3.12)–(3.14) gives that, for some constant $K > 0$ and for all $n \in \mathbb{N}$ and all $x \geq 0$,

$$\overline{V^{n*}}(x) \leq K\left((1+\delta)\mathrm{e}^{c\delta}\hat{V}(\alpha)\right)^n \overline{V}(x).$$

By setting $\delta$ to be small enough such that $(1+\delta)\mathrm{e}^{c\delta}\hat{V}(\alpha) \leq \hat{V}(\alpha) + \varepsilon$, we complete the proof. ■

The following lemma will be crucial in proving Theorem 2.1(b, c).

**Lemma 3.7** *Let $\{X, X_1, X_2, \ldots\}$ be a sequence of (arbitrarily dependent) random variables with common distribution $F$ on $\mathbb{R}$, let $\{Y, Y_1, Y_2, \ldots\}$ be another sequence of i.i.d. random variables with common distribution $G$ on $[0, \infty)$, and let the two sequences be mutually independent. Assume that there is some distribution $U \in \mathcal{R}^*_{-\alpha}$ for $\alpha > 0$ such that*

$$\overline{F}(x) + \overline{G}(x) = O(\overline{U}(x)).$$

*Assume also that $\mu_\alpha < 1$. Then*

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{1}{\overline{U}(x)} \Pr\left(\sum_{i=n+1}^{\infty} X_i \prod_{j=1}^{i} Y_j > x\right) = 0. \tag{3.15}$$

**Proof.** Choose some large constant $K_1 > 0$ such that the inequality $\overline{F}(x) \vee \overline{G}(x) \leq K_1\overline{U}(x)$ holds for all $x \in \mathbb{R}$, and then introduce a nonnegative random variable $X^*$ with a distribution

$$F^*(x) = \left(1 - K_1\overline{U}(x)\right)_+, \qquad x \geq 0.$$

Clearly, $\overline{F}(x) \leq \overline{F^*}(x) \leq K_1\overline{U}(x)$ for all $x \geq 0$ and $\overline{F^*}(x) = K_1\overline{U}(x)$ for all large $x$. The inequality $\overline{F}(x) \leq \overline{F^*}(x)$ for all $x \geq 0$ means that $X$ is stochastically not greater than $X^*$, denoted by $X \leq_{\mathrm{st}} X^*$. Moreover, since $U \in \mathcal{R}^*_{-\alpha}$, there is some large but fixed constant $t > 0$ such that $K_1 \int_t^\infty z^\alpha U(\mathrm{d}z) < 1 - \mu_\alpha$. For this fixed $t$, define

$$t_0 = \inf\{s \geq t : K_1\overline{U}(s) \leq \overline{G}(t)\},$$

and then introduce another nonnegative random variable $Y^*$ with a distribution

$$G^*(x) = G(x)\mathbf{1}_{(0\leq x<t)} + G(t)\mathbf{1}_{(t\leq x<t_0)} + \left(1 - K_1\overline{U}(x)\right)\mathbf{1}_{(x\geq t_0)}.$$

Clearly, $\mathrm{E}\,(Y^*)^\alpha < 1$, $\overline{G}(x) \le \overline{G^*}(x) \le K_1\overline{U}(x)$ for all $x > 0$, and $\overline{G^*}(x) = K_1\overline{U}(x)$ for all $x \ge t_0$. Thus, $Y \le_{\mathrm{st}} Y^*$. Let $Y_1^*, Y_2^*, \ldots$ be i.i.d. copies of $Y^*$ independent of $X^*$.

Choose some $0 < \varepsilon < \alpha \wedge (1 - \mathrm{E}\,(Y^*)^\alpha)$ such that $\mathrm{E}\,(Y^*)^{\alpha-\varepsilon} < 1$. By Lemma 3.6, there is some constant $K_2 > 0$ such that, for all $i \in \mathbb{N}$ and all $x \ge 1$,

$$\Pr\left(\prod_{j=1}^{i} Y_j^* > x\right) = \Pr\left(\sum_{j=1}^{i} \ln Y_j^* > \ln x\right) \le K_2\left(\mathrm{E}\,(Y^*)^\alpha + \varepsilon\right)^i \overline{G^*}(x). \tag{3.16}$$

Noticeably, the derivation in (3.16) tacitly requires that $Y_1^*, \ldots, Y_j^*$ are positive. Nevertheless, in case $G^*$ assigns a mass at 0, the upper bound in (3.16) is still correct and can easily be verified by conditioning on $\bigcap_{j=1}^{i}\left(Y_j^* > 0\right)$. By Lemma 3.5,

$$\Pr\left(X^* Y^* > x\right) \sim K_1\left(\mathrm{E}\,(X^*)^\alpha + \mathrm{E}\,(Y^*)^\alpha\right)\overline{U}(x). \tag{3.17}$$

Moreover, by (3.1), there is some constant $x_0 > 0$ such that, for all $x > x_0$ and $xy > x_0$,

$$\overline{U}(xy) \le (1+\varepsilon)\left(y^{-\alpha-\varepsilon} \vee y^{-\alpha+\varepsilon}\right)\overline{U}(x). \tag{3.18}$$

Now we start to estimate the tail probability in (3.15). Choosing some large $n$ such that $\sum_{i=n+1}^{\infty} 1/i^2 \le 1$. Clearly, for all $x > x_0$,

$$\begin{aligned}
\Pr\left(\sum_{i=n+1}^{\infty} X_i \prod_{j=1}^{i} Y_j > x\right) &\le \Pr\left(\sum_{i=n+1}^{\infty} X_i \prod_{j=1}^{i} Y_j > \sum_{i=n+1}^{\infty} \frac{x}{i^2}\right) \\
&\le \sum_{i=n+1}^{\infty} \Pr\left(X_i \prod_{j=1}^{i} Y_j > \frac{x}{i^2}\right) \\
&\le \left(\sum_{i>\sqrt{x/x_0}} + \sum_{n<i\le\sqrt{x/x_0}}\right) \Pr\left(X^* \prod_{j=1}^{i} Y_j^* > \frac{x}{i^2}\right) \\
&= I_1(x) + I_2(n,x),
\end{aligned} \tag{3.19}$$

where $I_2(n,x)$ is understood as 0 in case $n+1 > \sqrt{x/x_0}$. First we deal with $I_1(x)$. By Chebyshev's inequality,

$$I_1(x) \le x^{-\alpha}\mathrm{E}\,(X^*)^\alpha \sum_{i>\sqrt{x/x_0}} i^{2\alpha}\left(\mathrm{E}\,(Y^*)^\alpha\right)^i.$$

This means that $I_1(x)$ converges to 0 at least semi-exponentially fast since $\mathrm{E}\,(Y^*)^\alpha < 1$. Thus,

$$\lim_{x\to\infty} \frac{I_1(x)}{\overline{U}(x)} = 0. \tag{3.20}$$

Next we deal with $I_2(n,x)$. We further decompose it into three parts as

$$\begin{aligned}
I_2(n,x) &= \sum_{n<i\le\sqrt{x/x_0}} \Pr\left(X^*\prod_{j=1}^{i}Y_j^* > \frac{x}{i^2}, 0 < X^* \le \frac{x}{i^2}\right) \\
&\quad + \sum_{n<i\le\sqrt{x/x_0}} \Pr\left(X^* > \frac{x}{i^2}, \prod_{j=1}^{i}Y_j^* > 1\right) \\
&\quad + \sum_{n<i\le\sqrt{x/x_0}} \Pr\left(X^*\prod_{j=1}^{i}Y_j^* > \frac{x}{i^2}, \prod_{j=1}^{i}Y_j^* \le 1\right) \\
&= I_{21}(n,x) + I_{22}(n,x) + I_{23}(n,x). \qquad (3.21)
\end{aligned}$$

By conditioning on $X^*$ and then applying (3.16)–(3.18), we obtain

$$\begin{aligned}
I_{21}(n,x) &\le K_2 \sum_{n<i\le\sqrt{x/x_0}} \left(\mathrm{E}\,(Y^*)^\alpha + \varepsilon\right)^i \Pr\left(X^*Y^* > \frac{x}{i^2}\right) \\
&\sim K_1K_2\left(\mathrm{E}\,(X^*)^\alpha + \mathrm{E}\,(Y^*)^\alpha\right) \sum_{n<i\le\sqrt{x/x_0}} \left(\mathrm{E}\,(Y^*)^\alpha + \varepsilon\right)^i \overline{U}\left(\frac{x}{i^2}\right) \\
&\le (1+\varepsilon)K_1K_2\left(\mathrm{E}\,(X^*)^\alpha + \mathrm{E}\,(Y^*)^\alpha\right)\overline{U}(x) \sum_{n<i\le\sqrt{x/x_0}} i^{2(\alpha+\varepsilon)}\left(\mathrm{E}\,(Y^*)^\alpha + \varepsilon\right)^i.
\end{aligned}$$

Since $\mathrm{E}\,(Y^*)^\alpha + \varepsilon < 1$, it follows that

$$\lim_{n\to\infty}\limsup_{x\to\infty}\frac{I_{21}(n,x)}{\overline{U}(x)} = 0. \qquad (3.22)$$

Applying both (3.16) and (3.18), we have

$$I_{22}(n,x) \le (1+\varepsilon)K_1K_2\overline{G^*}(1)\,\overline{U}(x) \sum_{n<i\le\sqrt{x/x_0}} i^{2(\alpha+\varepsilon)}\left(\mathrm{E}\,(Y^*)^\alpha + \varepsilon\right)^i,$$

which implies that

$$\lim_{n\to\infty}\limsup_{x\to\infty}\frac{I_{22}(n,x)}{\overline{U}(x)} = 0. \qquad (3.23)$$

Similarly, applying (3.18) twice,

$$\begin{aligned}
I_{23}(n,x) &\le K_1 \sum_{n<i\le\sqrt{x/x_0}} \int_0^1 \overline{U}\left(\frac{x}{i^2y}\right)\Pr\left(\prod_{j=1}^{i}Y_j^* \in \mathrm{d}y\right) \\
&\le (1+\varepsilon)K_1 \sum_{n<i\le\sqrt{x/x_0}} \overline{U}\left(\frac{x}{i^2}\right)\left(\mathrm{E}\,(Y^*)^{\alpha-\varepsilon}\right)^i \\
&\le (1+\varepsilon)^2K_1\overline{U}(x) \sum_{n<i\le\sqrt{x/x_0}} i^{2(\alpha+\varepsilon)}\left(\mathrm{E}\,(Y^*)^{\alpha-\varepsilon}\right)^i,
\end{aligned}$$

which, together with $\mathrm{E}\,(Y^*)^{\alpha-\varepsilon} < 1$, gives that

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{I_{23}(n,x)}{\overline{U}(x)} = 0. \tag{3.24}$$

A combination of relations (3.19)–(3.24) completes the proof. ■

# 4 Proofs

## 4.1 Proof of Theorem 2.1(a)

We first prove relation (2.4). It is easy to verify that

$$M_n \overset{\mathrm{d}}{=} (X_n + M_{n-1})_+ Y_n, \qquad n \in \mathbb{N}, \tag{4.1}$$

where $\overset{\mathrm{d}}{=}$ denotes equality in distribution; see also Theorem 2.1 of Tang and Tsitsiashvili (2003). We proceed with induction. For $n = 1$, it follows from Lemma 3.5 that

$$\Pr(M_1 > x) = \Pr(X_{1,+}Y_1 > x) \sim \mu_\alpha \overline{F}(x) + \mathrm{E}X_+^\alpha \overline{G}(x) = A_1\overline{F}(x) + B_1\overline{G}(x). \tag{4.2}$$

Thus, relation (2.4) holds for $n = 1$. Now we assume by induction that relation (2.4) holds for $n - 1 \geq 1$ and prove it for $n$. By this induction assumption and Assumption 2.1, we know that every convex combination of the distributions of $X_n$ and $M_{n-1}$ belongs to the class $\mathcal{R}^*_{-\alpha} \subset \mathcal{S}(0)$. Applying Lemma 3.3 with $\alpha = 0$, we have

$$\Pr(X_n + M_{n-1} > x) \sim (1 + A_{n-1})\,\overline{F}(x) + B_{n-1}\overline{G}(x),$$

which, together with Assumption 2.1, implies that every convex combination of the distributions of $X_n + M_{n-1}$ and $Y_n$ belongs to the class $\mathcal{R}^*_{-\alpha}$. Applying Lemma 3.5, we obtain

$$\begin{aligned}\Pr(M_n > x) &= \Pr\big((X_n + M_{n-1})_+ Y_n > x\big)\\ &\sim \mu_\alpha \Pr(X_n + M_{n-1} > x) + \mathrm{E}\,(X_n + M_{n-1})_+^\alpha\, \overline{G}(x)\\ &\sim A_n\overline{F}(x) + B_n\overline{G}(x).\end{aligned}$$

Therefore, relation (2.4) holds for $n$.

Next we turn to relation (2.5). Introduce a sequence of random variables $\{T_n; n \in \mathbb{N}\}$ through the recursive equation

$$T_n = (X_n + T_{n-1})\,Y_n, \qquad n \in \mathbb{N}, \tag{4.3}$$

equipped with $T_0 = 0$. It is easy to see that $S_n \overset{\mathrm{d}}{=} T_n$ for $n \in \mathbb{N}$. Then the proof of relation (2.5) can be done by using the recursive equation (4.3) and going along the same lines as in the proof of relation (2.4) above.

## 4.2 Proof of Theorem 2.1(b)

Note that $A_n$ and $B_n$ increasingly converge to the finite constants $A_\infty$ and $B_\infty$. Also recall Lemma 3.7. Hence, for every $\delta > 0$, there is some large integer $n_0$ such that both

$$(A_\infty - A_{n_0}) + (B_\infty - B_{n_0}) \leq \delta \tag{4.4}$$

and

$$\Pr\left(\sum_{i=n_0+1}^{\infty} X_{i,+} \prod_{j=1}^{i} Y_j > x\right) \lesssim \delta \left(\overline{F}(x) + \overline{G}(x)\right) \tag{4.5}$$

hold. Now we start to deal with $\Pr(M_\infty > x)$. On the one hand, for every $\varepsilon > 0$, by Theorem 2.1(a), relation (4.5), and Assumption 2.1, in turn, we obtain

$$\begin{aligned}
\Pr(M_\infty > x) &\leq \Pr(M_{n_0} > (1-\varepsilon)x) + \Pr\left(\sum_{i=n_0+1}^{\infty} X_{i,+} \prod_{j=1}^{i} Y_j > \varepsilon x\right) \\
&\lesssim A_{n_0}\overline{F}((1-\varepsilon)x) + B_{n_0}\overline{G}((1-\varepsilon)x) + \delta\left(\overline{F}(\varepsilon x) + \overline{G}(\varepsilon x)\right) \\
&\sim (1-\varepsilon)^{-\alpha}\left(A_{n_0}\overline{F}(x) + B_{n_0}\overline{G}(x)\right) + \delta\varepsilon^{-\alpha}\left(\overline{F}(x) + \overline{G}(x)\right) \\
&\leq \left((1-\varepsilon)^{-\alpha} A_\infty + \delta\varepsilon^{-\alpha}\right)\overline{F}(x) + \left((1-\varepsilon)^{-\alpha} B_\infty + \delta\varepsilon^{-\alpha}\right)\overline{G}(x).
\end{aligned} \tag{4.6}$$

On the other hand, by Theorem 2.1(a) and relation (4.4),

$$\Pr(M_\infty > x) \geq \Pr(M_{n_0} > x) \gtrsim (A_\infty - \delta)\overline{F}(x) + (B_\infty - \delta)\overline{G}(x). \tag{4.7}$$

By the arbitrariness of $\delta$ and $\varepsilon$ in (4.6) and (4.7), we obtain relation (2.4) for $n = \infty$.

## 4.3 Proof of Theorem 2.1(c)

First we establish an asymptotic upper bound for $\Pr(S_\infty > x)$. As in the proof of Theorem 2.1(b), for every $\delta > 0$, suitably choose some large integer $n_0$ such that relations (4.4), (4.5), and the relation

$$-\delta \leq C_\infty - C_{n_0} \leq \delta \tag{4.8}$$

hold simultaneously. For every $\varepsilon > 0$, by Theorem 2.1(a), relation (4.5), Assumption 2.1, and relation (4.8), in turn, we obtain

$$\begin{aligned}
\Pr(S_\infty > x) &\leq \Pr(S_{n_0} > (1-\varepsilon)x) + \Pr\left(\sum_{i=n_0+1}^{\infty} X_{i,+} \prod_{j=1}^{i} Y_j > \varepsilon x\right) \\
&\lesssim \left(A_{n_0}\overline{F}((1-\varepsilon)x) + C_{n_0}\overline{G}((1-\varepsilon)x)\right) + \delta\left(\overline{F}(\varepsilon x) + \overline{G}(\varepsilon x)\right) \\
&\sim (1-\varepsilon)^{-\alpha}\left(A_{n_0}\overline{F}(x) + C_{n_0}\overline{G}(x)\right) + \delta\varepsilon^{-\alpha}\left(\overline{F}(x) + \overline{G}(x)\right) \\
&\leq \left((1-\varepsilon)^{-\alpha} A_\infty + \delta\varepsilon^{-\alpha}\right)\overline{F}(x) + \left((1-\varepsilon)^{-\alpha}(C_\infty + \delta) + \delta\varepsilon^{-\alpha}\right)\overline{G}(x).
\end{aligned}$$

Since $\delta$ and $\varepsilon$ are arbitrary positive constants, it follows that

$$\Pr(S_\infty > x) \lesssim A_\infty \overline{F}(x) + C_\infty \overline{G}(x).$$

For the corresponding asymptotic lower bound, as analyzed in the proof of Theorem 2.1(a), it suffices to prove that

$$\Pr(T_\infty > x) \gtrsim A_\infty \overline{F}(x) + C_\infty \overline{G}(x), \tag{4.9}$$

where $T_\infty$ is the weak limit of the sequence $\{T_n; n \in \mathbb{N}\}$ defined by (4.3). We apply the method developed by Grey (1994) to prove (4.9). Consider the stochastic difference equation

$$T_\infty \overset{\mathrm{d}}{=} (X + T_\infty) Y, \tag{4.10}$$

which inherits a stochastic structure from (4.3). Note that the weak solution of (4.10) exists and is unique. Furthermore, the limit distribution of $T_n$ is identical to this unique solution and, hence, it does not depend on the starting random variable $T_0$. See Vervaat (1979) and Goldie (1991) for these and related statements.

It is easy to check that $q = \Pr(T_\infty > 0) > 0$; see the proof of Theorem 1 of Grey (1994) for a similar argument. Construct a new starting random variable $\tilde{T}_0$ independent of $\{X_1, X_2, \ldots; Y_1, Y_2, \ldots\}$ with tail

$$\Pr\left(\tilde{T}_0 > x\right) = q \Pr(XY > x) \mathbf{1}_{(x \geq 0)} + \Pr(T_\infty > x) \mathbf{1}_{(x<0)}. \tag{4.11}$$

Starting with $\tilde{T}_0$, the recursive equation (4.3) generates the sequence $\{\tilde{T}_n; n \in \mathbb{N}\}$ correspondingly. Comparing (4.11) with (4.10), we see that $\tilde{T}_0$ and, hence, every $\tilde{T}_n$ are stochastically not greater than $T_\infty$; namely, it holds for all $x \in \mathbb{R}$ and all $n \in \{0\} \cup \mathbb{N}$ that

$$\Pr(T_\infty > x) \geq \Pr\left(\tilde{T}_n > x\right). \tag{4.12}$$

Furthermore, it holds that

$$\Pr\left(\tilde{T}_0 > x\right) \sim q \Pr(X_+ Y > x) \sim q\mu_\alpha \overline{F}(x) + q\mathrm{E}X_+^\alpha \overline{G}(x),$$

where the last step is analogous to (4.2). Thus, by Assumption 2.1, the distribution of $\tilde{T}_0$ belongs to the class $\mathcal{R}_{-\alpha}^*$. Then, by going along the same lines of the proof of Theorem 2.1(a) and using equation (4.3) starting with $\tilde{T}_0$, we obtain

$$\Pr\left(\tilde{T}_n > x\right) \sim \tilde{A}_n \overline{F}(x) + \tilde{C}_n \overline{G}(x) \tag{4.13}$$

with

$$\tilde{A}_n = \sum_{i=1}^{n} \mu_\alpha^i + q\mu_\alpha^{n+1}, \qquad \tilde{C}_n = \sum_{i=1}^{n} \mu_\alpha^{i-2} \mathrm{E}\tilde{T}_{n-i+1,+}^\alpha + q\mu_\alpha^n \mathrm{E}X_+^\alpha.$$

Since $\tilde{T}_n$ weakly converges to $T_\infty \overset{\mathrm{d}}{=} S_\infty$ and $\mu_\alpha < 1$, it is easy to see that $\lim_{n\to\infty} \tilde{A}_n = A_\infty$ and $\lim_{n\to\infty} \tilde{C}_n = C_\infty$, with the latter subject to a straightforward application of the dominated convergence theorem. Thus, substituting (4.13) into (4.12) and letting $n \to \infty$ on the right-hand side of the resulting formula, we arrive at relation (4.9) as desired.

### 4.4 Sketch of the proof of Corollary 2.1

Clearly, the recursive equations (4.1), (4.3), and the identity $S_n \overset{\mathrm{d}}{=} T_n$ for $n \in \mathbb{N}$ still hold since $\{(X_1, Y_1), (X_2, Y_2), \ldots\}$ is a sequence of i.i.d. random pairs. Introduce four independent random variables $X'$, $\check{X}'$, $Y'$, and $\check{Y}'$ with distributions $F$, $F^2$, $G$, and $G^2$, respectively, and let them be independent of $\{(X_1, Y_1), (X_2, Y_2), \ldots\}$. Using decomposition (2.9), we have

$$\begin{aligned} \Pr(M_n > x) &= \Pr\big((X_n + M_{n-1})_+ Y_n > x\big) \\ &= (1+\theta) \Pr\big((X' + M_{n-1})_+ Y' > x\big) - \theta \Pr\Big(\big(\check{X}' + M_{n-1}\big)_+ Y' > x\Big) \\ &\quad - \theta \Pr\big((X' + M_{n-1})_+ \check{Y}' > x\big) + \theta \Pr\Big(\big(\check{X}' + M_{n-1}\big)_+ \check{Y}' > x\Big). \end{aligned} \tag{4.14}$$

When $n = 1$, applying Lemma 3.5 to each term on the right-hand side of (4.14) gives

$$\Pr(M_1 > x) = \Pr(X_{1,+} Y_1 > x) \sim A_1' \overline{F}(x) + B_1' \overline{G}(x). \tag{4.15}$$

Then, as in the proof of Theorem 2.1(a), proceeding with induction according to (4.14) leads to (2.10). Relation (2.11) can be derived similarly. This proves Corollary 2.1(a).

Corollary 2.1(b, c) can be verified by the similar ideas used in proving Theorem 2.1(b, c). The key ingredient is establishing a relation similar to (3.15). Write $Z = XY$, $Z_1 = X_1 Y_1$, $Z_2 = X_2 Y_2$, and so on. It follows from (4.15) that

$$\Pr(Z > x) + \overline{G}(x) \asymp \overline{F}(x) + \overline{G}(x).$$

As in the proof of Lemma 3.7, we can construct independent random variables $Z^*$ and $Y^*$ both with tails equal to $K_1\big(\overline{F}(x) + \overline{G}(x)\big)$ for all large $x$ such that $Z \le_{\mathrm{st}} Z^*$, $Y \le_{\mathrm{st}} Y^*$, and $\mathrm{E}(Y^*)^\alpha < 1$. For some large $n$ such that $\sum_{i=n+1}^{\infty} 1/i^2 \le 1$, we write

$$\begin{aligned} \Pr\left(\sum_{i=n+1}^{\infty} X_i \prod_{j=1}^{i} Y_j > x\right) &\le \Pr\left(\sum_{i=n+1}^{\infty} X_i \prod_{j=1}^{i} Y_j > \sum_{i=n+1}^{\infty} \frac{x}{i^2}\right) \\ &= \sum_{i=n+1}^{\infty} \Pr\left(Z_i \prod_{j=1}^{i-1} Y_j > \frac{x}{i^2}\right) \\ &\le \sum_{i=n+1}^{\infty} \Pr\left(Z^* \prod_{j=1}^{i-1} Y_j^* > \frac{x}{i^2}\right). \end{aligned}$$

Then, going along the same lines of the rest of the proof of Lemma 3.7, we obtain

$$\lim_{n\to\infty}\limsup_{x\to\infty}\frac{1}{\overline{F}(x)+\overline{G}(x)}\Pr\left(\sum_{i=n+1}^{\infty}X_i\prod_{j=1}^{i}Y_j>x\right)=0,$$

which suffices for our purpose.

**Acknowledgments.** The authors are very grateful to the two reviewers and the associate editor for their insightful comments and constructive suggestions, which have helped significantly improve this work. Li's research was supported by the National Natural Science Foundation of China (Grant No. 11201245) and the Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20110031120003). Tang's research was supported by a Centers of Actuarial Excellence (CAE) Research Grant (2013–2016) from the Society of Actuaries.

# References

[1] Bankovsky, D.; Klüppelberg, C.; Maller, R. On the ruin probability of the generalised Ornstein-Uhlenbeck process in the Cramér case. Journal of Applied Probability 48A (2011), 15–28.

[2] Bingham, N. H.; Goldie, C. M.; Teugels, J. L. Regular Variation. Cambridge University Press, Cambridge, 1987.

[3] Blanchet, J. H.; Sigman, K. On exact sampling of stochastic perpetuities. Journal of Applied Probability 48A (2011), 165–182.

[4] Borovkov, A. A. Large deviations probabilities for random walks in the absence of finite expectations of jumps. Probability Theory and Related Fields 125 (2003), no. 3, 421–446.

[5] Breiman, L. On some limit theorems similar to the arc-sin law. Theory of Probability and Its Applications 10 (1965), 323–331.

[6] Chen, Y. The finite-time ruin probability with dependent insurance and financial risks. Journal of Applied Probability 48 (2011), no. 4, 1035–1048.

[7] Chen, Y.; Xie, X. The finite time ruin probability with the same heavy-tailed insurance and financial risks. Acta Mathematicae Applicatae Sinica (English Series) 21 (2005), no. 1, 153–156.

[8] Cline, D. B. H. Convolution tails, product tails and domains of attraction. Probability Theory and Related Fields 72 (1986), no. 4, 529–557.

[9] Cline, D. B. H. Convolutions of distributions with exponential and subexponential tails. Journal of the Australian Mathematical Society (Series A) 43 (1987), no. 3, 347–365.

[10] Cline, D. B. H.; Samorodnitsky, G. Subexponentiality of the product of independent random variables. Stochastic Processes and Their Applications 49 (1994), no. 1, 75–98.

[11] Collamore, J. F. Random recurrence equations and ruin in a Markov-dependent stochastic economic environment. Annals of Applied Probability 19 (2009), no. 4, 1404–1458.

[12] Denisov, D.; Foss, S.; Korshunov, D. Tail asymptotics for the supremum of a random walk when the mean is not finite. Queueing Systems 46 (2004), no. 1-2, 15–33.

[13] Doney, R. A.; Kyprianou, A. E. Overshoots and undershoots of Lévy processes. Annals of Applied Probability 16 (2006), no. 1, 91–106.

[14] Embrechts, P. A property of the generalized inverse Gaussian distribution with some applications. Journal of Applied Probability 20 (1983), no. 3, 537–544.

[15] Enriquez, N.; Sabot, C.; Zindy, O. A probabilistic representation of constants in Kesten's renewal theorem. Probability Theory and Related Fields 144 (2009), no. 3-4, 581–613.

[16] Feller, W. An Introduction to Probability Theory and Its Applications. Vol. II. Second edition John Wiley & Sons, Inc., New York-London-Sydney 1971.

[17] Foss, S.; Korshunov, D.; Zachary, S. An Introduction to Heavy-tailed and Subexponential Distributions. Springer, New York, 2011.

[18] Frolova, A.; Kabanov, Y.; Pergamenshchikov, S. In the insurance business risky investments are dangerous. Finance and Stochastics 6 (2002), no. 2, 227–235.

[19] Goldie, C. M. Implicit renewal theory and tails of solutions of random equations. Annals of Applied Probability 1 (1991), no. 1, 126–166.

[20] Goldie, C. M.; Grübel, R. Perpetuities with thin tails. Advances in Applied Probability 28 (1996), no. 2, 463–480.

[21] Grey, D. R. Regular variation in the tail behaviour of solutions of random difference equations. Annals of Applied Probability 4 (1994), no. 1, 169–183.

[22] Griffin, P. S. Convolution equivalent Lévy processes and first passage times. Annals of Applied Probability 23 (2013), no. 4, 1506–1543.

[23] Griffin, P. S.; Maller, R. A. Path decomposition of ruinous behavior for a general Lévy insurance risk process. Annals of Applied Probability 22 (2012), no. 4, 1411–1449.

[24] Griffin, P. S.; Maller, R. A.; van Schaik, K. Asymptotic distributions of the overshoot and undershoots for the Lévy insurance risk process in the Cramér and convolution equivalent cases. Insurance: Mathematics and Economics 51 (2012), no. 2, 382–392.

[25] Hao, X.; Tang, Q. Asymptotic ruin probabilities for a bivariate Lévy-driven risk model with heavy-tailed claims and risky investments. Journal of Applied Probability 49 (2012), no. 4, 939–953.

[26] Heyde, C. C.; Wang, D. Finite-time ruin probability with an exponential Lévy process investment return and heavy-tailed claims. Advances in Applied Probability 41 (2009), no. 1, 206–224.

[27] Hitczenko, P.; Wesołowski, J. Renorming divergent perpetuities. Bernoulli 17 (2011), no. 3, 880–894.

[28] Hult, H.; Lindskog, F. Ruin probabilities under general investments and heavy-tailed claims. Finance and Stochastics 15 (2011), no. 2, 243–265.

[29] Hult, H.; Samorodnitsky, G. Tail probabilities for infinite series of regularly varying random vectors. Bernoulli 14 (2008), no. 3, 838–864.

[30] Kalashnikov, V.; Norberg, R. Power tailed ruin probabilities in the presence of risky investments. Stochastic Processes and Their Applications 98 (2002), no. 2, 211–228.

[31] Kesten, H. Random difference equations and renewal theory for products of random matrices. Acta Mathematica 131 (1973), 207–248.

[32] Klüppelberg, C. Subexponential distributions and integrated tails. Journal of Applied Probability 25 (1988), no. 1, 132–141.

[33] Klüppelberg, C.; Kostadinova, R. Integrated insurance risk models with exponential Lévy investment. Insurance: Mathematics and Economics 42 (2008), no. 2, 560–577.

[34] Klüppelberg, C.; Kyprianou, A. E.; Maller, R. A. Ruin probabilities and overshoots for general Lévy insurance risk processes. Annals of Applied Probability 14 (2004), no. 4, 1766–1801.

[35] Korshunov, D. On distribution tail of the maximum of a random walk. Stochastic Processes and Their Applications 72 (1997), no. 1, 97–103.

[36] Li, J.; Tang, Q. A note on max-sum equivalence. Statistics & Probability Letters 80 (2010), 1720–1723.

[37] Norberg, R. Ruin problems with assets and liabilities of diffusion type. Stochastic Processes and Their Applications 81 (1999), no. 2, 255–269.

[38] Nyrhinen, H. On the ruin probabilities in a general economic environment. Stochastic Processes and Their Applications 83 (1999), no. 2, 319–330.

[39] Nyrhinen, H. Finite and infinite time ruin probabilities in a stochastic economic environment. Stochastic Processes and Their Applications 92 (2001), no. 2, 265–285.

[40] Pakes, A. G. Convolution equivalence and infinite divisibility. Journal of Applied Probability 41 (2004), no. 2, 407–424.

[41] Paulsen, J. Risk theory in a stochastic economic environment. Stochastic Processes and Their Applications 46 (1993), no. 2, 327–361.

[42] Paulsen, J. Ruin models with investment income. Probability Surveys 5 (2008), 416–434.

[43] Pergamenshchikov, S.; Zeitouny, O. Ruin probability in the presence of risky investments. Stochastic Processes and Their Applications 116 (2006), no. 2, 267–278.

[44] Tang, Q.; Tsitsiashvili, G. Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks. Stochastic Processes and Their Applications 108 (2003), no. 2, 299–325.

[45] Tang, Q.; Tsitsiashvili, G. Finite- and infinite-time ruin probabilities in the presence of stochastic returns on investments. Advances in Applied Probability 36 (2004), no. 4, 1278–1299.

[46] Tang, Q.; Wei, L. Asymptotic aspects of the Gerber-Shiu function in the renewal risk model using Wiener-Hopf factorization and convolution equivalence. Insurance: Mathematics and Economics 46 (2010), no. 1, 19–31.

[47] Veraverbeke, N. Asymptotic behaviour of Wiener-Hopf factors of a random walk. Stochastic Processes and Their Applications 5 (1977), no. 1, 27–37.

[48] Vervaat, W. On a stochastic difference equation and a representation of nonnegative infinitely divisible random variables. Advances in Applied Probability 11 (1979), no. 4, 750–783.