

A Kesten-type Bound for Sums of Randomly Weighted Subexponential Random Variables

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Abstract

Sums of randomly weighted subexponential random variables have become an important research topic, but most works on the topic consider randomly weighted sums of finitely many terms. To extend the study to the case of infinitely many terms, we establish a Kesten-type upper bound for the tail probabilities of sums of randomly weighted subexponential random variables. As an application, we derive a precise asymptotic formula for the tail probability of the aggregate present value of subexponential claims, where the present value factor is determined according to the zero-coupon bond price.

Keywords: randomly weighted sum; subexponentiality; Kesten-type bound; renewal process; zero-coupon bond

1 Introduction

Throughout the paper, denote by $\{X_1, X_2, \dots\}$ a sequence of independent and identically distributed (i.i.d.) random variables with common distribution function F , and denote by $\{\theta_1, \theta_2, \dots\}$ another sequence of nonnegative and uniformly bounded random variables independent of $\{X_1, X_2, \dots\}$. The target of this study is the randomly weighted sums

$$S_n^\theta = \sum_{k=1}^n \theta_k X_k, \quad n \in \mathbb{N}. \quad (1.1)$$

In this stochastic structure, X_1, X_2, \dots serve as primary variables while $\theta_1, \theta_2, \dots$ as random weights.

Due to their important applications to various areas including in particular insurance and finance, randomly weighted sums have become an important research topic. Originating from Tang and Tsitsiashvili (2003), an active strand of this literature focuses on

the tail behavior of randomly weighted sums of heavy-tailed random variables. See Chen et al. (2006), Zhu and Gao (2008), Zhang et al. (2009), Yi et al. (2011), Wang (2011), Yang et al. (2012, 2016), Hazra and Maulik (2012), Cheng (2014), Tang and Yuan (2014), Mao et al. (2015), Li (2018), Cheng and Cheng (2018), Geng et al. (2019), and Chen and Yang (2019), among many others.

A random variable X or its distribution function F with $\overline{F}(x) > 0$ for all x is said to be heavy tailed to the right if $Ee^{\gamma X} = \infty$ for all $\gamma > 0$. One of the most important classes of heavy-tailed distributions is the subexponential class. By definition, a distribution F on $[0, \infty)$ is subexponential, denoted by $F \in \mathcal{S}$, if

$$\lim_{x \rightarrow \infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n \quad (1.2)$$

holds for all (or, equivalently, for some) $n = 2, 3, \dots$, where F^{n*} denotes the n -fold convolution of F . More generally, a distribution F on $(-\infty, \infty)$ is still said to be subexponential to the right if $F_+(x) = F(x)1_{(0 \leq x < \infty)}$ is subexponential. The reader is referred to Embrechts et al. (1997) and Foss et al. (2011) for textbook treatments of subexponential distributions with applications to insurance and finance.

Most of the the references cited above consider the tail behavior of the randomly weighted sums of finitely many terms. In particular, Tang and Yuan (2014) obtained the following result:

Theorem 1.1 *Let X_1, \dots, X_n be n i.i.d. random variables with common distribution function $F \in \mathcal{S}$, let $\theta_1, \dots, \theta_n$ be n nonnegative, bounded, and not-degenerate-at-zero random variables independent of $\{X_1, \dots, X_n\}$. Then*

$$P\left(\sum_{k=1}^n \theta_k X_k > x\right) \sim \sum_{k=1}^n P(\theta_k X_k > x). \quad (1.3)$$

Throughout this paper, all limit relationships are for $x \rightarrow \infty$ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$. Moreover, for two positive bivariate functions $a(\cdot, y)$ and $b(\cdot, y)$, we say that $a(x, y) \sim b(x, y)$ holds uniformly for $y \in \Delta$ if

$$\lim_{x \rightarrow \infty} \sup_{y \in \Delta} \left| \frac{a(x, y)}{b(x, y)} - 1 \right| = 0.$$

To extend the study to randomly weighted sums of infinitely many terms, a main difficulty exists in proving that the tail probability of the total sum is dominated by that of a large partial sum while the contribution of the residual tends to be negligible. This is especially true for the subexponential case. We overcome this difficulty by establishing a Kesten-type upper bound for the tail probabilities of randomly weighted sums.

Theorem 1.2 *Let $\{X_1, X_2, \dots\}$ be a sequence of i.i.d. and real-valued random variables with common distribution function $F \in \mathcal{S}$, let $\{\theta_1, \theta_2, \dots\}$ be another sequence of non-negative and uniformly bounded random variables independent of $\{X_1, X_2, \dots\}$. Then for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$P\left(\sum_{k=1}^n \theta_k X_k > x\right) \leq C_\varepsilon (1 + \varepsilon)^n \sum_{k=1}^n P(\theta_k X_k > x) \quad (1.4)$$

holds for all $n \in \mathbb{N}$ and all $x \geq 0$.

In the rest of this paper, we present the proof of Theorem 1.2 after preparing two lemmas in Section 2, and show an application of Theorem 1.2 to risk theory in Section 3.

2 Proof of Theorem 1.2

2.1 A Kesten-type bound for deterministically weighted sums

Recall Proposition 5.1 of Tang and Tsitsiashvili (2003) regarding a uniform asymptotics for the sum of deterministically weighted subexponential random variables:

Lemma 2.1 *Let X_1, \dots, X_n be n i.i.d. random variables with common distribution function $F \in \mathcal{S}$. Then for any fixed $0 < a \leq b < \infty$, the relation*

$$P\left(\sum_{k=1}^n c_k X_k > x\right) \sim \sum_{k=1}^n \bar{F}(x/c_k) \quad (2.1)$$

holds uniformly for $(c_1, \dots, c_n) \in [a, b]^n$.

Before the proof of Theorem 1.2, we first establish a Kesten-type upper bound for the tail probabilities of deterministically weighted sums:

Lemma 2.2 *Let $\{X_1, X_2, \dots\}$ be a sequence of i.i.d. and real-valued random variables with common distribution function $F \in \mathcal{S}$, and let $0 < a \leq b < \infty$ be arbitrarily fixed constants. Then for any $\varepsilon > 0$, there exists a constant $C_\varepsilon = C_\varepsilon(a, b) > 0$ such that*

$$P\left(\sum_{k=1}^n c_k X_k > x\right) \leq C_\varepsilon (1 + \varepsilon)^n \sum_{k=1}^n P(c_k X_k > x) \quad (2.2)$$

holds for all $n \in \mathbb{N}$, all $x \geq 0$, and all $c_k \in [a, b]$ for $k = 1, 2, \dots$

Proof. For notational convenience, we write the fact $c_k \in [a, b]$ for $k = 1, \dots, n$ as $\underline{c}_n = (c_1, \dots, c_n) \in [a, b]^n$. By the facts that $P(\sum_{k=1}^n c_k X_k > x) \leq P(\sum_{k=1}^n c_k X_k^+ > x)$ for each $n \in \mathbb{N}$ and that $P(c_k X_k > x) = P(c_k X_k^+ > x)$ for each $k \in \mathbb{N}$ and each $x \geq 0$, without

loss of generality we can assume that $\{X_1, X_2, \dots\}$ are nonnegative random variables, so that their common distribution F is supported on $[0, \infty)$. Here and throughout the paper, $v^+ = \max\{v, 0\}$ for a real number v . Write

$$\alpha_n = \sup_{\underline{c}_n \in [a, b]^n} \sup_{x \geq 0} \frac{P\left(\sum_{k=1}^n c_k X_k > x\right)}{\sum_{k=1}^n P(c_k X_k > x)}.$$

Consider the tail probability of the weighted sum $\sum_{k=1}^{n+1} c_k X_k$. We can assume that $c_{n+1} = \min\{c_1, \dots, c_n, c_{n+1}\}$ because otherwise we may rearrange the weights without changing the tail probability. We derive

$$\begin{aligned} & P\left(\sum_{k=1}^{n+1} c_k X_k > x\right) \\ &= P\left(\sum_{k=1}^{n+1} c_k X_k > x, c_{n+1} X_{n+1} \leq x\right) + P(c_{n+1} X_{n+1} > x) \\ &\leq \alpha_n \sum_{k=1}^n P(c_k X_k + c_{n+1} X_{n+1} > x, c_{n+1} X_{n+1} \leq x) + P(c_{n+1} X_{n+1} > x) \\ &= \alpha_n \sum_{k=1}^n (P(c_k X_k + c_{n+1} X_{n+1} > x) - P(c_{n+1} X_{n+1} > x)) + P(c_{n+1} X_{n+1} > x), \end{aligned}$$

where in the second step we deal with the first probability by conditioning on X_{n+1} and applying the definition of α_n . By Lemma 2.1, uniformly for $(c_k, c_{n+1}) \in [a, b]^2$,

$$P(c_k X_k + c_{n+1} X_{n+1} > x) \sim P(c_k X_k > x) + P(c_{n+1} X_{n+1} > x).$$

Therefore, for any $\varepsilon > 0$, there is a constant $A = A(a, b) > 0$ irrespective of n such that

$$P(c_k X_k + c_{n+1} X_{n+1} > x) \leq \left(1 + \frac{\varepsilon}{2}\right) (P(c_k X_k > x) + P(c_{n+1} X_{n+1} > x))$$

holds for all $x > A$ and all $(c_k, c_{n+1}) \in [a, b]^2$. It follows that, uniformly for all $x > A$ and all $\underline{c}_{n+1} \in [a, b]^{n+1}$ but $c_{n+1} = \min\{c_1, \dots, c_n, c_{n+1}\}$,

$$\begin{aligned} & P\left(\sum_{k=1}^{n+1} c_k X_k > x\right) \\ &\leq \alpha_n \sum_{k=1}^n \left(\left(1 + \frac{\varepsilon}{2}\right) P(c_k X_k > x) + \frac{\varepsilon}{2} P(c_{n+1} X_{n+1} > x)\right) + P(c_{n+1} X_{n+1} > x) \\ &\leq (1 + \varepsilon) \alpha_n \sum_{k=1}^n P(c_k X_k > x) + P(c_{n+1} X_{n+1} > x) \\ &\leq ((1 + \varepsilon) \alpha_n + 1) \sum_{k=1}^{n+1} P(c_k X_k > x). \end{aligned}$$

This proves that

$$\sup_{\underline{c}_{n+1} \in [a, b]^{n+1}} \sup_{x > A} \frac{P\left(\sum_{k=1}^{n+1} c_k X_k > x\right)}{\sum_{k=1}^{n+1} P(c_k X_k > x)} \leq (1 + \varepsilon)\alpha_n + 1.$$

When $x \leq A$, it holds uniformly for $\underline{c}_{n+1} \in [a, b]^{n+1}$ that

$$\frac{P\left(\sum_{k=1}^{n+1} c_k X_k > x\right)}{\sum_{k=1}^{n+1} P(c_k X_k > x)} \leq \frac{1}{P(aX_1 > A)}.$$

Therefore,

$$\begin{aligned} \alpha_{n+1} &= \left(\sup_{\underline{c}_{n+1} \in [a, b]^{n+1}} \sup_{x > A} + \sup_{\underline{c}_{n+1} \in [a, b]^{n+1}} \sup_{0 \leq x \leq A} \right) \frac{P\left(\sum_{k=1}^{n+1} c_k X_k > x\right)}{\sum_{k=1}^{n+1} P(c_k X_k > x)} \\ &\leq (1 + \varepsilon)\alpha_n + 1 + \frac{1}{P(aX_1 > A)}. \end{aligned}$$

By the recursive inequality

$$\alpha_{n+1} \leq (1 + \varepsilon)\alpha_n + 1 + \frac{1}{P(aX_1 > A)},$$

we can deduce the Kesten-type upper bound (2.2) with a suitably chosen constant C_ε . ■

2.2 Proof of Theorem 1.2

We can assume that θ_1 is not degenerate at 0 because otherwise we may simply discard the trivial term $\theta_1 X_1$. Then we can find a small constant $\delta > 0$ such that $P(\theta_1 > \delta) > 0$. For an arbitrary index set $\mathcal{I} \subset \{1, \dots, n\}$, denote

$$\Delta_{\mathcal{I}}(\delta) = \{\omega : \theta_i(\omega) \leq \delta \text{ whenever } i \in \mathcal{I} \text{ while } \theta_j(\omega) > \delta \text{ whenever } j \notin \mathcal{I}\}.$$

These sets $\Delta_{\mathcal{I}}(\delta)$ indexed by \mathcal{I} are disjoint and form a partition of the whole probability space:

$$\Omega = \bigcup_{\mathcal{I} \subset \{1, \dots, n\}} \Delta_{\mathcal{I}}(\delta).$$

We derive

$$\begin{aligned} &P\left(\sum_{k=1}^n \theta_k X_k > x\right) \\ &\leq P\left(\sum_{k=1}^n \theta_k X_k^+ > x\right) \\ &= P\left(\sum_{k=1}^n \theta_k X_k^+ > x, \theta_1 > \delta, \dots, \theta_n > \delta\right) + \sum_{\mathcal{I} \subset \{1, \dots, n\}, \mathcal{I} \neq \emptyset} P\left(\sum_{k=1}^n \theta_k X_k^+ > x, \Delta_{\mathcal{I}}(\delta)\right) \end{aligned}$$

$$= I_1 + I_2. \quad (2.3)$$

Note that all random weights in I_1 are bounded away from both 0 and ∞ . Thus by Lemma 2.2, for arbitrarily given $\varepsilon > 0$, we arbitrarily choose some $\epsilon \in (0, \varepsilon)$ and infer that there exists a constant $C_{\epsilon, \delta} > 0$ such that

$$\begin{aligned} I_1 &\leq C_{\epsilon, \delta} (1 + \epsilon)^n \sum_{k=1}^n P(\theta_k X_k^+ > x, \theta_1 > \delta, \dots, \theta_n > \delta) \\ &\leq C_{\epsilon, \delta} (1 + \epsilon)^n \sum_{k=1}^n P(\theta_k X_k > x). \end{aligned} \quad (2.4)$$

We deal with I_2 as

$$I_2 \leq \sum_{\mathcal{I}: \mathcal{I} \subset \{1, \dots, n\}, \mathcal{I} \neq \emptyset} P(\Delta_{\mathcal{I}}(\delta)) P\left(\delta \sum_{k \in \mathcal{I}} X_k^+ + \sum_{k \notin \mathcal{I}} \theta_k X_k^+ > x \mid \Delta_{\mathcal{I}}(\delta)\right).$$

The sum in each tail probability above can be regarded as a randomly weighted sum with random weights bounded away from both 0 and ∞ . Thus, by Lemma 2.2 again, we have

$$\begin{aligned} I_2 &\leq \sum_{\mathcal{I}: \mathcal{I} \subset \{1, \dots, n\}, \mathcal{I} \neq \emptyset} P(\Delta_{\mathcal{I}}(\delta)) C_{\epsilon, \delta} (1 + \epsilon)^n \left(\sum_{k \in \mathcal{I}} P(\delta X_k > x) + \sum_{k \notin \mathcal{I}} P(\theta_k X_k > x \mid \Delta_{\mathcal{I}}(\delta)) \right) \\ &= C_{\epsilon, \delta} (1 + \epsilon)^n \sum_{\mathcal{I}: \mathcal{I} \subset \{1, \dots, n\}, \mathcal{I} \neq \emptyset} \left(P(\Delta_{\mathcal{I}}(\delta)) \sum_{k \in \mathcal{I}} P(\delta X_k > x) + \sum_{k \notin \mathcal{I}} P(\theta_k X_k > x, \Delta_{\mathcal{I}}(\delta)) \right) \\ &= C_{\epsilon, \delta} (1 + \epsilon)^n (I_{21} + I_{22}). \end{aligned} \quad (2.5)$$

By changing the order of the two sums in I_{21} , we obtain

$$\begin{aligned} I_{21} &= \sum_{\mathcal{I}: \mathcal{I} \subset \{1, \dots, n\}, \mathcal{I} \neq \emptyset} P(\Delta_{\mathcal{I}}(\delta)) \sum_{k \in \mathcal{I}} P(\delta X_k > x) \\ &= \sum_{k=1}^n P(\delta X_k > x) \sum_{\mathcal{I}: k \in \mathcal{I} \subset \{1, \dots, n\}} P(\Delta_{\mathcal{I}}(\delta)) \\ &\leq \sum_{k=1}^n P(\delta X_k > x) \\ &= n \frac{P(\delta X_1 > x, \theta_1 > \delta)}{P(\theta_1 > \delta)} \\ &\leq n \frac{P(\theta_1 X_1 > x)}{P(\theta_1 > \delta)} \\ &\leq \frac{n}{P(\theta_1 > \delta)} \sum_{k=1}^n P(\theta_k X_k > x), \end{aligned} \quad (2.6)$$

where the third step is due to $\sum_{\mathcal{I}:k \in \mathcal{I} \subset \{1, \dots, n\}} P(\Delta_{\mathcal{I}}(\delta)) \leq P(\Omega) = 1$ since the sets $\Delta_{\mathcal{I}}(\delta)$ for $\mathcal{I} \subset \{1, \dots, n\}$ are disjoint. Similarly,

$$\begin{aligned}
I_{22} &= \sum_{\mathcal{I}: \mathcal{I} \subset \{1, \dots, n\}, \mathcal{I} \neq \emptyset} \sum_{k \notin \mathcal{I}} P(\theta_k X_k > x, \Delta_{\mathcal{I}}(\delta)) \\
&= \sum_{k=1}^n P(\theta_k X_k > x, \theta_k > \delta, \theta_i \leq \delta \text{ for some } i = 1, \dots, n) \\
&\leq \sum_{k=1}^n P(\theta_k X_k > x). \tag{2.7}
\end{aligned}$$

A simple combination of (2.3)–(2.7) gives

$$P\left(\sum_{k=1}^n \theta_k X_k > x\right) \leq C_{\epsilon, \delta} (1 + \epsilon)^n \left(\frac{n}{P(\theta_1 > \delta)} + 2\right) \sum_{k=1}^n P(\theta_k X_k > x).$$

Since $\epsilon \in (0, \varepsilon)$, it is easy to see that there is some absolute constant C_ε large enough such that, uniformly for all $n \in \mathbb{N}$,

$$C_{\epsilon, \delta} (1 + \epsilon)^n \left(\frac{n}{P(\theta_1 > \delta)} + 2\right) \leq C_\varepsilon (1 + \varepsilon)^n.$$

The desired inequality (1.4) follows.

3 An application to risk theory

Consider the renewal risk model in which claims of i.i.d. random sizes X_1, X_2, \dots successively arrive at renewal epochs $0 < \tau_1 < \tau_2 < \dots$, so that the number of claims up to time $t \geq 0$, namely,

$$N_t = \sup\{n \in \mathbb{N} : \tau_n \leq t\}, \tag{3.1}$$

is an ordinary renewal counting process. Assume that the two sequences $\{X_1, X_2, \dots\}$ and $\{\tau_1, \tau_2, \dots\}$ are mutually independent, and denote by X a generic random variable of $\{X_1, X_2, \dots\}$.

As usual, for $0 \leq u \leq T < \infty$, denote by $p(u, T)$ the price at time u of a zero-coupon bond paying \$1 at maturity date T . Assuming the absence of arbitrage of the bond market, we have

$$p(u, T) = E^Q \left[e^{-\int_u^T r_s ds} \middle| \mathcal{F}_u \right],$$

where Q is a risk-neutral pricing measure under which the expectation is taken, $\{r_s, s \geq 0\}$ is the underlying risk-free interest rate assumed to be nonnegative, and $\{\mathcal{F}_s, s \geq 0\}$ is the corresponding filtration. The following properties become obvious:

- $p(u, u) = 1$ for all $u \geq 0$;

- $0 < p(u, T) \leq 1$ for all $0 \leq u \leq T < \infty$;
- $p(u, T)$ is non-increasing in $T \geq u$.

In the case of a constant force of interest $r > 0$, we simply have $p(u, T) = e^{-r(T-u)}$, but in this application we allow the price function $p(u, T)$ to be completely general. For more details of zero-coupon bonds, see Chapter 1 of Cairns (2004) or Chapter 22 of Björk (2009).

Thus, the aggregate present value of claim amounts up to time t is

$$S_t = \sum_{k=1}^{N_t} p(0, \tau_k) X_k, \quad t \geq 0, \quad (3.2)$$

where in case $N_t = 0$ the sum is understood as zero. The idea of using the bond price function to discount future values is commonly used in the finance literature; see e.g. Section 2 of Liang and Zariphopoulou (2017).

The following result gives a precise asymptotic formula for the tail probability of S_t , which is consistent with, but neither implies nor is implied by, Theorem 2.1 of Li et al. (2010). The assumption $P(\tau_1 \leq t) > 0$ is merely to avoid triviality; otherwise, $E[N_t] = 0$.

Theorem 3.1 *Consider the aggregate present value (3.2) under the renewal framework described above. If $F \in \mathcal{S}$, then for any $t > 0$ such that $P(\tau_1 \leq t) > 0$, we have*

$$P(S_t > x) \sim \int_0^t \bar{F}\left(\frac{x}{p(0, s)}\right) dE[N_s].$$

Proof. For an arbitrarily fixed positive integer M , we divide the tail probability $P(S_t > x)$ into two parts as

$$P(S_t > x) = \left(\sum_{n=1}^M + \sum_{n=M+1}^{\infty} \right) P\left(\sum_{k=1}^n p(0, \tau_k) X_k > x, N_t = n \right) = J_1 + J_2. \quad (3.3)$$

By the independence between $\{X_1, X_2, \dots\}$ and $\{\tau_1, \tau_2, \dots\}$, we can apply Theorem 1.1 to obtain

$$\begin{aligned} J_1 &\sim \sum_{n=1}^M \sum_{k=1}^n P(p(0, \tau_k) X_k > x, N_t = n) \\ &= \left(\sum_{n=1}^{\infty} \sum_{k=1}^n - \sum_{n=M+1}^{\infty} \sum_{k=1}^n \right) P(p(0, \tau_k) X_k > x, N_t = n) \\ &= J_{11} - J_{12}. \end{aligned} \quad (3.4)$$

By changing the order of the two sums in J_{11} , we have

$$J_{11} = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} P(p(0, \tau_k) X_k > x, N_t = n)$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} P(p(0, \tau_k)X_k > x, N_t \geq k) \\
&= \sum_{k=1}^{\infty} P(p(0, \tau_k)X_k > x, \tau_k \leq t) \\
&= \sum_{k=1}^{\infty} \int_0^t \bar{F}\left(\frac{x}{p(0, s)}\right) P(\tau_k \in ds) \\
&= \int_0^t \bar{F}\left(\frac{x}{p(0, s)}\right) dE[N_s], \tag{3.5}
\end{aligned}$$

where the last step is due to $E[N_s] = \sum_{k=1}^{\infty} P(\tau_k \leq s)$ for $s \geq 0$. For $x > 0$, by the monotonicity of the zero-coupon bond price $p(0, \cdot)$, we have

$$\begin{aligned}
J_{12} &= \sum_{n=M+1}^{\infty} \sum_{k=1}^n P(p(0, \tau_k)X_k > x, N_t = n) \\
&\leq \sum_{n=M+1}^{\infty} \sum_{k=1}^n P(p(0, \tau_1)X_k > x, N_t = n) \\
&= \sum_{n=M+1}^{\infty} \sum_{k=1}^n \int_0^t P(p(0, s)X_k > x) P(N_{t-s} = n-1) P(\tau_1 \in ds) \\
&= \int_0^t \bar{F}\left(\frac{x}{p(0, s)}\right) \left(\sum_{n=M+1}^{\infty} n P(N_{t-s} = n-1)\right) P(\tau_1 \in ds) \\
&= \int_0^t \bar{F}\left(\frac{x}{p(0, s)}\right) (E(N_{t-s} + 1) 1_{(N_{t-s} \geq M)}) P(\tau_1 \in ds) \\
&\leq E(N_t + 1) 1_{(N_t \geq M)} \int_0^t \bar{F}\left(\frac{x}{p(0, s)}\right) dE[N_s].
\end{aligned}$$

Since $EN_t < \infty$, for arbitrarily fixed small $\delta > 0$ we can find some $M \in \mathbb{N}$ large enough such that

$$J_{12} \leq \delta \int_0^t \bar{F}\left(\frac{x}{p(0, s)}\right) dE[N_s]. \tag{3.6}$$

Next we deal with J_2 in (3.3). For arbitrarily fixed small $\varepsilon > 0$, by the independence between $\{X_1, X_2, \dots\}$ and $\{\tau_1, \tau_2, \dots\}$, we apply Theorem 1.2 to obtain

$$J_2 \leq C_\varepsilon \sum_{n=M+1}^{\infty} (1 + \varepsilon)^n \sum_{k=1}^n P(p(0, \tau_k)X_k > x, N_t = n).$$

Then following the treatment on J_{12} above, we derive

$$J_2 \leq C_\varepsilon \sum_{n=M+1}^{\infty} (1 + \varepsilon)^n \sum_{k=1}^n P(p(0, \tau_1)X_k > x, N_t = n)$$

$$\begin{aligned}
&= C_\varepsilon \sum_{n=M+1}^{\infty} (1+\varepsilon)^n \sum_{k=1}^n \int_0^t P(p(0,s)X_k > x) P(N_{t-s} = n-1) P(\tau_1 \in ds) \\
&= C_\varepsilon \int_0^t \bar{F}\left(\frac{x}{p(0,s)}\right) \left(\sum_{n=M+1}^{\infty} n(1+\varepsilon)^n P(N_{t-s} = n-1) \right) P(\tau_1 \in ds) \\
&= C_\varepsilon \int_0^t \bar{F}\left(\frac{x}{p(0,s)}\right) (E(N_{t-s} + 1) (1+\varepsilon)^{N_{t-s}+1} 1_{(N_{t-s} \geq M)}) P(\tau_1 \in ds) \\
&\leq C_\varepsilon (E(N_t + 1) (1+\varepsilon)^{N_t+1} 1_{(N_t \geq M)}) \int_0^t \bar{F}\left(\frac{x}{p(0,s)}\right) dE[N_s].
\end{aligned}$$

By Theorem 1 of Kočetova et al. (2009), there is always some $b > 1$ such that $Eb^{N_t} < \infty$. Thus, for small $\varepsilon, \delta > 0$, we can find some $M \in \mathbb{N}$ large enough such that

$$J_2 \leq \delta \int_0^t \bar{F}\left(\frac{x}{p(0,s)}\right) dE[N_s]. \quad (3.7)$$

Finally, a simple combination of (3.3)–(3.7), by the arbitrariness of $\delta > 0$, gives the desired result. ■

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