

# Reducing risk by merging counter-monotonic risks

Ka Chun Cheung;<sup>\*†</sup> Jan Dhaene;<sup>‡</sup> Ambrose Lo;<sup>§</sup> Qihe Tang<sup>¶</sup>

## Abstract

In this article, we show that some important implications concerning comonotonic couples and corresponding convex order relations for their sums cannot be translated to counter-monotonicity in general. In a financial context, it amounts to saying that merging counter-monotonic positions does not necessarily reduce the overall level of risk. We propose a simple necessary and sufficient condition for such a merge to be effective. Natural interpretations and various characterizations of this condition are given. As applications, we develop cancellation laws for convex order and identify desirable structural properties of insurance indemnities that make an insurance contract universally marketable, in the sense that it is appealing to both the policyholder and the insurer.

*KEYWORDS:* comonotonicity; counter-monotonicity; convex order; Tail Value-at-Risk

## 1 Introduction

Consider a portfolio with random value  $X$  at the end of a given reference period. In order to have a better control of the risk involved, it is often desirable that an additional asset  $Z$  is added to the position  $X$  so that the overall risk is reduced in the sense that

$$\rho(X + Z) \leq \rho(X + \mathbb{E}(Z)),$$

where  $\rho$  is an appropriate risk measure. Commonly used risk measures include Value-at-Risk, Tail Value-at-Risk (TVaR), and other distortion risk measures. We refer to Denuit et al. (2005),

---

<sup>\*</sup>Corresponding author.

<sup>†</sup>Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong. E-mail: kccg@hku.hk

<sup>‡</sup>KU Leuven, Department of Accountancy, Finance and Insurance, Naamsestraat 69, B-3000 Leuven, Belgium. E-mail: Jan.Dhaene@kuleuven.be

<sup>§</sup>Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong. E-mail: amblo@hku.hk

<sup>¶</sup>Department of Statistics and Actuarial Science, The University of Iowa, 241 Schaeffer Hall, Iowa City, IA 52242, USA. E-mail: qihe-tang@uiowa.edu

Dowd and Blake (2006), Dhaene et al. (2006) and Föllmer and Schied (2011) for a modern and systematic treatment of the theory of (distortion) risk measures and their relevance to finance and insurance. It is well-known that many popular risk measures are consistent with convex order in that if  $\rho$  is such a risk measure, then

$$Y_1 \leq_{cx} Y_2 \implies \rho(Y_1) \leq \rho(Y_2).$$

Here  $Y_1 \leq_{cx} Y_2$  means that the random variable  $Y_1$  is smaller than the random variable  $Y_2$  in convex order in the sense that  $\mathbb{E}(Y_1) = \mathbb{E}(Y_2)$  and  $\mathbb{E}[(Y_1 - d)_+] \leq \mathbb{E}[(Y_2 - d)_+]$  for all real numbers  $d$ . It is known that  $Y_1 \leq_{cx} Y_2$  if and only if  $\mathbb{E}[f(Y_1)] \leq \mathbb{E}[f(Y_2)]$  for any convex function  $f$  for which the two expectations exist. From the definition of convex order, it follows that we may interpret the order relation  $Y_1 \leq_{cx} Y_2$  as that  $Y_1$  is less variable than  $Y_2$ . For more information about convex order, we refer to Denuit et al. (2005) and Shaked and Shanthikumar (2007).

If one wants to reduce the risk of a position by merging it with an additional asset, it is important to know what properties this additional asset should possess. The first property that comes to mind is that this additional asset and the original position should “move in different directions”. For instance, the risk of a stock position can be reduced by combining with a put option written on the stock (see Ahn et al. (1999) and Deelstra et al. (2007)). The intuitive reason behind this strategy is that such an instrument can offset the variability of the stock price at the end of the reference period. To formalize this idea mathematically, we recall the concepts of comonotonicity and counter-monotonicity, and we will also take this opportunity to introduce the notation and convention used throughout this paper.

All random variables discussed in this paper are defined on a common probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and are assumed to be integrable, i.e. they have finite expectations. By a standard extension argument (see, for instance, page 112 of Kallenberg (2002)), the underlying probability space can be taken to be atomless without loss of generality. A random vector  $(Y_1, \dots, Y_n)$  is *comonotonic* if there exist a random variable  $V$  and  $n$  non-decreasing functions  $g_1, \dots, g_n$  such that

$$(Y_1, \dots, Y_n) \stackrel{d}{=} (g_1(V), \dots, g_n(V)).$$

In particular, we may choose  $V$  to be any uniform(0, 1) random variable, and  $g_i$  to be  $F_{Y_i}^{-1}$ , the left-continuous inverse of the distribution function  $F_{Y_i}$ . Furthermore, by Corollary 6.11 of Kallenberg (2002), there exists a uniform(0, 1) random variable  $U$  such that an apparently stronger condition holds true:

$$(Y_1, \dots, Y_n) = (F_{Y_1}^{-1}(U), \dots, F_{Y_n}^{-1}(U)) \quad \text{almost surely.}$$

Equivalently,  $(Y_1, \dots, Y_n)$  is comonotonic if there is a null set  $N$  such that for any  $i, j \in \{1, \dots, n\}$ ,

$$(Y_i(\omega) - Y_i(\omega'))(Y_j(\omega) - Y_j(\omega')) \geq 0, \quad \omega, \omega' \in \Omega \setminus N.$$

A pair of random variables  $(X, Y)$  is said to be *counter-monotonic* if  $(X, -Y)$  is comonotonic.

We are now ready to formalize the idea of reducing risk by merging positions as mentioned above.

**Definition 1** A random variable  $Z$  is said to be a variance reducer for the random variable  $X$  if

$$\text{Var}(X + Z) \leq \text{Var}(X).$$

Moreover,  $Z$  is said to be a risk reducer for  $X$  if

$$X + Z \leq_{cx} X + \mathbb{E}(Z). \quad (1)$$

If, in addition,  $(X, Z)$  is counter-monotonic,  $Z$  is said to be a counter-monotonic risk reducer for  $X$ .

One of the main objectives of this paper is to identify conditions for  $Z$  to be a risk reducer for  $X$ , and to explain why counter-monotonicity is relevant. Using the stock-put option case as an example, we would like to build a firm mathematical framework to explain the risk reducing power of put options and to identify other risk reducing instruments as well.

Before investigating the notion of risk reducer, we recall the following well-known properties about comonotonic sums and convex order. For any integrable random variables  $X, Y$ , and  $Z$ , we have that

$$X \leq_{cx} Y, \quad (Y, Z) \text{ is comonotonic} \quad \implies \quad X + \mathbb{E}(Z) \leq_{cx} Y + Z \quad (2)$$

and

$$X \leq_{cx} Y, \quad (Y, Z) \text{ is comonotonic} \quad \implies \quad X + Z \leq_{cx} Y + Z. \quad (3)$$

Both (2) and (3) are special cases of a more general result (see Lemma 4 below), which asserts that if  $X_i \leq_{cx} Y_i$  for  $i = 1, 2$  and if  $(Y_1, Y_2)$  is comonotonic, then  $X_1 + X_2 \leq_{cx} Y_1 + Y_2$ .

Since bivariate comonotonicity and counter-monotonicity are two opposite extremal dependence structures, it seems natural to wonder whether analogous results of (2) and (3) hold for counter-monotonicity. More precisely, one conjectures that

$$X \leq_{cx} Y, \quad (X, Z) \text{ is counter-monotonic} \quad \implies \quad X + Z \leq_{cx} Y + \mathbb{E}(Z) \quad (4)$$

and

$$X \leq_{cx} Y, \quad (X, Z) \text{ is counter-monotonic} \quad \implies \quad X + Z \leq_{cx} Y + Z. \quad (5)$$

A possible interpretation of statement (4) is that if position  $X$  is less risky (variable) than position  $Y$ , and asset  $Z$  can offset price movements of  $X$ , then position  $X + Z$  is again less risky (variable) than  $Y$ . In particular, according to Definition 1, the implication (4) with  $X \equiv Y$  would mean that any asset that is counter-monotonic with  $X$  is a risk reducer of  $X$ . For statement (5), an interpretation would be that the order of riskiness is preserved on the portfolio level when an asset that is counter-monotonic with the less risky sub-portfolio is added to both sub-portfolios.

Unfortunately, neither conjecture (4) nor (5) holds true in general, as demonstrated by the following counter-example.

**Example 1** Let  $W$  be a standard normal random variable, and define

$$(X, Y, Z) := (W/2, W, -2W).$$

Then  $X \leq_{cx} Y$ , and  $(X, Z)$  is counter-monotonic. However,  $X + Z = -3W/2 \geq_{cx} W = Y$ , which contradicts (4). Furthermore, if we define

$$(X, Y, Z) := (W/2, W, -W)$$

instead, then  $X + Z = -W/2 \geq_{cx} 0 = Y + Z$ , which contradicts (5).

The reason why  $X + Z$  is even riskier than  $Y + Z$  or  $Y + \mathbb{E}(Z)$  in the example above is that the asset  $Z$  not only eliminates the original risk of  $X$  but also introduces additional risk. Indeed, we can easily see that the counter-monotonicity of  $(X, Z)$  does not necessarily imply that  $Z$  is a variance reducer for  $X$ . To this end, we notice that

$$\text{Var}(X + Z) \leq \text{Var}(X) \iff \text{Var}(Z) \leq -2\text{Cov}(X, Z).$$

The counter-monotonicity of  $(X, Z)$  only guarantees the positivity of  $-2\text{Cov}(X, Z)$ , but in case  $\text{Var}(Z)$  is larger than  $-2\text{Cov}(X, Z)$ ,  $Z$  will not be a variance reducer, let alone being a risk reducer of  $X$ .

From the counter-example and the discussion above, one sees that for a given position  $X$ , not every counter-monotonic asset  $Z$  is a risk reducer because of the possibility of over-compensation. The objectives of this paper are multifold. In Section 3, we propose a simple sufficient condition to identify risk reducers and give natural interpretations of this additional condition. We also show that this simple condition is useful in establishing new additivity and cancellation laws for convex order. Different characterizations of a risk reducer are presented in Section 4, where we prove that the sufficient condition is also necessary for conjecture (5) to hold true, and illustrate the functional relationship between a given random variable and its risk reducers. In Section 5, we apply our results to the study of the marketability of insurance indemnities. We identify desirable properties that an indemnity schedule should possess in order for the insurance contracts to be beneficial to both the policyholder and the insurer. In particular, we show that in order for an insurance indemnity to be mutually beneficial regardless of the risk preference and risk profile of the policyholder, it is necessary and sufficient for the indemnity schedule to be 1-Lipschitz.

We end this section by emphasizing that the risk-reducing concepts studied in this paper do not take into consideration the (no-arbitrage) market price and the preference of individual decision makers. Indeed, while a risk reducer can reduce the risk of a given position, an expected utility maximizer is not always better off by buying a risk reducer. To see this, assume that his utility function is given by some non-decreasing and concave function  $u$ . He will purchase an additional asset  $Z$  only if the following inequality is fulfilled:

$$\mathbb{E}[u(X + Z - \pi(Z))] \geq \mathbb{E}[u(X)], \tag{6}$$

where  $\pi(Z)$  is the market price of  $Z$ . The assumption that  $Z$  is a risk reducer for  $X$  does not necessarily imply that (6) is fulfilled. Furthermore, it may happen that inequality (6) is fulfilled even if  $Z$  is not a risk reducer for  $X$ . These possibilities are illustrated in the following example.

**Example 2** (i) Suppose that  $Z$  is a risk reducer for  $X$ . If  $\pi(Z) > \mathbb{E}(Z)$  and  $u(x) = x$  for all  $x$ , then (6) is violated. By modifying the utility function slightly (considering  $u(x) = (1-\varepsilon)x + \varepsilon\tilde{u}(x)$  for some strictly concave utility function  $\tilde{u}$  and positive  $\varepsilon$  sufficiently close to zero), it is easy to see that (6) is violated for some strictly concave utility function as well.

(ii) Let  $X$  be a positive integrable random variable with support  $[0, \infty)$ . Take  $Z = -2X$  and assume that  $\pi(Z) = \mathbb{E}(Z)$ . Then  $X + Z - \pi(Z) = -X + 2\mathbb{E}(X)$ , and its support equals  $(-\infty, 2\mathbb{E}(X)]$ . Since

$$\mathbb{E}[(X + Z - \pi(Z) - d)_+] = 0 < \mathbb{E}[(X - d)_+] \quad \text{for any } d > 2\mathbb{E}(X)$$

and

$$\mathbb{E}[(X - d)_-] = 0 < \mathbb{E}[(X + Z - \pi(Z) - d)_-] \quad \text{for any } d < 0,$$

it follows that  $X + Z - \pi(Z)$  and  $X$  are not comparable in convex order. Thus  $Z$  is not a risk reducer of  $X$ . Furthermore, we can construct a decreasing and strictly convex function  $f$  such that  $\mathbb{E}[f(X + Z - \pi(Z))] < \mathbb{E}[f(X)]$ . Hence, inequality (6) is satisfied by the increasing and strictly concave utility function  $u := -f$ .

In spite of the observations made in Example 2, we will show in Section 5 that the concept of risk reducer and the results to be established in Sections 3 and 4 have an interesting application in the context of designing insurance indemnity schedules, in which the decision maker is assumed to be an expected utility maximizer.

## 2 Preliminaries

In this section, we recall the definition and properties of Tail Value-at-Risk (TVaR), and the relationship between convex order, comonotonic sums, and counter-monotonic sums.

For a distribution function  $F$ , we use the notation  $F^{-1}$  to denote its left-continuous inverse:

$$F^{-1}(p) := \inf\{x \in \mathbb{R} \mid F(x) \geq p\}, \quad 0 < p < 1.$$

For a random variable  $Y$ , its TVaR at probability level  $p$  is defined as

$$\text{TVaR}_p(Y) := \frac{1}{1-p} \int_p^1 F_Y^{-1}(t) dt, \quad 0 < p < 1.$$

**Lemma 1** *The random variables  $X$  and  $Y$  are ordered in convex order sense, i.e.  $X \leq_{cx} Y$ , if and only if  $\mathbb{E}(X) = \mathbb{E}(Y)$  and  $\text{TVaR}_p(X) \leq \text{TVaR}_p(Y)$  for all  $0 < p < 1$ .*

**Lemma 2** *TVaR is subadditive, i.e.*

$$\text{TVaR}_p(X + Y) \leq \text{TVaR}_p(X) + \text{TVaR}_p(Y) \quad \text{for all } 0 < p < 1.$$

For proofs of these two lemmas and other properties of TVaR, we refer to Dhaene et al. (2006).

**Lemma 3**  *$(X, Y)$  is comonotonic if and only if*

$$\text{TVaR}_p(X + Y) = \text{TVaR}_p(X) + \text{TVaR}_p(Y) \quad \text{for all } 0 < p < 1.$$

The “only if” is a well-known result, see Dhaene et al. (2006). The proof of the “if” part can be found in Cheung (2010).

The following lemma summarizes some fundamental relationships between comonotonicity (counter-monotonicity) and convex order. For proofs and further discussion, we refer to Dhaene and Goovaerts (1996) and Dhaene et al. (2002).

**Lemma 4** *(i) If  $X_i \leq_{cx} Y_i$  for  $i = 1, 2$  and if  $(Y_1, Y_2)$  is comonotonic, then*

$$X_1 + X_2 \leq_{cx} Y_1 + Y_2.$$

*(ii) If  $(X_1, X_2)$  and  $(Y_1, Y_2)$  have the same marginal distributions and the former is counter-monotonic, then*

$$X_1 + X_2 \leq_{cx} Y_1 + Y_2.$$

### 3 A sufficient condition for being a risk reducer of a given random variable

We saw that conjectures (4) and (5) fail to hold true because the counter-monotonic asset  $Z$  may contain leftover and unnecessary risk even after offsetting the risk of the original position  $X$ . In this section, we suggest an extra condition to be imposed on a counter-monotonic asset so that over-compensation will not arise. To motivate our discussion, we return to the stock-put option example.

**Example 3** *Suppose that we are holding one share of stock. Denote by  $X$  the price of the stock at a fixed future time point. A common strategy to reduce the risk (variability) of this position is to purchase a European put option on this stock. After adding  $Z := (K - X)_+$  to  $X$ , the new position is given by  $X + Z = \max(X, K)$ . Obviously, it is less variable than  $X$  because the*

lower tail of the original position  $X$  is replaced by a constant. Another strategy is to short one share of the same stock. In this case,  $Z := -X$  and thus the new position  $X + Z$  is identically zero. On the other hand, it is generally believed that longing two European put options on the stock is not a good strategy to adopt, because in this case it is not clear whether the new position  $X + 2(K - X)_+$  is less variable than  $X$ .

While all of the three strategies (holding one put, shorting one unit of stock, and holding two puts) considered in Example 3 consist of adding counter-monotonic assets, our intuition suggests that the first two strategies are truly risk reducing, while this is not necessarily the case for the third one. There are two different heuristic considerations that allow us to distinguish the first two strategies from the third:

- (1) In the first two strategies (holding a put or shorting a unit of stock), the merged position  $X + Z$  preserves the nature of the original position  $X$  in that both of them are bullish on the stock price. In other words,  $(X, X + Z)$  is comonotonic, and hence the belief in how the markets will move remains unchanged after  $Z$  is added. If  $X$  performs well, so does the merged position  $X + Z$ . However, the latter position (after  $Z$  is added) is less variable because of the offsetting nature of  $Z$ . For the third strategy (holding two puts), the position  $X + Z$  is no longer comonotonic with  $X$ .
- (2) While  $Z$  and  $X$  move in opposite directions, the decrement (or increment) of  $Z$  is always less than the increment (or decrement) of  $X$  in the first two strategies. In this way, adding  $Z$  to  $X$  will transfer the counter-monotonic relation of  $(X, Z)$  to a counter-monotonic relation of the new portfolio  $X + Z$  with  $Z$ . However, in the third strategy, the new portfolio  $X + Z$  is no longer counter-monotonic with  $Z$ .

To sum up, in order for a counter-monotonic position  $Z$  to be effective in reducing the risk of  $X$ , the first consideration suggests that  $(X, X + Z)$  has to be comonotonic, while the second consideration suggests that the decrement (or increment) of  $Z$  is always less than the increment (or decrement) of  $X$ , which leads to the counter-monotonicity of  $(Z, X + Z)$ . In the next result, we formalize these two considerations and show that they are indeed equivalent when the original position  $X$  is continuous.

**Lemma 5** *Suppose that  $(X, Z)$  is counter-monotonic. Consider the following statements:*

- (a)  $(X, X + Z)$  is comonotonic;
- (b)  $(Z, X + Z)$  is counter-monotonic;
- (c) for some null set  $N$ ,

$$|Z(\omega) - Z(\omega')| \leq |X(\omega) - X(\omega')| \quad \text{for any } \omega, \omega' \in \Omega \setminus N.$$

Then (b)  $\Leftrightarrow$  (c)  $\Rightarrow$  (a). Furthermore, the three statements are equivalent if either  $F_X$  is continuous or  $Z = f(X)$  almost surely for some measurable function  $f$ .

*Proof:* Assume that (c) is true. Fix some  $\omega, \omega' \in \Omega \setminus N$  and assume that  $X(\omega') \leq X(\omega)$ . Then

$$(X(\omega) - X(\omega'))(X(\omega) + Z(\omega) - X(\omega') - Z(\omega')) \geq 0,$$

which means that  $(X, X + Z)$  is comonotonic. Therefore, (a) holds true. Similarly, fix some  $\omega, \omega' \in \Omega \setminus N$  and assume that  $Z(\omega') < Z(\omega)$ . Then (c), together with the counter-monotonicity of  $(X, Z)$ , implies that  $X(\omega') > X(\omega)$  and  $Z(\omega) - Z(\omega') \leq X(\omega') - X(\omega)$ . Then

$$(Z(\omega) - Z(\omega'))(X(\omega) + Z(\omega) - X(\omega') - Z(\omega')) \leq 0,$$

which means that  $(Z, X + Z)$  is counter-monotonic. Therefore, (b) holds true.

Now we assume that (b) holds true so that  $(-Z, X + Z)$  is comonotonic. Extending this vector by including the sum of  $-Z$  and  $X + Z$  as a new component leads to  $(-Z, X + Z, X)$ , which is still comonotonic. So there is a uniform(0, 1) random variable  $U$  such that

$$(X, X + Z, -Z) = (F_X^{-1}(U), F_{X+Z}^{-1}(U), F_{-Z}^{-1}(U))$$

almost surely. Denote by  $N$  the corresponding null set. Outside this null set, we have

$$F_X^{-1}(U) - F_{X+Z}^{-1}(U) = X - (X + Z) = -Z = F_{-Z}^{-1}(U).$$

Therefore,

$$F_X^{-1}(\cdot) - F_{-Z}^{-1}(\cdot) = F_{X+Z}^{-1}(\cdot)$$

almost everywhere (with respect to the Lebesgue measure) on  $(0, 1)$ . By the left-continuity of inverse distribution functions, the equality actually holds everywhere on  $(0, 1)$ . From this, we obtain that  $F_X^{-1}(\cdot) - F_{-Z}^{-1}(\cdot)$  is a non-decreasing function.

For any  $\omega, \omega'$  outside  $N$ , assume without loss of generality that  $0 < U(\omega') < U(\omega) < 1$ . Then

$$F_X^{-1}(U(\omega')) - F_{-Z}^{-1}(U(\omega')) \leq F_X^{-1}(U(\omega)) - F_{-Z}^{-1}(U(\omega))$$

and hence

$$|F_{-Z}^{-1}(U(\omega)) - F_{-Z}^{-1}(U(\omega'))| \leq |F_X^{-1}(U(\omega)) - F_X^{-1}(U(\omega'))|,$$

from which (c) follows.

Next, we assume that (a) holds true and that  $F_X$  is continuous. There exist null sets  $N_1, N_2, N_3$  and a uniform(0, 1) random variable  $U$  such that

$$\begin{aligned} (X(\omega) - X(\omega'))(X(\omega) + Z(\omega) - X(\omega') - Z(\omega')) &\geq 0 \quad \text{for any } \omega, \omega' \in \Omega \setminus N_1, \\ (X(\omega) - X(\omega'))(Z(\omega') - Z(\omega)) &\geq 0 \quad \text{for any } \omega, \omega' \in \Omega \setminus N_2, \end{aligned}$$

and  $X(\omega) = F_X^{-1}(U(\omega))$  for any  $\omega \in \Omega \setminus N_3$  with  $U(\omega) \neq U(\omega')$  whenever  $\omega, \omega' \in \Omega \setminus N_3$ . As  $F_X$  is continuous,  $F_X^{-1}$  is strictly increasing and hence  $X(\omega) \neq X(\omega')$  whenever  $\omega, \omega' \in \Omega \setminus N_3$ . It then follows that for any  $\omega, \omega' \in \Omega \setminus (N_1 \cup N_2 \cup N_3)$ ,

$$\begin{aligned} & (Z(\omega') - Z(\omega))(X(\omega) + Z(\omega) - X(\omega') - Z(\omega')) \\ &= (X(\omega) - X(\omega'))(X(\omega) + Z(\omega) - X(\omega') - Z(\omega')) \frac{Z(\omega') - Z(\omega)}{X(\omega) - X(\omega')} \geq 0. \end{aligned}$$

This shows that  $(Z, X + Z)$  is counter-monotonic and hence (b) holds true.

Finally, we assume that (a) holds true and  $Z = f(X)$  on  $\Omega \setminus N_4$  for some measurable function  $f$  and null set  $N_4$ . Define  $N_1$  and  $N_2$  as in the previous paragraph. For any  $\omega, \omega' \in \Omega \setminus (N_1 \cup N_2 \cup N_4)$  with  $Z(\omega') > Z(\omega)$ , we have  $X(\omega') < X(\omega)$  by the definition of  $N_2$  and  $N_4$ , and hence  $X(\omega) + Z(\omega) - X(\omega') - Z(\omega') \geq 0$  by the definition of  $N_1$ . Combining these yields

$$(Z(\omega') - Z(\omega))(X(\omega) + Z(\omega) - X(\omega') - Z(\omega')) \geq 0,$$

and so (b) holds true. □

As a remark, we note from the above proof that statements (b) and (c) in Lemma 5 are also equivalent to the condition that the function  $F_X^{-1}(\cdot) - F_{-Z}^{-1}(\cdot)$  is non-decreasing. In the theory of stochastic orders, such a condition can be expressed in terms of the *dispersive order* as  $-Z \leq_{disp} X$ .

The following example illustrates why in general statement (a) is not sufficient for (b) and (c) in Lemma 5.

**Example 4** *Let  $U$  be a uniform(0, 1) random variable, and define*

$$X := 1_{\{U > 1/2\}}, \quad Z := (U - 1/2)1_{\{U \leq 1/2\}} - 1_{\{U > 1/2\}}.$$

*Then  $X + Z = (U - 1/2)1_{\{U \leq 1/2\}}$ , and it is clear that  $(X, Z)$  is counter-monotonic and that  $(X, X + Z)$  is comonotonic. However,  $(Z, X + Z)$  is not counter-monotonic.*

This counter-example indicates that comonotonicity is not transitive in general: for any random variables  $V_i$ ,  $i = 1, 2, 3$ ,

$$(V_1, V_2) \text{ is comonotonic, } (V_2, V_3) \text{ is comonotonic} \not\Rightarrow (V_1, V_3) \text{ is comonotonic.}$$

Lemma 5 provides two conditions that can guarantee the transitivity of comonotonicity: one is that the “linking” random variable  $V_2$  has no point mass, and the second is that either  $V_1$  or  $V_3$  is expressible as a measurable function of the “linking” random variable  $V_2$ .

The next result shows that the counter-monotonicity of  $(Z, X + Z)$ , which is condition (b) of Lemma 5, implies that  $(X, Z)$  is counter-monotonic as well.

**Lemma 6**  $(Z, X + Z)$  is counter-monotonic if and only if  $(-Z, X + Z, X)$  is comonotonic.

*Proof:* Suppose that  $(Z, X + Z)$  is counter-monotonic, or equivalently,  $(-Z, X + Z)$  is comonotonic. Extending this vector by including the sum of  $-Z$  and  $X + Z$  yields the comonotonicity of  $(-Z, X + Z, X)$ . The converse is trivial.  $\square$

From now on, we denote by  $H(X)$  the collection of all random variables  $Z$  which are counter-monotonic with the combined position  $X + Z$ :

$$H(X) = \{Z \mid (Z, X + Z) \text{ is counter-monotonic}\}.$$

As the main result of this section, we show in Theorem 1 below that both conjectures (4) and (5) hold true if  $Z \in H(X)$ . In other words, any  $Z \in H(X)$  is a risk reducer of  $X$  and possesses the convex order-preserving property.

**Theorem 1** *If  $X \leq_{cx} Y$  and  $Z \in H(X)$ , then:*

- (i)  $X + Z \leq_{cx} Y + \mathbb{E}(Z)$ . In particular, any element in  $H(X)$  is a counter-monotonic risk reducer of  $X$ .
- (ii)  $X + Z \leq_{cx} Y + Z$ .

*Proof:* Fix any  $p \in (0, 1)$ . Since  $Z \in H(X)$ ,  $(-Z, X + Z)$  is comonotonic. It then follows from Lemma 3 that

$$\text{TVaR}_p(X + Z) + \text{TVaR}_p(-Z) = \text{TVaR}_p(X).$$

Since  $X \leq_{cx} Y$  and  $-\mathbb{E}(Z) \leq_{cx} -Z$ , Lemma 1 implies that

$$\text{TVaR}_p(X) \leq \text{TVaR}_p(Y)$$

and

$$-\mathbb{E}(Z) = \text{TVaR}_p(-\mathbb{E}(Z)) \leq \text{TVaR}_p(-Z).$$

Combining all these, we have

$$\text{TVaR}_p(X + Z) = \text{TVaR}_p(X) - \text{TVaR}_p(-Z) \leq \text{TVaR}_p(Y) + \mathbb{E}(Z) = \text{TVaR}_p(Y + \mathbb{E}(Z)).$$

By Lemma 1 again,  $X + Z \leq_{cx} Y + \mathbb{E}(Z)$ , which proves (i).

Similarly, for any  $0 < p < 1$ , we also have

$$\text{TVaR}_p(X + Z) = \text{TVaR}_p(X) - \text{TVaR}_p(-Z) \leq \text{TVaR}_p(Y) - \text{TVaR}_p(-Z) \leq \text{TVaR}_p(Y + Z),$$

where the last inequality follows from the subadditivity of TVaR. Hence,  $X + Z \leq_{cx} Y + Z$  by Lemma 1.  $\square$

Theorem 1 provides new ‘‘additivity laws’’ for convex order. As a by-product, we deduce the following ‘‘cancellation laws’’.

**Proposition 1** *If  $X + Z \leq_{cx} Y + Z$  and  $Z \in H(Y)$ , then  $X \leq_{cx} Y$ .*

*Proof:* By definition,  $Z \in H(Y)$  means that  $(-Z, Y + Z)$  is comonotonic. Now it follows from (3) that we can add  $-Z$  to both sides of  $X + Z \leq_{cx} Y + Z$  to yield the desired result.  $\square$

Notice that Proposition 1 and Theorem 1 (ii) can be put together in a more symmetric form:

$$\begin{array}{ccc} X & \leq_{cx} & Y \\ Z \in H(X) \downarrow & & \uparrow Z \in H(Y) \\ X + Z & \leq_{cx} & Y + Z. \end{array}$$

As far as we are aware, the two cancellation laws in the following proposition are new. They can be regarded as the converse results of (2) and (3).

**Proposition 2** (i) *If  $X + Z \leq_{cx} Y + Z$  and  $(Z, X)$  is comonotonic, then  $X \leq_{cx} Y$ .*

(ii) *If  $X + Z \leq_{cx} Y + \mathbb{E}(Z)$  and  $(Z, X)$  is comonotonic, then  $X \leq_{cx} Y$ .*

*Proof:* Clearly, the comonotonicity of  $(Z, X)$  is equivalent to the counter-monotonicity of  $(-Z, X + Z - Z)$  and hence  $-Z \in H(X + Z)$ . Now (i) follows from Theorem 1 (ii) by adding  $-Z$  to both sides of  $X + Z \leq_{cx} Y + Z$ .

Similarly, (ii) follows from Theorem 1 (i) by adding  $-Z$  and  $-\mathbb{E}(Z)$  to the left and right hand sides of  $X + Z \leq_{cx} Y + \mathbb{E}(Z)$ , respectively.  $\square$

Statement (i) of Proposition 2 and property (3) can be put together in a more symmetric form:

$$\begin{array}{ccc} X & \leq_{cx} & Y \\ (Z, Y) \text{ is comonotonic} \downarrow & & \uparrow (Z, X) \text{ is comonotonic} \\ X + Z & \leq_{cx} & Y + Z. \end{array}$$

## 4 Characterizations of risk reducers

In this section, we reformulate the defining condition of  $H(X)$  into some interesting equivalent statements and further explain the properties and structure of risk reducers.

## 4.1 Convex-order preserving property

We first show that the convex order-preserving property in Theorem 1 is not only necessary but also sufficient for being a risk reducer of  $X$ .

**Theorem 2** *If  $X + Z \leq_{cx} Y + Z$  for any random variable  $Y$  satisfying  $X \leq_{cx} Y$ , then  $Z \in H(X)$ .*

*Proof:* By Lemma 5, we want to show (i)  $(X, Z)$  is counter-monotonic, and (ii) the function  $\Phi := F_X^{-1} - F_{-Z}^{-1}$  is non-decreasing on  $(0, 1)$  (see the remark after Lemma 5).

To show (i), let  $X^*$  be a random variable which is identically distributed as  $X$  and is counter-monotonic with  $Z$ . By Lemma 4 (ii), we have  $X^* + Z \leq_{cx} X + Z$ . By hypothesis, we also have  $X + Z \leq_{cx} X^* + Z$ , so  $X + Z$  has the same distribution as the corresponding counter-monotonic sum  $X^* + Z$ . Since a counter-monotonic random vector is characterized by the minimality of the sum of its components with respect to the convex order (see Theorem 1 of Cheung and Lo (2013)), we conclude that  $(X, Z)$  is counter-monotonic.

We break down the proof of (ii) into three steps.

*Step 1 (Constructing an appropriate random variable  $Y^*$ ):* Assume by contradiction that  $\Phi$  is not non-decreasing on  $(0, 1)$ . Then there exist  $\alpha, \beta \in (0, 1)$  with  $\alpha < \beta$  such that  $\Phi(\alpha) > \Phi(\beta)$ . In other words,  $F_{-Z}^{-1}(\beta) - F_{-Z}^{-1}(\alpha) > F_X^{-1}(\beta) - F_X^{-1}(\alpha)$ . Since  $F_{-Z+c}^{-1} = F_{-Z}^{-1} + c$  for any constant  $c$ , we may assume, without loss of generality, that  $F_{-Z}^{-1}$  (or equivalently the underlying random variable  $-Z$ ) has been shifted either upward or downward by a suitable amount such that

$$F_{-Z}^{-1}(\beta) > F_X^{-1}(\beta) \quad \text{and} \quad F_X^{-1}(\alpha) > F_{-Z}^{-1}(\alpha).$$

Define a function  $\tilde{Y}$  on  $(0, 1)$  by

$$\tilde{Y}(t) := \begin{cases} F_{-Z}^{-1}(t), & \text{if } t \in [\alpha - \epsilon_1, \alpha] \cup [\beta - \epsilon_2, \beta], \\ F_X^{-1}(t), & \text{otherwise,} \end{cases}$$

where  $\epsilon_1$  and  $\epsilon_2$  are positive numbers chosen in such a way that

$$\begin{aligned} F_X^{-1}(t) &> F_{-Z}^{-1}(t) & \text{for } t \in J_1 := [\alpha - \epsilon_1, \alpha], \\ F_{-Z}^{-1}(t) &> F_X^{-1}(t) & \text{for } t \in J_2 := [\beta - \epsilon_2, \beta], \\ \mathbb{E}[Y^*] &= \mathbb{E}[X], \end{aligned}$$

in which  $Y^*$  is the random variable defined as  $\tilde{Y}(U)$ , where  $U$  is some uniform $(0, 1)$  random variable. The existence of such  $\epsilon_1$  and  $\epsilon_2$  is guaranteed by the left continuity of  $F_{-Z}^{-1}$  and  $F_X^{-1}$ . Note that  $Y^*$  is obtained from  $F_X^{-1}(U)$  by decrementing its values when  $U \in J_1$  and incrementing its values when  $U \in J_2$ , while keeping the expected value unchanged (see Figure 1).

*Step 2 (Showing  $X \leq_{cx} Y^*$ ):* For  $t \in I_1 := (F_X^{-1}(\alpha), F_X^{-1}(\beta - \varepsilon_2))$ , we have that

$$\{u \in (0, 1) \mid F_X^{-1}(u) \leq t\} = \{u \in (0, 1) \mid \tilde{Y}(u) \leq t\},$$

while for  $t \in I_2 := (-\infty, F_X^{-1}(\alpha)]$ , it follows from our construction that

$$\{u \in (0, 1) \mid F_X^{-1}(u) \leq t\} \subset \{u \in (0, 1) \mid \tilde{Y}(u) \leq t\},$$

and for  $t \in I_3 := [F_X^{-1}(\beta - \varepsilon_2), \infty)$ ,

$$\{u \in (0, 1) \mid F_X^{-1}(u) \leq t\} \supset \{u \in (0, 1) \mid \tilde{Y}(u) \leq t\}.$$

Combining all these observations, we have

$$F_{Y^*} \begin{cases} \geq F_X, & \text{on } I_2, \\ = F_X, & \text{on } I_1, \\ \leq F_X, & \text{on } I_3. \end{cases}$$

So  $F_X$  up-crosses  $F_{Y^*}$  once, and since  $\mathbb{E}[X] = \mathbb{E}[Y^*]$  by construction, we conclude that  $X \leq_{cx} Y^*$  (see page 23 of Müller and Stoyan (2002)).

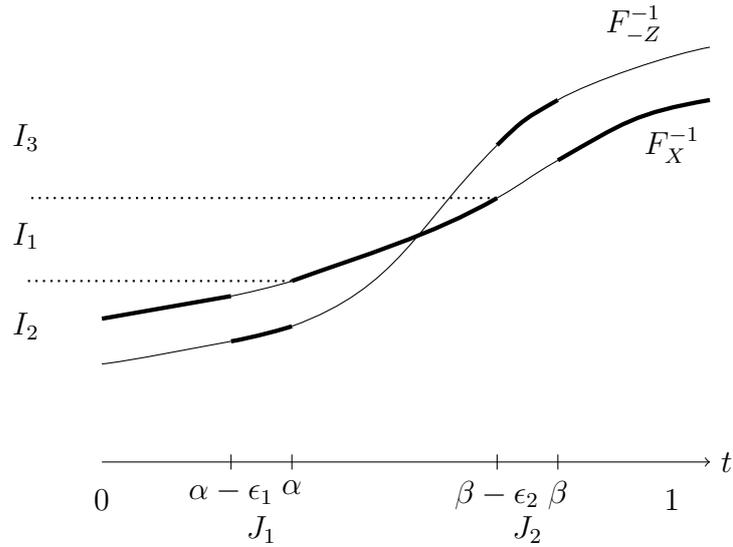


Figure 1: The construction of the functions and sets used in Theorem 2. The line in bold represents the function  $\tilde{Y}$ .

*Step 3 (Showing  $X + Z \not\leq_{cx} Y^* + Z$ ):* Note that  $X + Z$  and  $Y^* + Z$  differ only when  $U \in J_1 \cup J_2$ , in which case  $Y^* + Z \equiv 0$  but  $X + Z$  never vanishes. Therefore, we have

$$\mathbb{E}|X + Z| - \mathbb{E}|Y^* + Z| = \int_{J_1 \cup J_2} |F_X^{-1}(t) - F_{-Z}^{-1}(t)| dt > 0,$$

which implies that  $X + Z \not\leq_{cx} Y^* + Z$ .

In conclusion, we have constructed a random variable  $Y^*$  such that  $X \leq_{cx} Y^*$  but  $X + Z \not\leq_{cx} Y^* + Z$ , contradicting our hypothesis. Hence,  $\Phi$  must be non-decreasing on  $(0, 1)$ . This completes the proof.  $\square$

The condition  $X + Z \leq_{cx} Y + Z$  for all  $Y$  satisfying  $X \leq_{cx} Y$  can be regarded as a test on the risk-reducing capability of  $Z$ . Theorem 2 asserts that as long as  $Z$  preserves the order of variability with respect to any position riskier than  $X$ , then  $Z \in H(X)$ . Combining Theorems 1 and 2, we have established the following characterization of the collection  $H(X)$ .

**Corollary 1**  $Z \in H(X)$  if and only if  $X + Z \leq_{cx} Y + Z$  for all  $Y$  satisfying  $X \leq_{cx} Y$ .

## 4.2 The functional form of a risk reducer

The next proposition and its corollary describe a simple functional characterization of the elements of  $H(X)$ , which will be of use in Section 5. We note that results in the weaker form of a distributional representation using the language of dispersive order can be found in Müller and Stoyan (2002) and Shaked and Shanthikumar (2007); see also Bickel and Lehmann (1979) and Landsberger and Meilijson (1994). However, by virtue of the counter-monotonicity of  $(X, Z)$  in our current setting, we can draw a much stronger conclusion in the form of an almost sure equality.

**Proposition 3**  $Z \in H(X)$  if and only if there exists a function  $h$  such that  $Z = -h(X)$  almost surely and  $(h(X), X - h(X))$  is comonotonic.

*Proof:* If  $Z = -h(X)$  almost surely and  $(h(X), (\text{Id} - h)(X))$  is comonotonic, then  $(Z, X + Z) = (-h(X), (\text{Id} - h)(X))$  is counter-monotonic.

Conversely, suppose that  $(Z, X + Z)$  is counter-monotonic. We can write  $X$  as a comonotonic sum  $X = (X + Z) + (-Z)$ . By Proposition 4.5 of Denneberg (1994), there exists a function  $h$ , with both  $h$  and  $\text{Id} - h$  non-decreasing, such that  $X + Z = (\text{Id} - h)(X)$  and  $-Z = h(X)$  almost surely.  $\square$

Let  $D$  be a subset of  $\mathbb{R}$ . A function  $f : D \rightarrow \mathbb{R}$  is said to be 1-Lipschitz if  $|f(x) - f(y)| \leq |x - y|$  for any  $x, y \in D$ . If  $f$  is non-decreasing then, for  $x, y \in D$  with  $x < y$ ,

$$|f(x) - f(y)| \leq |x - y| \iff x - f(x) \leq y - f(y),$$

that is,  $\text{Id} - f$  is also non-decreasing.

**Lemma 7** A function  $f : D \rightarrow \mathbb{R}$  is non-decreasing and 1-Lipschitz if and only if  $(f(X), X - f(X))$  is comonotonic for any random variable  $X$  taking values in  $D$ .

*Proof:* If  $f$  is non-decreasing and 1-Lipschitz, then both  $f$  and  $\text{Id} - f$  are non-decreasing, and hence  $(f(X), (\text{Id} - f)(X))$  is comonotonic for any random variable  $X$  taking values in  $D$ . For the converse, suppose that  $f : D \rightarrow \mathbb{R}$  is a function such that  $(f(X), (\text{Id} - f)(X))$  is comonotonic for some random variable  $X$  taking values in  $D$ . It is clear that  $f$  must be non-decreasing on the support of  $X$ ; moreover, if  $x$  and  $y$  are inside the support of  $X$  with  $x \leq y$ , then  $(\text{Id} - f)(x) \leq (\text{Id} - f)(y)$ , which is equivalent to  $f(y) - f(x) \leq y - x$ , so  $f$  is 1-Lipschitz on the support of  $X$  as well.  $\square$

From this lemma, we can rephrase Proposition 3 and give a more explicit description of the function  $h$  there as follows.

**Corollary 2**  *$Z \in H(X)$  if and only if there exists a function  $h$  which is non-decreasing and 1-Lipschitz on the support of  $X$  such that  $Z = -h(X)$  almost surely.*

## 5 Implications in optimal insurance problems

Suppose that a risk bearer is exposed to a non-negative random loss  $X$ , and wishes to reduce the risk by purchasing an insurance for  $X$ . Let  $I$  be the indemnity purchased such that the risk bearer will receive  $I(x)$  from the insurer if  $x$  is the realization of  $X$ . In this section, we assume that the indemnity schedule  $I$  is non-decreasing and satisfies  $0 \leq I(x) \leq x$ , in order to avoid moral hazard.

The insurer, who relies on the law of large numbers and is risk-neutral (page 45 of Eeckhoudt et al. (2005)), usually sets the premium  $P$  at a level that is larger than or equal to  $\mathbb{E}[I(X)]$  to avoid a loss on average. Suppose that the risk bearer is an expected utility maximizer with a wealth  $w$  and a non-decreasing and concave utility function  $u$ . He will be better off after purchasing  $I$  if

$$\mathbb{E}[u(w - X + I(X) - P)] \geq \mathbb{E}[u(w - X)].$$

Let  $P^*(u, w, X, I)$  be the amount of premium that makes the inequality above an equality. This utility-equivalent premium is the maximum amount the risk bearer is willing to pay for the insurance protection  $I$ . In order for this insurance arrangement to be mutually beneficial to both the risk bearer and the insurer,  $P^*(u, w, X, I)$  has to be larger than or equal to  $\mathbb{E}[I(X)]$ .

As in many adverse selection models (see, for instance, Landsberger and Meilijson (1999) and Rothschild and Stiglitz (1976)), we assume that the insurer has to provide a menu of schedules  $I$  to all potential policyholders with different risk preferences (represented by their utility functions) and risk profiles (represented by the wealths and potential losses), rather than providing tailor-made schedules that are optimal to individual policyholders. In this situation, the schedules to be included in the menu have to be carefully designed so that they are not only beneficial and attractive to all potential policyholders but also profitable to the insurer. This leads us to propose the following new concept.

**Definition 2** *Let  $I$  be an indemnity schedule. If for any non-negative integrable random variable  $X$ , any wealth level  $w$ , and any non-decreasing and concave utility function  $u$ , the solution  $P^*(u, w, X, I)$  to the equation*

$$\mathbb{E}[u(w - X + I(X) - P)] = \mathbb{E}[u(w - X)],$$

*provided the expectations are well-defined, is larger than or equal to  $\mathbb{E}[I(X)]$ , then  $I$  is said to be universally marketable.*

From this definition, an indemnity schedule  $I$  is universally marketable if a mutually acceptable premium can be found for every risk bearer, regardless of the wealth, the utility function, and the underlying risk. This notion depends on how we judge the acceptability of a given premium. In our setting here, a premium is acceptable to the insurer if it is not less than the expected payment (from the risk-neutrality of the insurer), and is acceptable to the risk bearer if a higher level of expected utility can be achieved after purchasing the insurance.

Before proceeding to further analysis, it should be emphasized that a universally acceptable indemnity schedule may not be optimal from the policyholder's perspective; indeed, our concern here is on how an insurer should design policies that are acceptable by all policyholders, but not on finding the optimal choice of a particular policyholder. In fact, it is common in practice that insurance companies only provide a menu of policy schedules for potential policyholders to choose from, but not allowing them to design and purchase whatever they desire. Therefore, the notion of universal marketability and our subsequent analysis are not related to traditional (Pareto) optimal insurance models.

It is evident that a given indemnity schedule  $I$  may not be universally marketable without possessing some extra properties. The main objective of this section is to derive a necessary and sufficient condition for universal marketability; see Theorem 3 below.

Recall that all indemnity schedules considered here are assumed to be non-decreasing a priori.

**Theorem 3** *An indemnity schedule  $I$  is universally marketable if and only if it is 1-Lipschitz.*

*Proof:* Let  $X$  be an arbitrary non-negative integrable random variable, and  $I$  be a 1-Lipschitz indemnity schedule. By Lemma 7,  $(I(X), X - I(X))$  is comonotonic and hence  $-I(X) \in H(X)$  by definition. Theorem 1 then implies that

$$X - I(X) \leq_{cx} X - \mathbb{E}[I(X)].$$

Therefore, for any wealth level  $w$  and any non-decreasing and concave utility function  $u$ ,

$$\mathbb{E}[u(w - X + I(X) - \mathbb{E}[I(X)])] \geq \mathbb{E}[u(w - X)] = \mathbb{E}[u(w - X + I(X) - P^*(u, w, X, I))].$$

As  $u$  is non-decreasing,  $P^*(u, w, X, I) \geq \mathbb{E}[I(X)]$ , and hence  $I$  is universally marketable.

To prove the converse, assume that  $I$  is not 1-Lipschitz. Then there exist  $a > 0$  and  $\varepsilon > 0$  such that  $\delta := I(a + \varepsilon) - I(a) > \varepsilon$ . We want to show that the universal marketability condition  $\mathbb{E}[I(X)] \leq P^*(u, w, X, I)$  is violated by some particular choices of  $X$  and  $u$ . To this end, choose any  $p \in (0, 1 - \varepsilon/\delta)$ , and define a discrete random variable  $\tilde{X}$  by

$$\tilde{X} = \begin{cases} a, & \text{with probability } p, \\ a + \varepsilon, & \text{with probability } 1 - p. \end{cases}$$

Since  $(a - I(a)) - (a + \varepsilon - I(a + \varepsilon)) = \delta - \varepsilon > 0$ , the maximum value that  $\tilde{X} - I(\tilde{X})$  can attain equals  $a - I(a)$ . On the other hand, the maximum value that  $\tilde{X} - \mathbb{E}[I(\tilde{X})]$  can attain equals  $a + \varepsilon - \mathbb{E}[I(\tilde{X})]$ , but

$$a + \varepsilon - \mathbb{E}[I(\tilde{X})] = a + \varepsilon - (I(a)p + I(a + \varepsilon)(1 - p)) = a - I(a) + \varepsilon - (1 - p)\delta < a - I(a),$$

where the last step follows from our choice of  $p$ . It then follows that, for any  $d \in [a + \varepsilon - \mathbb{E}[I(\tilde{X})], a - I(a))$ ,

$$\mathbb{E}[(\tilde{X} - I(\tilde{X}) - d)_+] = (a - I(a) - d)p > 0 = \mathbb{E}[(\tilde{X} - \mathbb{E}[I(\tilde{X})] - d)_+],$$

and hence we can conclude that  $\tilde{X} - I(\tilde{X}) \not\leq_{sl} \tilde{X} - \mathbb{E}[I(\tilde{X})]$ . This means that for any wealth level  $w$ , there exist some non-decreasing and concave utility function  $\tilde{u}$  such that

$$\mathbb{E}[\tilde{u}(w - \tilde{X} + I(\tilde{X}) - \mathbb{E}[I(\tilde{X})])] < \mathbb{E}[\tilde{u}(w - \tilde{X})] = \mathbb{E}[\tilde{u}(w - \tilde{X} + I(\tilde{X}) - P^*(\tilde{u}, w, \tilde{X}, I))].$$

In particular, this implies that  $\mathbb{E}[I(\tilde{X})] > P^*(\tilde{u}, w, \tilde{X}, I)$ , and hence  $I$  is not universally marketable.  $\square$

By Lemma 7, the 1-Lipschitz property of an indemnity  $I$  is equivalent to the comonotonicity of the share of the loss borne by the insurer,  $I(X)$ , and the share of the loss borne by the policyholder,  $X - I(X)$ , for any loss variable  $X$ . Many commonly used indemnity schedules are 1-Lipschitz, such as

- (i) quota-share contracts  $I(x) = cx$ ,  $c \in [0, 1]$ ;
- (ii) stop-loss contracts  $I(x) = (x - d)_+$ ,  $d \geq 0$ ;
- (iii) limited loss contracts  $I(x) = \max(x, a)$ ,  $a \geq 0$ ;
- (iv) insurance layer contracts  $I(x) = (x - d)_+ - (x - a)_+$ ,  $a \geq d \geq 0$ ; and
- (v) contracts that are non-decreasing and convex with on  $\mathbb{R}_+$ .

The 1-Lipschitz property of contracts in (i) to (iv) above is obvious. For (v), suppose that  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is non-decreasing and convex, with  $0 \leq f(x) \leq x$  for all  $x \geq 0$ , but is not 1-Lipschitz. Then there exist  $0 \leq x < y$  such that

$$s := \frac{f(y) - f(x)}{y - x} > 1.$$

In this case, we have

$$h > \frac{y - f(y)}{s - 1} \implies f(y + h) \geq f(y) + sh > y + h,$$

which contradicts the condition that  $0 \leq f(x) \leq x$  for all  $x \geq 0$ . Theorem 3 provides a possible explanation of the popularity of indemnity schedules stated above in the insurance market, as noted in Denuit and Vermandele (1998) and Young (1999).

## Acknowledgments

The authors would like to thank Dr. Yichun Chi at the Central University of Finance and Economics, China, for helpful comments. Ka Chun Cheung was supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. HKU 701213). Jan Dhaene acknowledges the financial support of the Onderzoeksfonds KU Leuven (GOA/13/002). Qihe Tang acknowledges the financial support by a Centers of Actuarial Excellence (CAE) Research Grant (2013–2016) from the Society of Actuaries.

## References

- [1] Ahn, D.H., Boudoukh, J., Richardson, M., Whitelaw, R., 1999. Optimal risk management using options. *Journal of Finance* 54, 359-375.
- [2] Bäuerle, N., Müller, M., 2006. Stochastic orders and risk measures: Consistency and bounds. *Insurance: Mathematics and Economics* 38, 132-148.
- [3] Bickel, P.J., Lehmann, E.L., 1979. Descriptive Statistics for Non-Parametric Model IV: Spread. *Contributions to Statistics*, Jane Juresckova (Ed.), Reidel, Amsterdam.
- [4] Cheung, K.C., 2010. Characterizing a comonotonic random vector by the distribution of the sum of its components. *Insurance: Mathematics and Economics* 47, 130-136.
- [5] Cheung, K.C., Lo, A., 2013. Characterizations of counter-monotonicity and upper comonotonicity by (tail) convex order. *Insurance: Mathematics and Economics* 53, 334-342.
- [6] Deelstra, G., Ezzine, A., Heyman, D., Vanmaele, M., 2007. Managing Value-at-Risk for a bond using bond put options. *Computational Economics* 29, 139-149.
- [7] Denneberg, D., 1994. *Non-additive measure and integral*. Kluwer Academic Publishers, Netherlands.
- [8] Denuit, M., Dhaene, J., Goovaerts, M.J., Kaas, R., 2005. *Actuarial theory for dependent risks*. Wiley.

- [9] Denuit, M., Vermandele, C., 1998. Optimal reinsurance and stop-loss order. *Insurance: Mathematics and Economics* 22, 229-233.
- [10] Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R., Vyncke, D., 2002. The concept of comonotonicity in actuarial science and finance: theory. *Insurance: Mathematics and Economics* 31, 3-33.
- [11] Dhaene, J., Goovaerts, M.J., 1996. Dependency of risks and stop-loss order. *Astin Bulletin* 26, 201-212.
- [12] Dhaene, J., Vanduffel, S., Goovaerts, M.J., Kaas, R., Tang, Q., Vyncke, D., 2006. Risk measures and comonotonicity: a review. *Stochastic Models* 22, 573-606.
- [13] Dowd, K. Blake, D., 2006. After VaR: The theory, estimation and insurance applications of quantile-based risk measures. *Journal of Risk and Insurance* 73, 193-229.
- [14] Eeckhoudt, L., Gollier, C., Schlesinger, H., 2005. *Economic and financial decisions under risk*. Princeton University Press, New Jersey.
- [15] Föllmer, H., Schied, A., 2011. *Stochastic finance: An introduction in discrete time*. Walter de Gruyter, Berlin.
- [16] Kallenberg, O., 2002. *Foundations of modern probability (Second Edition)*. Springer-Verlag, New York.
- [17] Landsberger, M., Meilijson, I., 1994. Comontone Allocations, Bickel-Lehmann Dispersion and the Arrow-Pratt Measure of Risk Aversion. *Annals of Operations Research* 52, 97-106.
- [18] Landsberger, M., Meilijson, I., 1999. A general model of insurance under adverse selection. *Economic Theory* 14, 331-352.
- [19] Müller, A., Stoyan, D., 2002. *Comparison methods for stochastic models and risks*. John Wiley & Sons, Ltd., Chichester.
- [20] Rothschild, M., Stiglitz, J., 1976. Equilibrium in competitive insurance markets: An essay on the economics of imperfect information. *The Quarterly Journal of Economics* 90, 629-649.
- [21] Shaked, M., Shanthikumar, J.G., 2007. *Stochastic Orders*. Springer, New York.
- [22] Young, V. R., 1999. Optimal insurance under Wang's premium principle. *Insurance: Mathematics and Economics* 25, 109-122.