

Budget-constrained optimal reinsurance design under coherent risk measures

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Abstract

Reinsurance is a versatile risk management strategy commonly employed by insurers to optimize their risk profile. In this paper, we study an optimal reinsurance design problem minimizing a general law-invariant coherent risk measure of the net risk exposure of a generic insurer, in conjunction with a general law-invariant comonotonic additive convex reinsurance premium principle and a premium budget constraint. Due to its intrinsic generality, this contract design problem encompasses a wide body of optimal reinsurance models commonly encountered in practice. A three-step solution scheme is presented. Firstly, the objective and constraint functions are exhibited in the so-called Kusuoka's integral representations. Secondly, the mini-max theorem for infinite dimensional spaces is applied to interchange the infimum on the space of indemnities and the supremum on the space of probability measures. Thirdly, the recently developed Neyman–Pearson methodology due to Lo (2017a) is adopted to solve the resulting infimum problem. Analytic and transparent expressions for the optimal reinsurance policy are provided, followed by illustrative examples.

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1 Introduction

Reinsurance, which is essentially insurance purchased by insurance companies, has been one of the most burgeoning research areas in actuarial science over the past decade. Its popularity stems from its versatility in achieving various strategic objectives of an insurer and the increasingly frequent occurrence of catastrophes in today's complex business environment. Pioneered

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in Borch (1960) and Arrow (1963) concerned with the variance minimization and expected utility maximization from the perspective of an insurer, the development in the theory of optimal reinsurance has recently extended to a modern risk management framework and predominantly taken two directions.

One thriving line of research is devoted to the consideration of more diverse objective functionals than the traditional variance and expected utility and more general reinsurance premium principles than the oft-used expectation premium principle, also known as the actuarial pricing principle. This direction has attracted a number of recent research work, with the majority centering on the use of common risk measures in the banking and insurance industries. Prominent examples of risk measures that have been studied include Value-at-Risk, Tail Value-at-Risk (TVaR), distortion risk measures, and law-invariant coherent and convex risk measures. See, e.g., Cai et al. (2008), Cheung et al. (2014), Balbás et al. (2015), Cheung et al. (2015), Zhuang et al. (2017), Asimit and Boonen (2018). The technical sophistication of the optimal reinsurance problem in question depends not only on the generality of the optimization functional, but also on whether the optimization functional admits a convenient representation that facilitates mathematical derivations as well as verbal interpretation. Optimal reinsurance contracts that range from stop-loss, quota-share policies to general insurance layers have been identified, depending on the exact specification of the reinsurance model.

Another recent advance in the theory of optimal reinsurance has been the recognition that reinsurance contracts should not be devised in a theoretical vacuum devoid of the externalities faced by the insurer, but should actively embrace the practical considerations of insurers and reinsurers in the market via the imposition of various optimization constraints. This ensures that the resulting optimal reinsurance contract is in line with practicalities and thus marketable in practice. Typical constraints that can be embedded in the insurer's decision-making process include reinsurance premium budget constraints (see, e.g., Tan et al. (2011), Cui et al. (2013), Zhuang et al. (2016), Cheung and Lo (2017)) that take into account the financial limitations of the insurer, and reinsurer's risk constraints (see, e.g., Lo (2017b), Lo and Tang (2018)) that reflect the reinsurer's risk-bearing capacity, ensuring that the reinsurance policy is optimal to the insurer and simultaneously acceptable to the reinsurer. The incorporation of such constraints, however, substantially raises the technical complications of the otherwise unconstrained optimal reinsurance problem.

Interweaving the two emerging lines of research above, this paper aims to study the design of the optimal reinsurance contract when the insurer perceives risk via a general law-invariant coherent risk measure, the treatment of which in the reinsurance arena is still in its infancy, in conjunction with a reinsurance premium calibrated by a general law-invariant comonotonic additive convex risk measure and a premium budget constraint. The adoption of law-invariant coherent and comonotonic additive risk measures as the objective functional of the insurer and the reinsurance premium principle is made on both operational and theoretical grounds. From an operational point of view, there is no general consensus about universally desirable risk measures. It is therefore practically and mathematically appealing that our analysis holds true for as large a family of risk measures as possible. In this regard, the abstraction offered by our constrained reinsurance model based on law-invariant coherent risk measures means that

numerous common risk measures fall under the umbrella of our treatment. From a theoretical point of view, coherent risk measures possess, in addition to a number of properties favored by practitioners, various tractable representation results that put the optimization problem in perspective and substantially ease the technical derivations in the later part of this paper. Among the wide spectrum of external constraints that can be imposed on the model, we select the reinsurance premium budget constraint due to its practical interest to business corporations and ease of interpretation, although our methodology works equally well for constraints of a much more general nature (see (5)). The presence of the budget constraint allows us to quantify the effects of the premium budget on the design of the optimal reinsurance contract.

Our solution scheme comprises three key steps. The starting point of our analysis is the use of the so-called Kusuoka's TVaR representation for law-invariant coherent risk measures and comonotonic additive convex risk measures (see Kusuoka (2001)) coupled with TVaR's integral representation as a particular distortion risk measure to exhibit the objective and constraint functions of our optimal reinsurance problem in integral forms conducive to further investigation. The constrained optimization problem is then displayed in the form of a mini-max two-stage problem, which arises ubiquitously in robust optimization studies, see, e.g., Polak et al. (2010), Shapiro (2012), and Asimit et al. (2017) among others. With these convenient integral representations at our disposal, the second step involves interchanging the supremum that emanates from the Kusuoka's representations and the infimum that our risk minimization problem entails by virtue of a version of Sion's mini-max theorem for infinite dimensional spaces. Justifying the applicability of the mini-max theorem to our context is by no means an easy task and relies heavily on functional analytic arguments. Following the supremum-infimum swap transforming the mini-max problem to a maxi-min problem, the third and final step of our solution scheme lies in analytically solving the inner constrained TVaR minimization problem via an application of the Neyman–Pearson technique recently introduced in Lo (2017a), which lends itself to tackling a wide body of constrained optimal reinsurance problems. It is shown that the optimal contract obeys the likelihood ratio test structure that is commonly observed in reinsurance studies. However, the precise design depends on the interaction between the law-invariant coherent risk measure and the law-invariant comonotonic additive convex risk measure that represent the insurer's risk and the reinsurance premium, respectively.

In view of the vast literature on optimal reinsurance, it is imperative to point out the connections between the current work and a few recent papers with a similar theme. With respect to the reinsurance premium principle, our work is a substantial generalization of Cheung et al. (2014), where the expectation premium principle was adopted, and Chen et al. (2016) involving the use of the TVaR premium principle. Both the expectation premium principle and TVaR premium principle are special cases of the law-invariant comonotonic additive convex premium principle employed in this paper. In terms of methodology, our three-step solution scheme also differs radically from the geometric argument in Cheung et al. (2014) and the convoluted algebraic manipulations in Chen et al. (2016), both of which first treat the TVaR minimization problem to deduce *a priori* the form of the optimal solution(s), whose uniqueness issue is left unsettled. In contrast, our approach is the first of its kind to draw upon the Neyman–Pearson nature implicit in the optimal reinsurance problem based on general law-invariant

coherent and convex risk measures. Such a methodological difference renders a transparent and explicit construction of the optimal solutions possible. With regard to optimization constraints, our work also extends Cheung et al. (2014) and Chen et al. (2016) and complements Cai et al. (2017), who considered the case of multiple reinsurers but without any restriction on the amount spent on reinsurance, by incorporating the reinsurance premium budget constraint and studying its effect on the structure of the optimal contract.

The remainder of this paper is organized as follows. Section 2 assembles a series of integral representations that are germane to our subsequent analysis and formulates the constrained optimal reinsurance problem that is the linchpin of this paper. Our three-step solution scheme unfolds in Section 3, where we express various quantities pertaining to the optimal reinsurance problem in integral forms. In Section 4, we specialize the mini-max theorem for infinite dimensional spaces to our setting and exchange the order of infimum and supremum that constitute the objective function. The justification of such an interchange is deferred to the appendix. The third step is performed in Section 5, in which the inner infimum problem is expeditiously solved by an application of Lo (2017a)'s Neyman–Pearson technique. Section 6 collects the intermediate results from Sections 3 to 5 and exhibits the ultimate solutions of our optimal reinsurance problem explicitly. These solutions are illustrated in Section 7 via [two](#) concrete examples. Finally, Section 8 concludes the paper.

2 Preliminaries and problem formulation

In this section, we lay the mathematical foundation for this paper by recollecting several useful integral representation results and setting up formulation of the budget-constrained optimal reinsurance problem. Throughout this paper, all random variables are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By a standard extension procedure (see, e.g., page 112 of Kallenberg (2002)), we assume without loss of generality that this probability space is atomless.

2.1 Representation results for risk measures

In this paper, the risk faced by the insurer and the reinsurance premium are evaluated via law-invariant risk measures with certain properties. We first recall several definitions for risk measures which will be intensively used in the sequel.

Definition 2.1. Let $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be the space of all bounded random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A risk measure $\rho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}$ is called *convex* if it satisfies translation invariance, monotonicity, and convexity: for any $Y, Z \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$,

- (i) (Translation invariance) for any $c \in \mathbb{R}$, $\rho(Y + c) = \rho(Y) + c$;
- (ii) (Monotonicity) if $Y \leq Z$, then $\rho(Y) \leq \rho(Z)$; and
- (iii) (Convexity) for any $\beta \in [0, 1]$, $\rho(\beta Y + (1 - \beta)Z) \leq \beta\rho(Y) + (1 - \beta)\rho(Z)$.

The convex risk measure ρ is called *coherent* if it further satisfies positive homogeneity:

- (iv) (Positive homogeneity) for any $\beta \geq 0$ and $Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, $\rho(\beta Y) = \beta\rho(Y)$.

A risk measure ρ is called *comonotonic additive*, *law-invariant*, and *continuous from above* if it satisfies respectively:

- (v) (Comonotonic additivity) if Y and Z are comonotonic, then $\rho(Y + Z) = \rho(Y) + \rho(Z)$;
- (vi) (Law-invariance) if Y has the same distribution as Z , then $\rho(Y) = \rho(Z)$;
- (vii) (Continuity from above) if $\{Y_n\} \subseteq L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is such that $Y_n \downarrow Y$, then $\rho(Y_n) \downarrow \rho(Y)$.

A prominent example of a law-invariant risk measure is *Tail Value-at-Risk* (TVaR), also known variously as expected shortfall, conditional tail expectation, and conditional Value-at-Risk (although there are subtle differences between these terms). For thorough references on TVaR, see, e.g., Rockafellar and Uryasev (2002) and Denuit et al. (2005).

Definition 2.2. Let Y be a random variable whose distribution function is F_Y and survival function is S_Y .

- (i) The generalized left-continuous inverse function of F_Y is defined by

$$F_Y^{-1}(p) \triangleq \inf\{y \in \mathbb{R} \mid F_Y(y) \geq p\}, \quad \text{for all } p \in (0, 1],$$

with the convention that $\inf \emptyset = \infty$. In particular, the essential supremum of Y is $\text{ess sup}(Y) \triangleq F_Y^{-1}(1)$.

- (ii) The TVaR of the random variable Y at the probability level $\alpha \in [0, 1)$ is defined by

$$\text{TVaR}_\alpha(Y) \triangleq \frac{1}{1 - \alpha} \int_\alpha^1 F_Y^{-1}(p) \, dp.$$

A distinguishing characteristic of any law-invariant risk measures, TVaR in particular, is that they possess some convenient integral representations that not only shed light on the perception of risk they reflect, but are also crucial to theoretical derivations. As will be seen later, these representation results play a vital role in our analysis. For TVaR, the following Lebesgue-Stieltjes integral representation, taken from Lemma 2.1 of Cheung and Lo (2017), shows that TVaR belongs to the class of distortion risk measures introduced in Wang (1996).

Lemma 2.1. Let Y be a non-negative random variable. For any $\alpha \in [0, 1)$ and non-decreasing absolutely continuous function h with $h(0) = 0$, we have

$$\text{TVaR}_\alpha(h(Y)) = \int_0^{\text{ess sup}(Y)} g_\alpha(S_Y(y)) \, dh(y),$$

where $g_\alpha : [0, 1] \rightarrow [0, 1]$ is defined by

$$g_\alpha(\lambda) \triangleq \min\left(\frac{\lambda}{1 - \alpha}, 1\right).$$

The next two lemmas provide TVaR-representations for any law-invariant coherent risk measure which is continuous-from-above and any law-invariant comonotonic additive convex risk

measure, and highlight the importance of TVaR as the fundamental building block of these risk measures. Their proofs can be found in Corollary 4.63 and Theorem 4.93 of Föllmer and Schied (2016).

Lemma 2.2. A coherent risk measure ρ is law-invariant and continuous-from-above if and only if there exists a set $\mathcal{M} \subseteq \mathcal{M}_1([0, 1])$, where $\mathcal{M}_1([0, 1])$ is the set of all probability measures defined on $[0, 1]$, such that

$$\rho(Y) = \sup_{\mu \in \mathcal{M}} \int_0^1 \text{TVaR}_\alpha(Y) \mu(d\alpha)$$

for any non-negative bounded random variable Y .

Lemma 2.3. A risk measure ρ is law-invariant, comonotonic additive, and convex if and only if there exists a measure $\mu \in \mathcal{M}_1([0, 1])$, such that

$$\rho(Y) = \int_0^1 \text{TVaR}_\alpha(Y) \mu(d\alpha)$$

for any non-negative bounded random variable Y .

A simple but important example of a law-invariant, comonotonic additive, and convex risk measure is TVaR, which, at the probability level α , is captured by setting the probability measure μ to the Dirac measure concentrated on α . For more details about these risk measures, readers are referred, for example, to Section 4.7 of Föllmer and Schied (2016).

2.2 Problem formulation

Building upon the mathematical underpinnings in the preceding subsection, we now describe the budget-constrained optimal reinsurance model studied in this paper. The problem of interest revolves around a non-negative bounded random variable X modeling the insurable loss of an insurer over a fixed period of time. Denote by F_X and S_X the (non-degenerate) distribution function and the survival function of X , respectively. To mitigate the risk exposure it faces, the insurer may elect to purchase a reinsurance policy from a reinsurer, splitting the ground-up loss X as follows. Corresponding to any realized value x of X , the reinsurer pays $I(x)$ to the insurer, which in turn retains the residual loss $x - I(x)$. We call the function I the *ceded loss function*, which is the decision variable of interest in a reinsurance design problem. In return for incurring the ceded loss $I(X)$, the reinsurer receives from the insurer the reinsurance premium $\rho(I(X))$, which is a risk measure of $I(X)$. The resulting net risk exposure of the insurer is then altered from the ground-up loss X to

$$T_I(X) \triangleq X - I(X) + \rho(I(X)). \tag{1}$$

In essence, optimal reinsurance is concerned with the design of the ceded loss function to achieve a certain sense of optimality. This in turn hinges upon the decision of how risk should be optimally ceded and retained, and how the cost and benefit associated with reinsurance should be traded off.

In practice, any reasonable ceded loss function should satisfy the following properties:

- (i) If the insurer does not report any loss, then the ceded loss is zero.
- (ii) To avoid moral hazard, a unit increment in the ground-up loss should give rise to less than a unit increment in the ceded loss.

Properties (i) and (ii) together imply that the ceded loss is non-negative and no greater than the ground-up loss suffered by the insurer. These practical considerations translate into the following mathematical statements:

- (i) $I(0) = 0$;
- (ii) $0 \leq I(x) - I(y) \leq x - y$, for any $x, y \in [0, \text{ess sup}(X)]$ with $y \leq x$.

In the reinsurance literature, condition (ii) is usually referred to as the 1-Lipschitz condition. In the sequel, we assume that any feasible ceded loss function I lies in the set \mathcal{I} with conditions (i) and (ii) imposed:

$$\mathcal{I} \triangleq \left\{ I : [0, \text{ess sup}(X)] \rightarrow [0, \text{ess sup}(X)] \left| \begin{array}{l} I(0) = 0, \\ 0 \leq I(x) - I(y) \leq x - y \\ \text{for all } x, y \in [0, \text{ess sup}(X)] \text{ with } y \leq x \end{array} \right. \right\}.$$

By condition (ii), for any $I \in \mathcal{I}$, both I and $\text{Id} - I$ are non-decreasing functions, where Id is the identity function. Since every $I \in \mathcal{I}$ is Lipschitz continuous, it is absolutely continuous as well.

We are now in a position to formally state the insurer's *budget-constrained risk minimization problem*:

$$\begin{cases} \inf_{I \in \mathcal{I}} & \rho_1(T_I(X)) \\ \text{s.t.} & \rho_2(I(X)) \leq \pi, \end{cases} \quad (2)$$

where ρ_1 is a law-invariant continuous-from-above coherent risk measure prescribed by the insurer, ρ_2 is a law-invariant comonotonic additive convex risk measure adopted by the reinsurer to calibrate the reinsurance premium, and $\pi \geq 0$ is the exogenously given budget allocated to reinsurance. In this budget-constrained problem, the insurer seeks to design the reinsurance contract that minimizes its risk exposure as quantified by the coherent risk measure ρ_1 , while complying with the ρ_2 -based reinsurance premium budget constraint, which in reality is often in place due to operational concerns. Again, it deserves mention that our methodology is applicable to constraints more general than the premium budget constraint, which is selected for the ease of interpretation. In the following sections, we will direct our efforts towards solving Problem (2) analytically for all non-negative values of π and investigating how the presence of the budget constraint impacts on the design of the optimal reinsurance contract.

3 Step 1: Kusuoka's integral representations of objective and constraint functions

The first crucial step of solving Problem (2) is to express the risk measures that constitute its objective and constraint functions in appropriate integral forms, which will be central to

subsequent analysis. This step relies on several representation results recalled in Subsection 2.1.

To begin with, note that by the definition of $T_I(X)$ in (1) and the translation invariance of ρ_1 , we have

$$\rho_1(T_I(X)) = \rho_1(X - I(X) + \rho_2(I(X))) = \rho_1((\text{Id} - I)(X)) + \rho_2(I(X)).$$

By Lemma 2.1 (applicable since every $I \in \mathcal{I}$ is absolutely continuous with $I(0) = 0$), Lemma 2.2, the comonotonic additivity of TVaR, along with Fubini's Theorem, we have

$$\begin{aligned} \rho_1((\text{Id} - I)(X)) &= \sup_{\mu_1 \in \mathcal{M}} \left(\int_0^1 \text{TVaR}_\alpha((\text{Id} - I)(X)) \mu_1(d\alpha) \right) \\ &= \sup_{\mu_1 \in \mathcal{M}} \left(\int_0^1 \text{TVaR}_\alpha(X) \mu_1(d\alpha) - \int_0^1 \text{TVaR}_\alpha(I(X)) \mu_1(d\alpha) \right) \\ &= \sup_{\mu_1 \in \mathcal{M}} \left(\int_0^1 \text{TVaR}_\alpha(X) \mu_1(d\alpha) - \int_0^1 \left(\int_0^{\text{ess sup}(X)} g_\alpha(S_X(x)) dI(x) \right) \mu_1(d\alpha) \right) \\ &= \sup_{\mu_1 \in \mathcal{M}} \left(\int_0^1 \text{TVaR}_\alpha(X) \mu_1(d\alpha) - \int_0^{\text{ess sup}(X)} \left(\int_0^1 g_\alpha(S_X(x)) \mu_1(d\alpha) \right) dI(x) \right), \end{aligned}$$

for some $\mathcal{M} \subseteq \mathcal{M}_1([0, 1])$. Similarly, by Lemmas 2.1 and 2.3, and Fubini's Theorem,

$$\begin{aligned} \rho_2(I(X)) &= \int_0^1 \text{TVaR}_\alpha(I(X)) \mu_2(d\alpha) \\ &= \int_0^1 \left(\int_0^{\text{ess sup}(X)} g_\alpha(S_X(x)) dI(x) \right) \mu_2(d\alpha) \\ &= \int_0^{\text{ess sup}(X)} \left(\int_0^1 g_\alpha(S_X(x)) \mu_2(d\alpha) \right) dI(x), \end{aligned}$$

for some $\mu_2 \in \mathcal{M}_1([0, 1])$. Therefore, with the definitions

$$G_1(\mu_1, x) \triangleq \int_0^1 g_\alpha(S_X(x)) \mu_1(d\alpha) \quad \text{for all } \mu_1 \in \mathcal{M} \text{ and } x \in [0, \text{ess sup}(X)]$$

and

$$G_2(x) \triangleq \int_0^1 g_\alpha(S_X(x)) \mu_2(d\alpha) \quad \text{for all } x \in [0, \text{ess sup}(X)],$$

the objective and constraint functions of Problem (2) can be written respectively as

$$\begin{aligned} \rho_1(T_I(X)) &= \sup_{\mu_1 \in \mathcal{M}} \left(\int_0^1 \text{TVaR}_\alpha(X) \mu_1(d\alpha) - \int_0^{\text{ess sup}(X)} \left(\int_0^1 g_\alpha(S_X(x)) \mu_1(d\alpha) \right) dI(x) \right. \\ &\quad \left. + \int_0^{\text{ess sup}(X)} \left(\int_0^1 g_\alpha(S_X(x)) \mu_2(d\alpha) \right) dI(x) \right) \\ &= \sup_{\mu_1 \in \mathcal{M}} \left(\int_0^1 \text{TVaR}_\alpha(X) \mu_1(d\alpha) + \int_0^{\text{ess sup}(X)} (G_2(x) - G_1(\mu_1, x)) dI(x) \right) \end{aligned}$$

and

$$\rho_2(I(X)) = \int_0^{\text{ess sup}(X)} G_2(x) dI(x).$$

Define $\mathcal{I}(\pi) \triangleq \left\{ I \in \mathcal{I} : \int_0^{\text{ess sup}(X)} G_2(x) dI(x) \leq \pi \right\}$. Hence, Problem (2) is equivalent to

$$\inf_{I \in \mathcal{I}(\pi)} \sup_{\mu_1 \in \mathcal{M}} \left(\int_0^1 \text{TVaR}_\alpha(X) \mu_1(d\alpha) + \int_0^{\text{ess sup}(X)} (G_2(x) - G_1(\mu_1, x)) dI(x) \right), \quad (3)$$

where the decision variable I is collected as the integrator of the second integral.

4 Step 2: application of Sion's Mini-max Theorem

With the tractable form of the objective function developed in Step 1 in place, the second important step of solving Problem (2), or equivalently Problem (3), is to interchange the infimum and supremum in (3). To this end, we apply one version of the mini-max theorem, known as *Sion's Mini-max Theorem* (see, e.g., Theorem 2.132 in Barbu and Precupanu (2012)).

Theorem 4.1 (Mini-max Theorem for Infinite Dimensional Spaces). Let \mathcal{X} and \mathcal{Y} be Hausdorff vector spaces, $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$ be non-empty compact convex subsets, and F be a real-valued function on $A \times B$ such that

- (i) for each $a \in A$, $F(a, \cdot)$ is upper semi-continuous and quasi-concave on B ;
- (ii) for each $b \in B$, $F(\cdot, b)$ is lower semi-continuous and quasi-convex on A .

Then F satisfies the mini-max equality on $A \times B$, in the sense that

$$\min_{a \in A} \max_{b \in B} F(a, b) = \max_{b \in B} \min_{a \in A} F(a, b).$$

Note that Theorem 4.1 simply states the mini-max equality of the two extrema, while our ultimate aim is to solve for the optimal ceded loss function I^* in Problem (2), or equivalently Problem (3). To this end, we need the following equivalences (see, e.g., Section 2.3.1 in Barbu and Precupanu (2012) and Lemma 36.2 in Rockafellar (1997)):

Proposition 4.2. Let A and B be non-empty sets, and F be a real-valued function on $A \times B$. Then the following statements are equivalent:

- (i) F satisfies the mini-max equality on $A \times B$, i.e.,

$$\min_{a \in A} \max_{b \in B} F(a, b) = \max_{b \in B} \min_{a \in A} F(a, b);$$

- (ii) F has a saddle point, i.e., there exists a pair $(a^*, b^*) \in A \times B$ such that

$$F(a^*, b) \leq F(a^*, b^*) \leq F(a, b^*), \quad \text{for all } (a, b) \in A \times B.$$

Moreover, a pair $(a^*, b^*) \in A \times B$ is a saddle point of F if and only if all of the following conditions hold:

(a) F has a saddle value, in the sense that

$$\inf_{a \in A} \sup_{b \in B} F(a, b) = \sup_{b \in B} \inf_{a \in A} F(a, b);$$

(b) the infimum in the expression $\inf_{a \in A} \sup_{b \in B} F(a, b)$ is attained at $a^* \in A$:

$$\inf_{a \in A} \sup_{b \in B} F(a, b) = \min_{a \in A} \sup_{b \in B} F(a, b) = \sup_{b \in B} F(a^*, b);$$

(c) the supremum in the expression $\sup_{b \in B} \inf_{a \in A} F(a, b)$ is attained at $b^* \in B$:

$$\sup_{b \in B} \inf_{a \in A} F(a, b) = \max_{b \in B} \inf_{a \in A} F(a, b) = \inf_{a \in A} F(a, b^*).$$

Assume, at the moment, that the conditions in Theorem 4.1 are satisfied. Then by virtue of Theorem 4.1 and Proposition 4.2, the optimal objective value of Problem (3) equals that of

$$\sup_{\mu_1 \in \mathcal{M}} \inf_{I \in \mathcal{I}(\pi)} \left(\int_0^1 \text{TVaR}_\alpha(X) \mu_1(d\alpha) + \int_0^{\text{ess sup}(X)} (G_2(x) - G_1(\mu_1, x)) dI(x) \right),$$

which involves solving the following two optimization problems:

(i) for each fixed $\mu_1 \in \mathcal{M}$, determine

$$V(\mu_1) \triangleq \inf_{I \in \mathcal{I}(\pi)} \left(\int_0^1 \text{TVaR}_\alpha(X) \mu_1(d\alpha) + \int_0^{\text{ess sup}(X)} (G_2(x) - G_1(\mu_1, x)) dI(x) \right), \quad (4)$$

with its set of minimizers denoted by $\mathcal{I}^*(\mu_1)$;

(ii) $\sup_{\mu_1 \in \mathcal{M}} V(\mu_1)$.

Moreover, it follows from Proposition 4.2 that the optimal ceded loss function I^* of Problem (2) satisfies $I^* \in \mathcal{I}^*(\mu_1^*)$, where μ_1^* is a maximizer that gives rise to $\sup_{\mu_1 \in \mathcal{M}} V(\mu_1)$. Therefore, the third crucial step of solving Problem (2) is to tackle Problem (4), which, because of the absence of the supremum over the space of probability measures, is essentially a constrained optimal reinsurance design problem under law-invariant comonotonic additive convex risk measures. To maximize the readability of the paper, the detailed examination for the validity of the conditions in Theorem 4.1 is postponed to the appendix.

5 Step 3: Neyman–Pearson solution to the inner infimum problem

To solve Problem (4) systematically and expeditiously, we utilize the Neyman–Pearson solution scheme recently introduced in Lo (2017a), which draws upon the intrinsic Neyman–Pearson

nature of a wide body of constrained optimal reinsurance problems and allows for a unifying, transparent and thorough treatment of these problems. Loosely speaking, the Neyman–Pearson methodology identifies the unit-valued derivative of each ceded loss function in \mathcal{I} as the test function of an appropriate hypothesis test and transforms the problem of designing optimal reinsurance contracts to one resembling the search of optimal test functions, which in turn is achieved by the celebrated Neyman–Pearson Lemma in statistical hypothesis testing theory. Formally, the solution scheme treats the following abstract constrained functional minimization problem:

$$\begin{cases} \inf_{I \in \mathcal{I}} & \int_0^\infty f_1(x) \, dI(x), \\ \text{s.t.} & \int_0^\infty f_0(x) \, dI(x) \leq \pi, \end{cases} \quad (5)$$

where f_0 and f_1 are fixed integrable functions on \mathbb{R}_+ , and π is a fixed real constant. The solution of Problem (5) is given by the following theorem.

Theorem 5.1 (Theorem 3.2 in Lo (2017a)). Define a non-decreasing function $\mathcal{G} : [-\infty, 0] \rightarrow \mathbb{R}$ by

$$\mathcal{G}(c) \triangleq \int_{\{f_1 < cf_0\}} f_0(x) \, dx.$$

- (i) If $\mathcal{G}(0) = \int_{\{f_1 < 0\}} f_0(x) \, dx \leq \pi$, then the optimal solutions for Problem (5) must be in the form of

$$I^*(x) = \int_0^x \mathbb{1}_{\{f_1(\cdot) < 0\}}(t) \, dt + \int_0^x \mathbb{1}_{\{f_1(\cdot) = 0\}}(t) \, d\tilde{I}(t)$$

for any $\tilde{I} \in \mathcal{I}$ such that

$$\int_0^\infty f_0(x) \, dI^*(x) = \int_{\{f_1 < 0\}} f_0(x) \, dx + \int_{\{f_1 = 0\}} f_0(x) \, d\tilde{I}(x) \leq \pi;$$

- (ii) If $\mathcal{G}(-\infty) = \int_{\{f_0 < 0\}} f_0(x) \, dx \leq \pi < \mathcal{G}(0) = \int_{\{f_1 < 0\}} f_0(x) \, dx$, then the optimal solutions for Problem (5) must be in the form of

$$I^*(x) = \int_0^x \mathbb{1}_{\{f_1(\cdot) < c^* f_0(\cdot)\}}(t) \, dt + \int_0^x \mathbb{1}_{\{f_1(\cdot) = c^* f_0(\cdot)\}}(t) \, d\tilde{I}(t)$$

where $c^* = \mathcal{G}^{-1}(\pi)$ ⁱ and \tilde{I} is any function in \mathcal{I} such that

$$\int_0^\infty f_0(x) \, dI^*(x) = \int_{\{f_1 < c^* f_0\}} f_0(x) \, dx + \int_{\{f_1 = c^* f_0\}} f_0(x) \, d\tilde{I}(x) = \pi;$$

- (iii) If $\pi < \mathcal{G}(-\infty) = \int_{\{f_0 < 0\}} f_0(x) \, dx$, then Problem (5) has no solutions.

Applying Theorem 5.1 with $f_1(x) = G_2(x) - G_1(\mu_1, x)$, where $\mu_1 \in \mathcal{M}$ is *a priori* fixed, $f_0(x) = G_2(x)$, and $\mathcal{G}(c) = \int_{\{(1-c)G_2(\cdot) < G_1(\mu_1, \cdot)\}} G_2(x) \, dx$ immediately yields the solutions of Problem (4) as given in Theorem 5.2 below. **Note that because $f_0 = G_2 \geq 0$ and $\pi \geq 0$ in the context of Problem (4), only Cases (i) and (ii) above are possible.** From a wider perspective, this theorem also specifies the likelihood ratio test structure that governs all of the solutions

ⁱNote that $c^* = \inf \{c \in [-\infty, 0] \mid \mathcal{G}(c) > \pi\}$ always exists as a real negative number in Case (ii) because the set $\{c \in [-\infty, 0] \mid \mathcal{G}(c) > \pi\}$ is non-empty.

of Problem (2), irrespective of the precise choices of the two risk measures ρ_1 and ρ_2 . In the following, we write $\mathbf{1}_A$ for the indicator function of an event A , i.e., $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ otherwise.

Theorem 5.2 (Solutions of Problem (4)). *Let $\mu_1 \in \mathcal{M}$ be fixed.*

- (i) If $\mathcal{G}(0) = \int_{\{G_2(\cdot) < G_1(\mu_1, \cdot)\}} G_2(x) dx \leq \pi$, then the optimal ceded loss function for Problem (4) must be in the form of

$$I^*(x) = \int_0^x \mathbf{1}_{\{G_2(\cdot) < G_1(\mu_1, \cdot)\}}(t) dt + \int_0^x \mathbf{1}_{\{G_2(\cdot) = G_1(\mu_1, \cdot)\}}(t) d\tilde{I}(t)$$

for any $\tilde{I} \in \mathcal{I}$ such that

$$\int_0^{\text{ess sup}(X)} G_2(x) dI^*(x) = \int_{\{G_2(\cdot) < G_1(\mu_1, \cdot)\}} G_2(x) dx + \int_{\{G_2(\cdot) = G_1(\mu_1, \cdot)\}} G_2(x) d\tilde{I}(x) \leq \pi;$$

- (ii) If $\mathcal{G}(-\infty) = 0 \leq \pi < \mathcal{G}(0) = \int_{\{G_2(\cdot) < G_1(\mu_1, \cdot)\}} G_2(x) dx$, then the optimal ceded loss function for Problem (4) must be in the form of

$$I^*(x) = \int_0^x \mathbf{1}_{\{(1-c^*)G_2(\cdot) < G_1(\mu_1, \cdot)\}}(t) dt + \int_0^x \mathbf{1}_{\{(1-c^*)G_2(\cdot) = G_1(\mu_1, \cdot)\}}(t) d\tilde{I}(t)$$

for any $\tilde{I} \in \mathcal{I}$ such that

$$\begin{aligned} \int_0^{\text{ess sup}(X)} G_2(x) dI^*(x) &= \int_{\{(1-c^*)G_2(\cdot) < G_1(\mu_1, \cdot)\}} G_2(x) dx + \int_{\{(1-c^*)G_2(\cdot) = G_1(\mu_1, \cdot)\}} G_2(x) d\tilde{I}(x) \\ &= \pi. \end{aligned}$$

Moreover, the optimal objective value of Problem (4) is given by

$$c^* \pi + \int_0^{\text{ess sup}(X)} \min \{G_1(\mu_1, x), (1-c^*)G_2(x)\} dx.$$

Proof. Only the optimal objective value needs to be proved, since the optimal ceded loss functions in both cases are direct consequences of Theorem 5.1.

For Case (i), by Lemma 2.1 and Fubini's Theorem, the optimal objective value of Problem

(4) is given by

$$\begin{aligned}
& \int_0^1 \left(\int_0^{\text{ess sup}(X)} g_\alpha(S_X(x)) \, dx \right) \mu_1(d\alpha) + \int_0^{\text{ess sup}(X)} (G_2(x) - G_1(\mu_1, x)) \, dI^*(x) \\
&= \int_0^{\text{ess sup}(X)} G_1(\mu_1, x) \, dx + \int_{\{G_2(\cdot) < G_1(\mu_1, \cdot)\}} (G_2(x) - G_1(\mu_1, x)) \, dx \\
&= \int_{\{G_2 \geq G_1(\mu_1, \cdot)\}} G_1(\mu_1, x) \, dx + \int_{\{G_2(\cdot) < G_1(\mu_1, \cdot)\}} G_2(x) \, dx \\
&= \int_0^{\text{ess sup}(X)} \min \{G_1(\mu_1, x), G_2(x)\} \, dx \\
&= c^* \pi + \int_0^{\text{ess sup}(X)} \min \{G_1(\mu_1, x), (1 - c^*)G_2(x)\} \, dx,
\end{aligned}$$

where the last equality holds since $c^* = 0$ in case (i). Case (ii) proceeds along the same line as Case (i). \square

6 Complete solutions of Problem (2)

Given the development in Sections 3 to 5, it remains to assemble all the intermediate results thus far and formulate the full solutions of Problem (2). For any $\mu_1 \in \mathcal{M}$, define

$$\mathcal{G}(\mu_1, c) \triangleq \int_{\{(1-c)G_2(\cdot) < G_1(\mu_1, \cdot)\}} G_2(x) \, dx, \quad \text{for all } c \in [-\infty, 0],$$

and

$$c^*(\mu_1) \triangleq \inf \left\{ c \in [-\infty, 0] \mid \int_{\{(1-c)G_2(\cdot) < G_1(\mu_1, \cdot)\}} G_2(x) \, dx \geq \pi \right\}.$$

By Theorem 5.2 and the verification of the conditions for Theorem 4.1 in the appendix, the solutions for Problem (2), or equivalently [Problem \(3\)](#), are given by the following theorem.

Theorem 6.1. Assume that the ground-up loss X is continuous, and \mathcal{M} is convex and compact with respect to a weak*-topologyⁱⁱ. Let

$$\mu_1^* = \arg \max_{\mu_1 \in \mathcal{M}} \left(c^*(\mu_1) \pi + \int_0^{\text{ess sup}(X)} \min \{G_1(\mu_1, x), (1 - c^*(\mu_1))G_2(x)\} \, dx \right). \quad (6)$$

Then,

- (i) if $\mathcal{G}(\mu_1^*, 0) = \int_{\{G_2(\cdot) < G_1(\mu_1^*, \cdot)\}} G_2(x) \, dx \leq \pi$, then the optimal ceded loss function for [Problem \(2\)](#) **must be in the form of**

$$I^*(x) = \int_0^x \mathbb{1}_{\{G_2(\cdot) < G_1(\mu_1^*, \cdot)\}}(t) \, dt + \int_0^x \mathbb{1}_{\{G_2(\cdot) = G_1(\mu_1^*, \cdot)\}}(t) \, d\tilde{I}(t)$$

ⁱⁱFor one of the possible candidates, see the appendix.

for some $\tilde{I} \in \mathcal{I}$ such that

$$\int_0^{\text{ess sup}(X)} G_2(x) dI^*(x) = \int_{\{G_2(\cdot) < G_1(\mu_1^*, \cdot)\}} G_2(x) dx + \int_{\{G_2(\cdot) = G_1(\mu_1^*, \cdot)\}} G_2(x) d\tilde{I}(x) \leq \pi;$$

in particular, if the set

$$\{x \in [0, \text{ess sup}(X)] \mid G_2(x) = G_1(\mu_1^*, x)\}$$

has zero Lebesgue measure, then the optimal ceded loss function for Problem (2) is unique and is given by

$$I^*(x) = \int_0^x \mathbb{1}_{\{G_2(\cdot) < G_1(\mu_1^*, \cdot)\}}(t) dt;$$

(ii) if $\mathcal{G}(\mu_1^*, -\infty) = 0 \leq \pi < \mathcal{G}(\mu_1^*, 0) = \int_{\{G_2(\cdot) < G_1(\mu_1^*, \cdot)\}} G_2(x) dx$, then the optimal ceded loss function for Problem (2) **must be in the form of**

$$I^*(x) = \int_0^x \mathbb{1}_{\{(1-c^*(\mu_1^*))G_2(\cdot) < G_1(\mu_1^*, \cdot)\}}(t) dt + \int_0^x \mathbb{1}_{\{(1-c^*(\mu_1^*))G_2(\cdot) = G_1(\mu_1^*, \cdot)\}}(t) d\tilde{I}(t)$$

for some $\tilde{I} \in \mathcal{I}$ such that

$$\begin{aligned} \int_0^{\text{ess sup}(X)} G_2(x) dI^*(x) &= \int_{\{(1-c^*(\mu_1^*))G_2(\cdot) < G_1(\mu_1^*, \cdot)\}} G_2(x) dx \\ &\quad + \int_{\{(1-c^*(\mu_1^*))G_2(\cdot) = G_1(\mu_1^*, \cdot)\}} G_2(x) d\tilde{I}(x) \\ &= \pi; \end{aligned}$$

in particular, if the set

$$\{x \in [0, \text{ess sup}(X)] \mid (1 - c^*(\mu_1^*))G_2(x) = G_1(\mu_1^*, x)\}$$

has zero Lebesgue measure, then the optimal ceded loss function for Problem (2) is unique and is given by

$$I^*(x) = \int_0^x \mathbb{1}_{\{(1-c^*(\mu_1^*))G_2(\cdot) < G_1(\mu_1^*, \cdot)\}}(t) dt.$$

7 Illustrative examples

The general structure of the optimal reinsurance policy has been laid out in Theorem 6.1. Different distributions of the ground-up loss X give rise to optimal solutions sharing the same likelihood ratio test structure, but their precise construction requires solving Problem (6) for the optimal probability measure μ_1^* over the set \mathcal{M} . In general, tackling Problem (6) for arbitrary \mathcal{M} (equivalently, arbitrary ρ_1) and ρ_2 is a very challenging, if not impossible task due to the highly nonlinear, non-canonical, and infinite-dimensional nature of the problem. For this reason,

in this section we provide two concrete examples to illustrate the specific form of the optimal reinsurance contract under some commonly encountered coherent and comonotonic additive, convex risk measures. With the prescription of appropriate choices of the class \mathcal{M} and the probability measure μ_2 , the optimal contracts are shown to assume some appealing and easily comprehensible forms, shedding light on their properties.

The first illustrative example studies the concrete form of the optimal reinsurance contract for a general law-invariant, continuous-from-above coherent risk measure when the reinsurance premium is calibrated by the expectation premium principle, which is one of the most commonly used premium principles in the reinsurance literature. Our results show that sufficiently small losses should not be optimally ceded and that subject to a simple sufficient condition on the coherent risk measure, a stop-loss reinsurance policy is optimal. These findings substantially generalize existing results derived under the net premium principle (e.g., Example 3.2 of Cai et al. (2017)), which is a special case of the expectation premium principle. In what follows, we denote by δ_x the Dirac measure situated at a real point x and by $(\cdot)_+$ the positive part function.

Example 7.1 (General law-invariant, continuous-from-above coherent risk measure with the expectation premium principle). Let the ground-up loss X have a continuous and strictly increasing distribution function. Consider an arbitrary $\mathcal{M} \subseteq \mathcal{M}_1([0, 1])$ and $\mu_2 = (1 + \theta)\delta_0$ ⁱⁱⁱ for some given $\theta \geq 0$. In this case, ρ_1 is a general law-invariant continuous-from-above coherent risk measure, while

$$\rho_2(Y) = (1 + \theta) \int_0^1 \text{TVaR}_\alpha(Y) \delta_0(d\alpha) = (1 + \theta)\text{TVaR}_0(Y) = (1 + \theta)\mathbb{E}[Y]$$

corresponds to the expectation premium principle with a safety loading of θ . Moreover,

$$G_2(x) = (1 + \theta) \int_0^1 g_\alpha(S_X(x)) \delta_0(d\alpha) = (1 + \theta)g_0(S_X(x)) = (1 + \theta)S_X(x)$$

for all $x \in [0, \text{ess sup}(X)]$. To solve Problem (2) with the above choice of ρ_2 , we have to identify the subsets

$$\begin{aligned} &\{x \in [0, \text{ess sup}(X)] \mid (1 - c^*(\mu_1^*)) G_2(x) < G_1(\mu_1^*, x)\}, \\ &\{x \in [0, \text{ess sup}(X)] \mid (1 - c^*(\mu_1^*)) G_2(x) = G_1(\mu_1^*, x)\}, \end{aligned}$$

where μ_1^* and $c^*(\mu_1^*)$ are defined as in Theorem 6.1. To this end, we first examine, for different values of α , the shapes of the functions (see Figure 1)

$$g_\alpha(\lambda) = \min\left(\frac{\lambda}{1 - \alpha}, 1\right) \quad \text{and} \quad (1 - c^*(\mu_1^*)) (1 + \theta)g_0(\lambda) = (1 - c^*(\mu_1^*)) (1 + \theta)\lambda,$$

which give rise to $G_1(\mu_1^*, x)$ and $(1 - c^*(\mu_1^*)) G_2(x)$, respectively, upon the replacement of λ by $S_X(x)$ and integration with respect to μ_1^* . With the definition

$$d^* = d^*(\theta, c^*(\mu_1^*)) \triangleq F_X^{-1}\left(1 - \frac{1}{(1 + \theta)(1 - c^*(\mu_1^*))}\right)$$

ⁱⁱⁱAlthough μ_2 is not a probability measure, this does not affect the validity of the results in this paper.

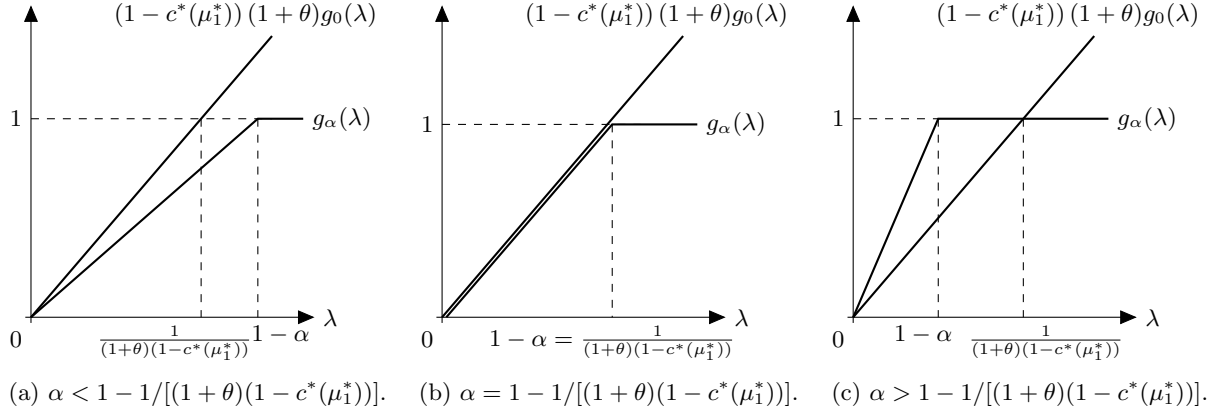


Figure 1: Comparison of $g_\alpha(\lambda)$ with $(1 - c^*(\mu_1^*)) (1 + \theta)g_0(\lambda)$ for different values of α .

we deduce that:

(a) If $\alpha < 1 - 1/[(1 + \theta)(1 - c^*(\mu_1^*))]$, then

$$g_\alpha(S_X(x)) < (1 - c^*(\mu_1^*)) (1 + \theta)g_0(S_X(x)) \quad \text{for all } x \in [0, \text{ess sup}(X)).$$

(b) If $\alpha = 1 - 1/[(1 + \theta)(1 - c^*(\mu_1^*))]$, then

$$g_\alpha(S_X(x)) \begin{cases} < (1 - c^*(\mu_1^*)) (1 + \theta)g_0(S_X(x)), & \text{for } 0 \leq x < d^*, \\ = (1 - c^*(\mu_1^*)) (1 + \theta)g_0(S_X(x)), & \text{for } d^* \leq x \leq \text{ess sup}(X). \end{cases}$$

(c) If $\alpha > 1 - 1/[(1 + \theta)(1 - c^*(\mu_1^*))]$, then

$$g_\alpha(S_X(x)) \begin{cases} < (1 - c^*(\mu_1^*)) (1 + \theta)g_0(S_X(x)), & \text{for } 0 \leq x < d^*, \\ = (1 - c^*(\mu_1^*)) (1 + \theta)g_0(S_X(x)), & \text{for } x = d^*, \\ > (1 - c^*(\mu_1^*)) (1 + \theta)g_0(S_X(x)), & \text{for } d^* < x < \text{ess sup}(X). \end{cases}$$

We now distinguish two ranges of values of x .

Case 1. If $0 \leq x < d^*$, then $g_\alpha(S_X(x)) < (1 - c^*(\mu_1^*)) (1 + \theta)g_0(S_X(x))$ for all $\alpha \in [0, 1)$, so

$$\begin{aligned} G_1(\mu_1^*, x) &= \int_0^1 g_\alpha(S_X(x)) \mu_1^*(d\alpha) \\ &< \int_0^1 (1 - c^*(\mu_1^*)) (1 + \theta)g_0(S_X(x)) \mu_1^*(d\alpha) \\ &= (1 - c^*(\mu_1^*)) (1 + \theta)g_0(S_X(x)) \\ &= (1 - c^*(\mu_1^*)) G_2(x). \end{aligned}$$

By Theorem 6.1, we have $I^*(x) = 0$, i.e., no reinsurance coverage should be purchased for sufficiently small losses.

Case 2. If $d^* \leq x \leq \text{ess sup}(X)$, then in general there is no definite order between $G_1(\mu_1^*, x)$

and $(1 - c^*(\mu_1^*)) G_2(x)$, and further information about μ_1^* and $c^*(\mu_1^*)$ is required to compare the two functions. For example, if μ_1^* is concentrated on the interval $I(\theta, c^*(\mu_1^*)) \triangleq [1 - 1/[(1 + \theta)(1 - c^*(\mu_1^*))], 1)$ with $\mu_1^*((1 - 1/[(1 + \theta)(1 - c^*(\mu_1^*))], 1)) > 0$, then

$$G_1(\mu_1^*, x) > (1 - c^*(\mu_1^*)) G_2(x) \quad \text{for all } d^* < x < \text{ess sup}(X)$$

By Theorem 6.1 along with Case 1, we have $I^*(x) = (x - d^*)_+$, which means that the optimal reinsurance contract is a stop-loss contract that entails full coverage for losses beyond d^* .

We remark that the sufficient condition that μ_1^* is concentrated on $I(\theta, c^*(\mu_1^*))$ becomes milder as the premium budget becomes less stringent, which is reflected by a $c^*(\mu_1^*)$ that is relatively small in magnitude and makes $I(\theta, c^*(\mu_1^*))$ wider. In particular, the condition is automatically fulfilled in the important case when $\theta = c^*(\mu_1^*) = 0$, which means that the reinsurance premium budget as calibrated by the net premium principle is not tight (or no premium budget is imposed *a priori*). In this case, $I(0, 0) = [0, 1)$ and the optimal reinsurance contract reduces to full coverage, a result that holds true for all μ_1^* and therefore for all law-invariant, continuous-from-above coherent risk measure ρ_1 .

The second illustrative example is an extended one that centers on the absolute semi-deviation risk measure in conjunction with the TVaR premium principle. This risk measure possesses a Kusuoka representation which can be captured by a convex combination of Dirac measures and allows us to identify \mathcal{M} with the $(0, 1)$ interval. A highlight of this example is that in the course of our analysis, we provide an illustration of how Problem (6) can be solved in a given setting. Even for a relatively simple risk measure, the solution procedure turns out to be highly non-trivial.

Example 7.2 (Absolute semi-deviation risk measure). In this example, let ρ_1 and ρ_2 be the *absolute semi-deviation risk measure* and the β -level TVaR, respectively:

$$\begin{aligned} \rho_1(Y) &= \mathbb{E}[Y] + c \mathbb{E}[(Y - \mathbb{E}[Y])_+] \\ \rho_2(Y) &= \text{TVaR}_\beta(Y) \end{aligned}$$

where $c \in (0, 1]$ and $\beta \in [0, 1)$ are fixed. Notice that ρ_1 is a law-invariant, coherent, and continuous-from-above risk measure, while ρ_2 is a law-invariant, comonotonic additive, and convex risk measure. Indeed, by Example 2 in Shapiro (2013), ρ_1 can be represented as

$$\rho_1(Y) = \sup_{\kappa \in (0, 1)} ((1 - c\kappa) \text{TVaR}_0(Y) + c\kappa \text{TVaR}_{1-\kappa}(Y)) = \sup_{\mu_1 \in \mathcal{M}} \int_0^1 \text{TVaR}_\alpha(Y) \mu_1(d\alpha),$$

where

$$\mathcal{M} = \{(1 - c\kappa) \delta_0 + c\kappa \delta_{1-\kappa} \mid \kappa \in (0, 1)\},$$

and ρ_2 can be written as $\rho_2(Y) = \int_0^1 \text{TVaR}_\alpha(Y) \mu_2(d\alpha)$ with $\mu_2 = \delta_\beta$. Since \mathcal{M} is not convex

or compact, its closed convex hull $\overline{\text{conv}(\mathcal{M})}$ should be considered:

$$\rho_1(Y) = \sup_{\mu_1 \in \mathcal{M}} \int_0^1 \text{TVaR}_\alpha(Y) \mu_1(d\alpha) = \sup_{\mu_1 \in \overline{\text{conv}(\mathcal{M})}} \int_0^1 \text{TVaR}_\alpha(Y) \mu_1(d\alpha),$$

where the second equality is due to Proposition 1 in Shapiro (2013). The closed convex hull $\overline{\text{conv}(\mathcal{M})}$ is compact. Indeed,

$$\mathcal{M} \subseteq \mathcal{M}_{[0,1]} \triangleq \{(1 - c\kappa) \delta_0 + c\kappa \delta_{1-\kappa} : \kappa \in [0, 1]\},$$

which is compact. Since $\mathcal{M}_{[0,1]}$ is a subset of $\mathcal{M}^S([0, 1])$, the set of finite signed measures on $[0, 1]$, which is a Banach space, we deduce that $\overline{\text{conv}(\mathcal{M}_{[0,1]})}$ is also compact (see, for example, Theorem 5.35 in Aliprantis and Border (2006)). Therefore, $\overline{\text{conv}(\mathcal{M})}$, as a subset of $\overline{\text{conv}(\mathcal{M}_{[0,1]})}$, is compact.

For simplicity, we hereby assume that the distribution function of the ground-up loss X is continuous and strictly increasing and consider the case that $\pi = \infty$, i.e., the budget constraint is void. Therefore, Problem (6) becomes

$$\max_{\mu_1 \in \overline{\text{conv}(\mathcal{M})}} \int_0^{\text{ess sup}(X)} \min \{G_1(\mu_1, x), G_2(x)\} dx.$$

Because the objective function is continuous in μ_1 , the problem is further equivalent to

$$\sup_{\mu_1 \in \overline{\text{conv}(\mathcal{M})}} \int_0^{\text{ess sup}(X)} \min \{G_1(\mu_1, x), G_2(x)\} dx, \quad (7)$$

where the convex hull of \mathcal{M} is given by

$$\text{conv}(\mathcal{M}) = \left\{ \left(1 - c \sum_{i=1}^m \beta_i \kappa_i \right) \delta_0 + c \sum_{i=1}^m \beta_i \kappa_i \delta_{1-\kappa_i} \mid \begin{array}{l} \kappa_1, \dots, \kappa_m \in (0, 1), \kappa_m < \dots < \kappa_1, \\ \beta_1, \dots, \beta_m \in [0, 1], \beta_1 + \dots + \beta_m = 1, \\ m = 1, 2, \dots \end{array} \right\}.$$

We now show that the objective function in (7) at a certain $\mu_1 \in \text{conv}(\mathcal{M})$ concentrated on $m + 1$ points (including 0) with $m \geq 2$ is always majorized by another element of $\text{conv}(\mathcal{M})$ concentrated on m points, so the supremum in (7) must be attained by an element of \mathcal{M} . To this end, we fix any given $\mu_1 = (1 - c \sum_{i=1}^m \beta_i \kappa_i) \delta_0 + c \sum_{i=1}^m \beta_i \kappa_i \delta_{1-\kappa_i}$ in $\text{conv}(\mathcal{M})$ with $m = 2, 3, \dots$ and define

$$\tilde{\mu}_1 \triangleq \left(1 - c \sum_{i=1}^{m-1} \tilde{\beta}_i \tilde{\kappa}_i \right) \delta_0 + c \sum_{i=1}^{m-1} \tilde{\beta}_i \tilde{\kappa}_i \delta_{1-\tilde{\kappa}_i},$$

where

$$\tilde{\beta}_1 = \beta_1, \quad \dots, \quad \tilde{\beta}_{m-2} = \beta_{m-2}, \quad \tilde{\beta}_{m-1} = \beta_{m-1} + \beta_m,$$

and

$$\tilde{\kappa}_1 = \kappa_1, \quad \dots, \quad \tilde{\kappa}_{m-2} = \kappa_{m-2}, \quad \tilde{\kappa}_{m-1} = \frac{\beta_{m-1}}{\tilde{\beta}_{m-1}} \kappa_{m-1} + \frac{\beta_m}{\tilde{\beta}_{m-1}} \kappa_m.$$

(note that $\tilde{\beta}_{m-1} \tilde{\kappa}_{m-1} = \beta_{m-1} \kappa_{m-1} + \beta_m \kappa_m$) Since $\tilde{\kappa}_1, \dots, \tilde{\kappa}_{m-1} \in (0, 1)$, $\tilde{\kappa}_{m-1} < \dots < \tilde{\kappa}_1$, $\tilde{\beta}_1, \dots, \tilde{\beta}_{m-1} \in [0, 1]$, and $\tilde{\beta}_1 + \dots + \tilde{\beta}_{m-1} = 1$, we have $\tilde{\mu}_1 \in \text{conv}(\mathcal{M})$. As

$$\begin{aligned} & G_1(\mu_1, x) \\ = & \left(1 - c \sum_{i=1}^m \beta_i \kappa_i\right) S_X(x) + c \sum_{i=1}^m \beta_i \min\{S_X(x), \kappa_i\} \\ = & \begin{cases} \left(1 - c \sum_{i=1}^m \beta_i \kappa_i\right) S_X(x) + c \sum_{i=1}^m \beta_i \kappa_i, & \text{if } \kappa_1 \leq S_X(x) \leq 1 \\ \left(1 + c\beta_1(1 - \kappa_1) - c \sum_{i=2}^m \beta_i \kappa_i\right) S_X(x) + c \sum_{i=2}^m \beta_i \kappa_i, & \text{if } \kappa_2 \leq S_X(x) \leq \kappa_1 \\ \vdots & \vdots \\ \left(1 + c \sum_{i=1}^{j-1} \beta_i (1 - \kappa_i) - c \sum_{i=j}^m \beta_i \kappa_i\right) S_X(x) + c \sum_{i=j}^m \beta_i \kappa_i, & \text{if } \kappa_j \leq S_X(x) \leq \kappa_{j-1} \\ \vdots & \vdots \\ \left(1 + c \sum_{i=1}^{m-2} \beta_i (1 - \kappa_i) - c \sum_{i=m-1}^m \beta_i \kappa_i\right) S_X(x) + c \sum_{i=m-1}^m \beta_i \kappa_i, & \text{if } \kappa_{m-1} \leq S_X(x) \leq \kappa_{m-2} \\ \left(1 + c \sum_{i=1}^{m-1} \beta_i (1 - \kappa_i) - c \sum_{i=m}^m \beta_i \kappa_i\right) S_X(x) + c \sum_{i=m}^m \beta_i \kappa_i, & \text{if } \kappa_m \leq S_X(x) \leq \kappa_{m-1} \\ \left(1 + c \sum_{i=1}^m \beta_i (1 - \kappa_i)\right) S_X(x), & \text{if } 0 \leq S_X(x) \leq \kappa_m \end{cases} \end{aligned}$$

and

$$\begin{aligned} & G_1(\tilde{\mu}_1, x) \\ = & \left(1 - c \sum_{i=1}^{m-1} \tilde{\beta}_i \tilde{\kappa}_i\right) S_X(x) + c \sum_{i=1}^{m-1} \tilde{\beta}_i \min\{S_X(x), \tilde{\kappa}_i\} \\ = & \begin{cases} \left(1 - c \sum_{i=1}^{m-1} \tilde{\beta}_i \tilde{\kappa}_i\right) S_X(x) + c \sum_{i=1}^{m-1} \tilde{\beta}_i \tilde{\kappa}_i, & \text{if } \tilde{\kappa}_1 \leq S_X(x) \leq 1 \\ \left(1 + c\tilde{\beta}_1(1 - \tilde{\kappa}_1) - c \sum_{i=2}^{m-1} \tilde{\beta}_i \tilde{\kappa}_i\right) S_X(x) + c \sum_{i=2}^{m-1} \tilde{\beta}_i \tilde{\kappa}_i, & \text{if } \tilde{\kappa}_2 \leq S_X(x) \leq \tilde{\kappa}_1 \\ \vdots & \vdots \\ \left(1 + c \sum_{i=1}^{m-2} \tilde{\beta}_i (1 - \tilde{\kappa}_i) - c \sum_{i=m-1}^{m-1} \tilde{\beta}_i \tilde{\kappa}_i\right) S_X(x) + c \sum_{i=m-1}^{m-1} \tilde{\beta}_i \tilde{\kappa}_i, & \text{if } \tilde{\kappa}_{m-1} \leq S_X(x) \leq \tilde{\kappa}_{m-2} \\ \left(1 + c \sum_{i=1}^{m-1} \tilde{\beta}_i (1 - \tilde{\kappa}_i)\right) S_X(x), & \text{if } 0 \leq S_X(x) \leq \tilde{\kappa}_{m-1} \end{cases}, \end{aligned}$$

we have

$$G_1(\tilde{\mu}_1, x) = G_1(\mu_1, x) \quad \text{if } 0 \leq S_X(x) \leq \kappa_m \text{ or } \kappa_{m-1} \leq S_X(x) \leq 1,$$

while

$$G_1(\tilde{\mu}_1, x) \geq G_1(\mu_1, x) \quad \text{if } \kappa_m \leq S_X(x) \leq \kappa_{m-1}.$$

Applying these arguments iteratively, we have

$$\begin{aligned} \sup_{\mu_1 \in \text{conv}(\mathcal{M})} \int_0^{\text{ess sup}(X)} \min \{G_1(\mu_1, x), G_2(x)\} dx &= \sup_{\mu_1 \in \mathcal{M}} \int_0^{\text{ess sup}(X)} \min \{G_1(\mu_1, x), G_2(x)\} dx \\ &= \sup_{\kappa \in (0,1)} \int_0^{\text{ess sup}(X)} \min \{G_1(\kappa, x), G_2(x)\} dx, \end{aligned}$$

where

$$G_1(\kappa, x) \triangleq \begin{cases} (1 - c\kappa) S_X(x) + c\kappa, & \text{if } \kappa < S_X(x) \leq 1 \\ (1 + c(1 - \kappa)) S_X(x), & \text{if } 0 \leq S_X(x) \leq \kappa \end{cases} \quad \text{and} \quad G_2(x) = \min \left\{ \frac{S_X(x)}{1 - \beta}, 1 \right\}.$$

Therefore, solving Problem (7) is equivalent to solving

$$\sup_{\kappa \in (0,1)} \int_0^{\text{ess sup}(X)} \min \{G_1(\kappa, x), G_2(x)\} dx$$

and identifying the point $\kappa^* \in (0, 1)$ at which the supremum is attained. Now we consider the following two ranges of values of κ .

- Suppose that $\left(1 - \frac{\beta}{c(1-\beta)}\right)_+ \leq \kappa < 1$. In this case $G_1(\kappa, x) \leq G_2(x)$ for any $x \in [0, \text{ess sup}(X)]$. Hence

$$\begin{aligned} & \int_0^{\text{ess sup}(X)} \min \{G_1(\kappa, x), G_2(x)\} dx \\ &= \int_0^{\text{ess sup}(X)} G_1(\kappa, x) dx \\ &= \int_0^{F_X^{-1}(1-\kappa)} [c\kappa + (1 - c\kappa)S_X(x)] dx + \int_{F_X^{-1}(1-\kappa)}^{\text{ess sup}(X)} [1 + c(1 - \kappa)]S_X(x) dx \\ &= \int_0^{\text{ess sup}(X)} S_X(x) dx + c \left[\kappa \int_0^{F_X^{-1}(1-\kappa)} F_X(x) dx + (1 - \kappa) \int_{F_X^{-1}(1-\kappa)}^{\text{ess sup}(X)} S_X(x) dx \right]. \end{aligned}$$

By the fundamental theorem of calculus,

$$\frac{d}{d\kappa} \int_0^{\text{ess sup}(X)} \min \{G_1(\kappa, x), G_2(x)\} dx = F_X^{-1}(1 - \kappa) - \mathbb{E}[X] \begin{cases} > 0, & \text{if } \kappa < S_X(\mathbb{E}[X]) \\ = 0, & \text{if } \kappa = S_X(\mathbb{E}[X]) \\ < 0, & \text{if } \kappa > S_X(\mathbb{E}[X]) \end{cases}$$

and hence

$$\begin{aligned} \kappa_1^* &\triangleq \arg \max_{\kappa \in \left[\left(1 - \frac{\beta}{c(1-\beta)}\right)_+, 1\right)} \int_0^{\text{ess sup}(X)} \min \{G_1(\kappa, x), G_2(x)\} dx \\ &= \max \left\{ S_X(\mathbb{E}[X]), \left(1 - \frac{\beta}{c(1-\beta)}\right)_+ \right\}. \end{aligned}$$

- Suppose that $0 < \kappa < \left(1 - \frac{\beta}{c(1-\beta)}\right)_+ \in (0, 1]$. In this case, (recall that F_X is continuous

and strictly increasing)

$$G_2(x) \begin{cases} > G_1(\kappa, x), & \text{for } 0 < x < F_X^{-1}\left(\frac{\beta/[c(1-\beta)]}{\beta/[c(1-\beta)]+\kappa}\right) \\ = G_1(\kappa, x), & \text{for } x = F_X^{-1}\left(\frac{\beta/[c(1-\beta)]}{\beta/[c(1-\beta)]+\kappa}\right) \\ < G_1(\kappa, x), & \text{for } F_X^{-1}\left(\frac{\beta/[c(1-\beta)]}{\beta/[c(1-\beta)]+\kappa}\right) < x < \text{ess sup}(X) \end{cases}.$$

Hence

$$\begin{aligned} & \int_0^{\text{ess sup}(X)} \min\{G_1(\kappa, x), G_2(x)\} dx \\ &= \int_0^{F_X^{-1}\left(\frac{\beta/[c(1-\beta)]}{\beta/[c(1-\beta)]+\kappa}\right)} G_1(\kappa, x) dx + \int_{F_X^{-1}\left(\frac{\beta/[c(1-\beta)]}{\beta/[c(1-\beta)]+\kappa}\right)}^{\text{ess sup}(X)} G_2(x) dx \\ &= \int_0^{F_X^{-1}\left(\frac{\beta/[c(1-\beta)]}{\beta/[c(1-\beta)]+\kappa}\right)} [c\kappa + (1 - c\kappa) S_X(x)] dx + \frac{1}{1 - \beta} \int_{F_X^{-1}\left(\frac{\beta/[c(1-\beta)]}{\beta/[c(1-\beta)]+\kappa}\right)}^{\text{ess sup}(X)} S_X(x) dx. \end{aligned}$$

For this objective function, there is no closed-form expression for

$$\kappa_2^* \triangleq \arg \max_{\kappa \in (0, 1 - \beta/[c(1-\beta)])} \int_0^{\text{ess sup}(X)} \min\{G_1(\kappa, x), G_2(x)\} dx.$$

Depending on whether $\kappa^* = \kappa_1^*$ or κ_2^* (which can be verified numerically when the distribution of X is specified), we deduce that:

Case 1. Suppose that $\kappa^* = \kappa_1^* \in \left[\left(1 - \frac{\beta}{c(1-\beta)}\right)_+, 1\right)$.

- (a) If $\left(1 - \frac{\beta}{c(1-\beta)}\right)_+ < S_X(\mathbb{E}[X])$, then $\kappa_1^* = S_X(\mathbb{E}[X])$ and $G_1(\kappa_1^*, x) < G_2(x)$ for any $x \in (0, \text{ess sup}(X))$, and hence $I^* \equiv 0$ is the unique optimal solution.
- (b) If $S_X(\mathbb{E}[X]) \leq \left(1 - \frac{\beta}{c(1-\beta)}\right)_+$, then $\kappa_1^* = \left(1 - \frac{\beta}{c(1-\beta)}\right)_+$ and $G_1(\kappa_1^*, x) = G_2(x)$ if and only if $x \in [F_X^{-1}(1 - \kappa_1^*), \text{ess sup}(X)]$, and hence I^* is of the form

$$I^*(x) = \left(\tilde{I}(x) - \tilde{I}(F_X^{-1}(1 - \kappa_1^*))\right)_+$$

for some $\tilde{I} \in \mathcal{I}$.

Case 2. Suppose that $\kappa^* = \kappa_2^* \in \left(0, 1 - \frac{\beta}{c(1-\beta)}\right)$. Then

$$I^*(x) = \left(x - F_X^{-1}\left(\frac{\beta/[c(1-\beta)]}{\beta/[c(1-\beta)] + \kappa_2^*}\right)\right)_+$$

is the unique optimal solution.

For further illustrations, we assume that X follows a right-censored exponential distribution with distribution function $F_X(x) = 1 - e^{-0.5x}$ if $0 \leq x < 10$ and $F_X(x) = 1$ if $x \geq 10$, and

that $c = 1$. The following table shows the maximizer κ^* of Problem (6), the optimal ceded loss function I^* , and the corresponding case for different values of β .

β	κ^*	Case	I^*
0.0	any value in $(0, 1)$	2	$I^*(x) = x$
0.2	0.7500	1 (b)	$I^*(x) = (\tilde{I}(x) - \tilde{I}(0.5754))_+$ for some $\tilde{I} \in \mathcal{I}$
0.4	0.3704	1 (a)	$I^*(x) = 0$
0.6	0.3704	1 (a)	$I^*(x) = 0$
0.8	0.3704	1 (a)	$I^*(x) = 0$

In fact, when $\beta > 0.5$, then $1 - \frac{\beta}{c(1-\beta)} < 0$ and only Case 1 (a) is possible. The fact that no coverage should be purchased in this case is a consequence of the high reinsurance premium in comparison to the ceded risk.

We end this example by remarking that the above analysis, which substantially reduces the search of the optimal measure μ_1^* from the convex hull of \mathcal{M} to \mathcal{M} itself works only when $\pi = \infty$. If π is a positive real value, then numerical methods should be resorted to solve for the optimal measure μ_1^* in the convex hull and hence the optimal ceded loss function I^* .

8 Concluding remarks

In this paper, we analytically identify and concretely illustrate the optimal reinsurance arrangement that minimizes a general law-invariant coherent risk measure of the net risk exposure of an insurer when the reinsurance premium is calculated based on a general law-invariant comonotonic additive convex risk measure. A reinsurance premium budget constraint is in place. By capitalizing on the inherent Neyman–Pearson character in the optimal reinsurance problem, our solution methodology results in optimal solutions that are explicit and transparent in nature.

It should be pointed out that with cosmetic adjustments, our analysis can be easily extended to the case when both the the objective functional and the reinsurance premium are based on a general law-invariant convex risk measure. Such an extension will be achieved, however, at the expense of a substantial increase in notational burden (two suprema and two penalty functions will be present) in exchange for a minimal gain in generality and insights.

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Appendix: Verification of the conditions for mini-max theorem

Let $BV[0, \text{ess sup}(X)]$ be the set of all real-valued functions I defined on $[0, \text{ess sup}(X)]$, normalized so that $I(0) = 0$ and is right continuous with bounded variation, i.e.,

$$\|I\|_{BV} \triangleq \sup_{n \in \mathbb{N}; 0=x_0 \leq x_1 \leq \dots \leq x_n \leq \text{ess sup}(X)} \sum_{i=1}^n |I(x_i) - I(x_{i-1})| < \infty.$$

To apply Theorem 4.1, we need to check that:

- (1) $\mathcal{I}(\pi)$ is a compact convex subset of the Hausdorff vector space $BV[0, \text{ess sup}(X)]$;
- (2) \mathcal{M} is a subset of the Hausdorff vector space $\mathcal{M}^S([0, 1])$, which is the set of finite signed measures on $[0, 1)$ (the convexity and compactness of \mathcal{M} with respect to a weak*-topology are assumed);
- (3) the function $V : \mathcal{I}(\pi) \times \mathcal{M} \rightarrow \mathbb{R}$ defined by

$$V(I, \mu_1) \triangleq \int_0^1 \text{TVaR}_\alpha(X) \mu_1(d\alpha) + \int_0^{\text{ess sup}(X)} (G_2(x) - G_1(\mu_1, x)) dI(x),$$

is continuous and linear in each argument.

We divide our proof into three parts accordingly.

- (1) Denote by

$$B_{BV}(\text{ess sup}(X)) \triangleq \{I \in BV[0, \text{ess sup}(X)] : \|I\|_{BV} \leq \text{ess sup}(X)\}$$

the closed ball in $BV[0, \text{ess sup}(X)]$ with radius $\text{ess sup}(X)$. As every $I \in \mathcal{I}$ is monotone, $\|I\|_{BV} = I(\text{ess sup}(X)) - I(0) \leq \text{ess sup}(X)$, and hence $\mathcal{I} \subseteq B_{BV}(\text{ess sup}(X))$. It is well-known that $BV[0, \text{ess sup}(X)]$ is a vector space endowed with the usual addition and scalar multiplication operations for real-valued functions, and that $\|\cdot\|_{BV}$ is a norm on $BV[0, \text{ess sup}(X)]$. Hence, $(BV[0, \text{ess sup}(X)], \|\cdot\|_{BV})$ is a normed vector space and thus a topological vector space, using the open balls as the base for the (strong) topology. Moreover, $(BV[0, \text{ess sup}(X)], \|\cdot\|_{BV})$ is a Banach and Hausdorff space.

Similarly, let $C[0, \text{ess sup}(X)]$ be the set of all real-valued continuous functions on $[0, \text{ess sup}(X)]$. As for $BV[0, \text{ess sup}(X)]$, $(C[0, \text{ess sup}(X)], \|\cdot\|_\infty)$ is a topological vector space and it is a separable, Banach, and Hausdorff space.

Denote by τ the weak*-topology on the topological dual space $C[0, \text{ess sup}(X)]^*$, which is the space of all continuous linear functionals on $C[0, \text{ess sup}(X)]$. By definition, for any $\phi \in C[0, \text{ess sup}(X)]$, the map

$$T_\phi(\psi) = \psi(\phi), \quad \text{for all } \psi \in C[0, \text{ess sup}(X)]^*$$

is continuous with respect to the weak*-topology τ . By Riesz representation theorem, for

any $\psi \in C[0, \text{ess sup}(X)]^*$, there exists a unique $I \in \text{BV}[0, \text{ess sup}(X)]$ such that

$$\psi(\phi) = \int_0^{\text{ess sup}(X)} \phi(x) dI(x), \quad \text{for all } \phi \in C[0, \text{ess sup}(X)].$$

Conversely, for any $I \in \text{BV}[0, \text{ess sup}(X)]$, we can define a linear functional on $C[0, \text{ess sup}(X)]$ as above. Hence, the topological dual space $C[0, \text{ess sup}(X)]^*$ of $C[0, \text{ess sup}(X)]$ can be identified with $\text{BV}[0, \text{ess sup}(X)]$, and the weak*-topology τ is defined on $\text{BV}[0, \text{ess sup}(X)]$ such that, for any $\phi \in C[0, \text{ess sup}(X)]$, the map

$$T_\phi(I) = \int_0^{\text{ess sup}(X)} \phi(x) dI(x), \quad \text{for all } I \in \text{BV}[0, \text{ess sup}(X)]$$

is continuous with respect to τ . Moreover, since $(C[0, \text{ess sup}(X)], \|\cdot\|_\infty)$ is a normed vector space, $(\text{BV}[0, \text{ess sup}(X)], \tau)$ is Hausdorff.

We want to show that (\mathcal{I}, τ) is a compact topological space. By Banach-Alaoglu theorem, the closed ball $B_{\text{BV}}(\text{ess sup}(X))$ is compact in the weak*-topology τ . Since $\mathcal{I} \subseteq B_{\text{BV}}(\text{ess sup}(X))$, it suffices to show that \mathcal{I} is closed in τ . Because $(C[0, \text{ess sup}(X)], \|\cdot\|_\infty)$ is separable, by Proposition 3.24 in Fabian et al. (2001), $(B_{\text{BV}}(\text{ess sup}(X)), \tau)$ is also metrizable. It therefore suffices to show that \mathcal{I} is sequentially closed in the weak*-topology τ . To this end, let $\{I_n\}$ be a sequence in \mathcal{I} such that $I_n \xrightarrow{*} I$ for some $I \in B_{\text{BV}}(\text{ess sup}(X))$, i.e., for any $\phi \in C[0, \text{ess sup}(X)]$,

$$\int_0^{\text{ess sup}(X)} \phi(x) dI_n(x) \rightarrow \int_0^{\text{ess sup}(X)} \phi(x) dI(x).$$

Since $I \in B_{\text{BV}}(\text{ess sup}(X)) \subseteq \text{BV}[0, \text{ess sup}(X)]$, $I(0) = 0$. Because each I_n is increasing on $[0, \text{ess sup}(X)]$, I is also increasing on $[0, \text{ess sup}(X)]$. Indeed, if there exist $a, b \in [0, \text{ess sup}(X)]$ such that $a < b$ but $I(a) > I(b)$, then one can choose a $\phi \in C[0, \text{ess sup}(X)]$ such that $\phi(x) \geq 0$ for any $x \in [a, b]$ while $\phi(x) = 0$ for $x \notin [a, b]$ and $\int_0^{\text{ess sup}(X)} \phi(x) dI(x) < 0$. However, for any $n \in \mathbb{N}$, $\int_0^{\text{ess sup}(X)} \phi(x) dI_n(x) \geq 0$, implying that $\int_0^{\text{ess sup}(X)} \phi(x) dI(x) \geq 0$, which leads to a contradiction. For instance, define

$$\phi_C(x) = \begin{cases} C(x-a) & \text{for } x \in [a, a + \frac{1}{C}], \\ 1 & \text{for } x \in [a + \frac{1}{C}, b - \frac{1}{C}], \\ C(b-x) & \text{for } x \in [b - \frac{1}{C}, b], \\ 0 & \text{for } x \notin [a, b], \end{cases}$$

for some constant $C > \frac{2}{b-a}$. Obviously, $\phi_C(x) \geq 0$ for any $x \in [a, b]$ while $\phi_C(x) = 0$ for $x \notin [a, b]$. Therefore,

$$\lim_{C \rightarrow \infty} \int_0^{\text{ess sup}(X)} \phi_C(x) dI(x) = \int_a^b dI(x) = I(b) - I(a).$$

In other words, there exists a large enough $C > \frac{2}{b-a}$ such that

$$\int_0^{\text{ess sup}(X)} \phi_C(x) dI(x) < I(b) - I(a) + \frac{I(a) - I(b)}{2} = \frac{I(b) - I(a)}{2} < 0.$$

Therefore, I must be increasing on $[0, \text{ess sup}(X)]$. By Portmanteau's theorem (see, for instance, Theorem 2.1 in Billingsley (2009)), since $I_n \xrightarrow{*} I$, we have $I_n(x) \rightarrow I(x)$ for all $x \in [0, \text{ess sup}(X)]$ at which I is continuous. As I is increasing on $[0, \text{ess sup}(X)]$, I is continuous on $[0, \text{ess sup}(X)]$ except at countably many points in $[0, \text{ess sup}(X)]$, say except on $E \subseteq [0, \text{ess sup}(X)]$. Then, I is continuous on $D = [0, \text{ess sup}(X)] \setminus E$, which is dense in $[0, \text{ess sup}(X)]$. Fix $x \in E$. By denseness, there exist $\{x_m\} \subseteq D$ such that $x_m \downarrow x$. Since each I_n is 1-Lipschitz, we get

$$0 \leq I_n(x_m) - I_n(x) \leq x_m - x.$$

Letting $n \rightarrow \infty$ and $m \rightarrow \infty$ and using the right continuity of I , we have $I_n(x) \rightarrow I(x)$ for any $x \in [0, \text{ess sup}(X)]$. As a result, the 1-Lipschitzity of each I_n easily implies that I is also 1-Lipschitz and thus lies in \mathcal{I} . This shows that \mathcal{I} is sequentially closed in the weak*-topology τ , and that (\mathcal{I}, τ) is a compact topological space. Now by the continuity of G_2 , it is clear that $\mathcal{I}(\pi)$ is a closed subset of \mathcal{I} and, by extension, a compact subset of the Hausdorff vector space $\text{BV}[0, \text{ess sup}(X)]$ in τ .

(2) It is immediate that \mathcal{M} is a subset of $\mathcal{M}^S([0, 1])$. Define

$$\|\mu\|_{\text{TV}} = |\mu|([0, 1]), \quad \text{for all } \mu \in \mathcal{M}^S([0, 1]).$$

It is well-known that $\mathcal{M}^S([0, 1])$ is a vector space endowed with the usual addition and scalar multiplication operations for signed measures and $\|\cdot\|_{\text{TV}}$ is a norm on $\mathcal{M}^S([0, 1])$. It follows that $(\mathcal{M}^S([0, 1]), \|\cdot\|_{\text{TV}})$ is a normed vector space, and thus is a topological vector space, using the open balls as the base for the (strong) topology. Moreover, $(\mathcal{M}^S([0, 1]), \|\cdot\|_{\text{TV}})$ is a Banach and Hausdorff space.

By the similar arguments as above and the one-to-one correspondence between signed measures and right-continuous functions of bounded variation, the weak*-topology $\tilde{\tau}$ is defined on $\mathcal{M}^S([0, 1])$ such that, for any $\phi \in C[0, 1]$, the map

$$\tilde{T}_\phi(\mu) = \int_0^1 \phi(\alpha) \mu(d\alpha), \quad \text{for all } \mu \in \mathcal{M}^S([0, 1])$$

is continuous with respect to $\tilde{\tau}$. Moreover, since $(C[0, 1], \|\cdot\|_\infty)$ is a normed vector space, $(\mathcal{M}^S([0, 1]), \tilde{\tau})$ is Hausdorff.

(3) Fix any $\mu_1 \in \mathcal{M}$. Because X is continuous on $[0, \text{ess sup}(X)]$, $G_2(\cdot) - G_1(\mu_1, \cdot)$ is continuous on $[0, \text{ess sup}(X)]$, and thus $V(\cdot, \mu_1)$ is continuous in $\mathcal{I}(\pi)$ with respect to the weak*-topology τ . It is also obvious that $V(\cdot, \mu_1)$ is linear on $\mathcal{I}(\pi)$.

Now fix any $I \in \mathcal{I}(\pi)$. Since $\text{TVaR}_\alpha((\text{Id} - I)(X))$ is continuous in $\alpha \in [0, 1]$, $V(I, \cdot)$ is continuous on \mathcal{M} with respect to the weak*-topology $\tilde{\tau}$. The linearity of $V(I, \cdot)$ on \mathcal{M} is

clear.

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