

Liquidation Risk in the Presence of Chapters 7 and 11 of the U.S. Bankruptcy Code

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Abstract

We aim at quantitatively measuring the liquidation risk of a firm subject to both Chapters 7 and 11 of the U.S. bankruptcy code. The firm value is modeled by a general time-homogeneous diffusion process in which the drift and volatility are level dependent and can be easily adjusted to reflect the state changes of the firm. An explicit formula for the probability of liquidation is established, based on which we gain a quantitative understanding of how the capital structures before and during bankruptcy affect the probability of liquidation.

Keywords: bankruptcy; liquidation; partial differential equation; time-homogeneous diffusion; two-sided exit problem

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1 Introduction

Stemming from Merton (1974) and Black and Cox (1976), numerous structural models have been proposed in which bankruptcy and liquidation are usually treated as the same

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event that the firm value reaches an absorbing low barrier. In the real world, however, the procedures of bankruptcy and liquidation as described in the U.S. bankruptcy code are rather complicated. When a firm is unable to service its debt or pay back its creditors but its fiscal situation is not severe, it is usually given the right to declare bankruptcy under Chapter 11 of reorganization rather than Chapter 7 of immediate liquidation. Chapter 11 allows the firm to remain in control of its business operations with a bankruptcy court providing oversight. The court grants the firm a certain observation period during which the firm manager can restructure its debt. The debtor usually proposes a plan of reorganization to keep its business alive and pay back the creditors over time. The reorganization plan may either succeed or fail. For the latter case, Chapter 11 will be converted to Chapter 7 governed by §1019 of the U.S. bankruptcy code and the firm may be forced to be liquidated.

Warren and Westbrook (2009) showed empirical evidences for the conclusion that the Chapter 11 system offers a realistic hope for troubled businesses to turn around their operations and rebuild their capital structures. Lee et al. (2011) found that entrepreneur-friendly bankruptcy laws such as Chapter 11 is significantly correlated with the rate of new firm entry. See also Hotchkiss (1995), Bris et al. (2006), Denis and Rodgers (2007), Annabi et al. (2012), and Christensen et al. (2013) for related empirical studies of the role of Chapter 11.

Under these practical considerations, many recent works in the literature of corporate finance have included Chapter 11 reorganization proceedings and made a distinction between bankruptcy and liquidation. In the works of Moraux (2004), François and Morellec (2004), Galai et al. (2007), Broadie and Kaya (2007), and Dai et al. (2013), among others, the liquidation time is modeled by the first time the firm value constantly/cumulatively stayed below the bankruptcy barrier over a grace period granted by the bankruptcy court. However, in these works only the bankruptcy barrier was considered and, hence, the firm is not necessarily liquidated even when its value is extremely low, which violates the principle of limited liability.

On the other hand, Paseka (2003) and Broadie et al. (2007) explicitly incorporated both the bankruptcy and the liquidation barriers in their models. In this paper, we shall follow the Chapters 7 and 11 bankruptcy and liquidation model proposed by Broadie et al. (2007).

We model the firm value by a general time-homogeneous diffusion process. A merit of this model is that the drift and volatility of the firm value are level dependent and can be easily adjusted to reflect the state changes of the firm. Our derivations are mainly based on the partial differential equations (PDE) and the perturbation techniques, the latter of which are recently developed for stochastic processes; see e.g., Dassios and Wu (2010), Landriault et al. (2011), and Li and Zhou (2013). Therefore, we expect that this study can be extended to other value processes, such as Lévy-driven models and Markov regime-switching models.

The major contribution of this paper is to provide an analytical formula for the prob-

ability of liquidation, which is at the core of financial risk management. We would like to remark that our results have immediate applications to pricing corporate bonds and other derivatives such as credit default swaps. In particular, in light of this work, it is anticipated that the recent study of Parisian options by Chesney et al. (1997), Chesney and Gauthier (2006), Anderluh and van der Weide (2009), Dassios and Wu (2010, 2011), and Albrecher et al. (2012), among others, can be extended to more elaborate options formulated in the current new framework.

Our paper is also relevant for the recent study of Parisian ruin in the context of risk theory originating from the study of Parisian options. The concept of Parisian ruin was first introduced by Dassios and Wu (2008) under the Brownian motion framework. Later on, the underlying structure was generalized to the Lévy framework by Dassios and Wu (2009), Czarna and Palmowski (2011), and Loeffen et al. (2013). However, these works only investigated the so-called Parisian stopping time, which corresponds to the liquidation time incurred by Chapter 11 of bankruptcy code in our framework. By further incorporating the feature of Chapter 7, our paper essentially studies the minimum of a Parisian stopping time and a first passage time. In other words, we monitor not only the length of excursions below the bankruptcy barrier but also the depth. Another related paper is by Gauthier (2002), who also studied such a mixture of the two stopping times. However, Gauthier's analytical results are only in the framework of Brownian motion processes and are rather involved. In addition, our methodology, which combines the PDE and perturbation techniques, is very different from Gauthier's.

There are several limitations of our paper. First, the perturbation techniques used in this paper cannot be easily extended to time inhomogeneous models. But if the firm value process is piecewise time homogeneous, one can first derive the Laplace transform of the liquidation time for each time period and then calculate the probability of liquidation by Laplace inversion techniques. Second, as a common drawback of structure models driven by Brownian motion, the short-term credit spread of our time-homogeneous diffusion model tends to zero as the maturity approaches zero. This can be overcome by incorporating downward jumps or considering stochastic bankruptcy barriers; see, e.g., Section 3.1.9 of Brigo et al. (2013). Third, calibration of the time-homogeneous diffusion structure is beyond our paper. It is difficult to calibrate the process from Credit Default Swap (CDS) data. However, recent studies show that one can calibrate time-homogeneous diffusion models from perpetual American options for different strikes; see, e.g., Ekström and Hobson (2011).

The rest of the paper is organized as follows. The probability of liquidation is defined and the time-homogeneous diffusion model is introduced in Section 2. Analytical results for the probability of liquidation are presented in Section 3. Some numerical studies are

implemented in Section 4. Finally, two major proofs of our paper are postponed to Section 5.

2 Modeling

2.1 The probability of liquidation

We follow Broadie et al. (2007) to describe the procedures of bankruptcy and liquidation by incorporating the Chapter 11 reorganization, the Chapter 7 liquidation, the conversion from Chapter 11 to Chapter 7, and the grace period in Chapter 11. Formally, suppose that the firm value is modeled by a general stochastic process $X = \{X_t, t \geq 0\}$ starting with $X_0 = x_0$. For a real number $x \leq x_0$, denote by T_x the first time when the process X down-crosses the level x ; that is,

$$T_x = \inf \{t > 0 : X_t < x\}.$$

Hereafter, we follow the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = 0$. For $x \geq x_0$, the first time when the process X up-crosses the level x can be defined similarly.

Let $a < b$ and $c > 0$ be three endogenously determined constants, with a interpreted as the Chapter 7 liquidation barrier, b as the Chapter 11 reorganization barrier, and c as the duration of a grace period in Chapter 11 granted by the bankruptcy court. Let $\tau_b(c)$ be the first time when the process X has continuously stayed below level b for c units of time, namely,

$$\tau_b(c) = \inf \{t > 0 : t - g_t \geq c\} \quad \text{with} \quad g_t = \sup \{s \leq t : X_s \geq b\}.$$

Then the liquidation time is defined by

$$T_a \wedge \tau_b(c).$$

Hereafter, we denote by $u \wedge v = \min\{u, v\}$ and $u \vee v = \max\{u, v\}$.

This definition of liquidation can be verbally explained as follows. Once the firm value is below the reorganization barrier b , the firm declares Chapter 11 to trigger the reorganization procedure and an invisible distress clock starts ticking. If the firm value rebounds and rises above the barrier b before the grace period runs out, Chapter 11 is resolved and the distress clock is reset to zero. However, the firm may be eventually liquidated under Chapter 7 in case the reorganization plan does not succeed. There are two scenarios of liquidation: either the firm value drops below the liquidation barrier a or the time the firm spends in bankruptcy exceeds the grace period c granted by the bankruptcy court. Figure 1 of Broadie et al. (2007) depicts these scenarios.

The probability of liquidation subject to Chapters 7 and 11 with the liquidation barrier a , the reorganization barrier b , and the duration c of a grace period, or, in short, the probability of liquidation subject to the triple (a, b, c) , is defined by

$$q(x_0; a, b, c) = \mathbb{P}_{x_0} \{T_a \wedge \tau_b(c) < \infty\}, \quad a < b \leq x_0, c > 0. \quad (2.1)$$

Throughout the paper, we denote by \mathbb{P}_x the law of X with $X_0 = x$. Note that this probability of liquidation is in the infinite-time horizon. It provides us with a quantitative understanding of the firm's liquidation risk in the long run. Besides, letting $a \downarrow -\infty$ in (2.1) yields

$$q(x_0; -\infty, b, c) = \mathbb{P}_{x_0} \{\tau_b(c) < \infty\}.$$

This is essentially the probability of liquidation introduced by François and Morellec (2004), Brodie and Kaya (2007), and Dai et al. (2013) in finance, or the Parisian ruin probability studied in Dassios and Wu (2009), Czarna and Palmowski (2011), and Loeffen et al. (2013) in risk theory.

We slightly extend the domain of the probability of liquidation $q(x; a, b, c)$ in (2.1) to $x > a$. It is sometimes more convenient to carry out our discussions on the corresponding survival probability defined by

$$p(x; a, b, c) = 1 - q(x; a, b, c) = \mathbb{P}_x \{T_a \wedge \tau_b(c) = \infty\}. \quad (2.2)$$

Hereafter, we often drop the arguments a, b, c from $p(x; a, b, c)$ and $q(x; a, b, c)$ whenever we proceed with general discussions without emphasis on them.

2.2 Time-homogeneous diffusion processes

We model the firm value by a general time-homogeneous diffusion process $X = \{X_t, t \geq 0\}$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with dynamics

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad (2.3)$$

where $X_0 = x_0$ is the initial value and $\{W_t, t \geq 0\}$ is a standard Brownian motion. Denote by $\{\mathcal{F}_t, t \geq 0\}$ the natural filtration generated by $\{W_t, t \geq 0\}$. As usual, assume that $\mu(\cdot)$ and $\sigma(\cdot) > 0$ satisfy the conditions of the existence and uniqueness theorem for the stochastic differential equation (2.3); namely, there exists a constant $K > 0$ such that, for all $x, y \in \mathbb{R}$,

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq K|x - y|, \quad \mu^2(x) + \sigma^2(x) \leq K(1 + x^2). \quad (2.4)$$

Then the unique strong solution of (2.3) possesses the strong Markov property; see G̃ihman and Skorohod (1972, Pages 40 and 107).

Here and thereafter, we restrict X to the time-homogeneous diffusion process (2.3). Since X has continuous sample paths, the two first passage times from above or below a barrier $x \in \mathbb{R}$ are identical to the first hitting time

$$T_x = \inf \{t > 0 : X_t = x\}.$$

Besides, since the probability of liquidation $q(x_0; a, b, c)$ is obviously monotone decreasing in c , letting $c \downarrow 0$ in (2.1) yields

$$q(x_0; a, b, 0) = \mathbb{P}_{x_0} \{T_b < \infty\}, \quad (2.5)$$

while letting $c \uparrow \infty$ yields

$$q(x_0; a, b, \infty) = \mathbb{P}_{x_0} \{T_a < \infty\}. \quad (2.6)$$

Relation (2.5) is due to $\tau_b(c) \downarrow T_b$ as $c \downarrow 0$ and (2.6) is due to $\tau_b(c) \geq c$ a.s. Hence, the duration c serves as a bridge connecting the two traditional probabilities of bankruptcy.

The two-sided exit problem for the diffusion process X has been well studied. It is well known that, for $u < x < v$,

$$\mathbb{P}_x \{T_u < T_v\} = \frac{\int_x^v G(y)dy}{\int_u^v G(y)dy} \quad \text{and} \quad \mathbb{P}_x \{T_u > T_v\} = \frac{\int_u^x G(y)dy}{\int_u^v G(y)dy}, \quad (2.7)$$

where

$$G(x) = \exp \left\{ - \int^x \frac{2\mu(y)}{\sigma^2(y)} dy \right\};$$

see, e.g. Page 110 of G̃ihman and Skorohod (1972) or Section 6.4 of Klebaner (2005). By the homogeneity of the diffusion process, the lower limit of the integral above can be set to any point in the domain. The function $S(x) = \int^x G(y)dy$ is referred to as the scale function of X . Throughout the paper, to avoid triviality we assume that

$$S(\infty) = \int^\infty G(y)dy < \infty.$$

Thus, letting $v = \infty$ in the second relation in (2.7) yields

$$\mathbb{P}_x \{T_u = \infty\} = \frac{\int_u^x G(y)dy}{\int_u^\infty G(y)dy} \in (0, 1). \quad (2.8)$$

3 Analytical results

The following proposition is a key step in establishing our main result. In its proof, which is postponed to Section 5, instead of using probabilistic arguments we employ a PDE technique. The advantage of this approach is that it can be applied to other underlying structures such as jump-diffusion processes. This proposition forms a theoretical basis for the perturbation approach used in the proof of Proposition 3.2.

Proposition 3.1 For $a < b$ and $c > 0$, the following limits exist and are equal:

$$\lim_{\varepsilon \downarrow 0} \frac{\mathbb{P}_{b-\varepsilon} \{T_b > T_a \wedge c\}}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{P}_b \{T_{b+\varepsilon} > T_a \wedge c\}}{\varepsilon}.$$

We introduce an auxiliary quantity

$$A(a, b, c) = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{P}_{b-\varepsilon} \{T_b > T_a \wedge c\}}{\varepsilon}, \quad (3.1)$$

which also equals $\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{P}_b \{T_{b+\varepsilon} > T_a \wedge c\}$ by Proposition 3.1. We know that $A(a, b, c)$ can be considered as the boundary derivative of the solution of a linear parabolic PDE, i.e.,

$$A(a, b, c) = \lim_{\varepsilon \downarrow 0} \frac{u(b - \varepsilon, c)}{\varepsilon}, \quad (3.2)$$

where $u(x, t)$ solves

$$\begin{cases} u_t(x, t) = \frac{\sigma^2(x)}{2} u_{xx}(x, t) + \mu(x) u_x(x, t), & a < x < b \text{ and } t > 0, \\ u(a, t) = 1, & t \geq 0, \\ u(b, t) = 0, & t \geq 0, \\ u(x, 0) = 1, & a < x < b. \end{cases}$$

Hence, its value can be determined by standard numerical PDE approaches; see Section 4 for more discussions.

The following result expresses the probability of liquidation/survival at level b defined by (2.2). Note that level b is a critical threshold as it starts or resets the distress clock. The proof of Proposition 3.2 is also postponed to Section 5.

Proposition 3.2 For $b > a$ and $c > 0$, we have

$$p(b) = \frac{G(b)}{G(b) + A(a, b, c) \int_b^\infty G(y) dy}. \quad (3.3)$$

By the strong Markov property, for $x_0 \geq b$ the survival probability satisfies

$$\begin{aligned} p(x_0) &= \mathbb{P}_{x_0} \{T_a \wedge \tau_b(c) = \infty, T_b = \infty\} + \mathbb{P}_{x_0} \{T_a \wedge \tau_b(c) = \infty, T_b < \infty\} \\ &= \mathbb{P}_{x_0} \{T_b = \infty\} + \mathbb{P}_{x_0} \{T_b < \infty\} p(b). \end{aligned}$$

Using (2.8) and (3.3), we obtain our main result:

Theorem 3.1 For $a < b \leq x_0$ and $c > 0$, we have

$$q(x_0) = \frac{A(a, b, c)}{A(a, b, c) \int_b^\infty G(y) dy + G(b)} \int_{x_0}^\infty G(y) dy. \quad (3.4)$$

Some remarks follow below:

1) The methodology for the proof of (3.4) can also be used to derive the Laplace transform of the liquidation time $T_a \wedge \tau_b(c)$, that is

$$\mathbb{P}_{x_0} \{T_a \wedge \tau_b(c) < e_\lambda\},$$

where e_λ is an independent exponential random variable with parameter $\lambda > 0$. By a numerical inverse Laplace transform approach, we can further determine the probability of liquidation in a finite-time horizon. However, since the calculations and the results will be much more complicated, we present the probability of liquidation in infinite-time horizon only.

2) We can rewrite (3.4) a bit more explicitly as

$$q(x_0) = \frac{A(a, b, c)}{A(a, b, c) \int_b^\infty \exp \left\{ - \int_b^y \frac{2\mu(z)}{\sigma^2(z)} dz \right\} dy + 1} \int_{x_0}^\infty \exp \left\{ - \int_b^y \frac{2\mu(z)}{\sigma^2(z)} dz \right\} dy.$$

Also recall the definition of the auxiliary quantity $A(a, b, c)$ in (3.1). We see that the two integrals above involve the values of $\mu(\cdot)$ and $\sigma(\cdot)$ over the range (b, ∞) only while the auxiliary quantity $A(a, b, c)$ involves the values of $\mu(\cdot)$ and $\sigma(\cdot)$ over the range $[a, b]$ only. Through this formula, we can gain a quantitative understanding of how the capital structures before and during bankruptcy affect the probability of liquidation $q(x_0)$.

3) Recall the definition of $A(a, b, c)$ in (3.1). It is easy to see that

$$\lim_{c \downarrow 0} A(a, b, c) = \infty \quad \text{and} \quad \lim_{c \uparrow \infty} A(a, b, c) = \frac{G(b)}{\int_a^b G(y) dy}. \quad (3.5)$$

Substituting (3.5) into (3.4) leads to explicit expressions for the two traditional probabilities of bankruptcy defined by (2.5) and (2.6), which are identical to the expressions obtained by directly applying (2.8).

4) The analytical formula for the probability of liquidation (3.4) has advantage in numerical computation. Haber et al. (1999) showed that the PDE approach easily outperforms Monte Carlo simulation for the calculation of Parisian option price, which is similar to the calculation of the probability of liquidation. However, without any simplification, the probability of liquidation satisfies a high-dimensional PDE (Haber et al. (1999)). Thanks to (3.4), not only we reduce the problem to one dimension (the auxiliary function $A(a, b, c)$) but also we gain a lot of intuition of the probability of liquidation.

Unfortunately, there seems to exist no simple expression for the auxiliary quantity $A(a, b, c)$ even for a linear Brownian motion, for which case $A(a, b, c)$ can be expressed as an infinite series by using formula 2.15.4(1) of Borodin and Salminen (2002). See also

Lin (1998), who derived such series expressions for the density functions of hitting times of a (geometric) Brownian motion.

Nevertheless, the quantity

$$A(-\infty, b, c) = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{P}_{b-\varepsilon} \{T_b > c\}}{\varepsilon}$$

has a simple expression for a few diffusion processes such as a Brownian motion with positive drift. More precisely, it is well known that the hitting time T_y at level $y > 0$ of the drifted Brownian motion $X_t = \mu t + \sigma W_t$, $\mu, \sigma, t > 0$, follows an inverse Gaussian distribution with cumulative distribution function

$$\mathbb{P} \{T_y \leq c\} = \Phi \left(\frac{\mu c - y}{\sigma \sqrt{c}} \right) + \exp \left\{ \frac{2\mu y}{\sigma^2} \right\} \Phi \left(-\frac{\mu c + y}{\sigma \sqrt{c}} \right), \quad (3.6)$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function; see, e.g. Chhikara and Folks (1989). By relations (3.4) and (3.6), it is easy to derive the following formulas for the corresponding Parisian ruin probabilities:

Corollary 3.1 *Let $x_0 \geq b$, $c > 0$, and $\rho = 2\mu/\sigma^2$.*

(i) *If X is a Brownian motion with positive drift, i.e., X is given by (2.3) with $\mu(x) \equiv \mu > 0$ and $\sigma(x) \equiv \sigma > 0$, then*

$$\mathbb{P}_{x_0} \{\tau_b(c) < \infty\} = \frac{\sqrt{2} \exp \left\{ -\frac{\mu \rho c}{4} \right\} - \sigma \rho \sqrt{\pi c} \Phi \left(-\frac{\sigma \rho \sqrt{c}}{2} \right)}{\sqrt{2} \exp \left\{ -\frac{\mu \rho c}{4} \right\} + \sigma \rho \sqrt{\pi c} \Phi \left(\frac{\sigma \rho \sqrt{c}}{2} \right)} \exp \{-\rho(x_0 - b)\}. \quad (3.7)$$

(ii) *If X is a geometric Brownian motion, i.e., X is given by (2.3) with $\mu(x) = \mu x$ and $\sigma(x) = \sigma x$ for some $\sigma > 0$, and if $b > 0$ and $\rho > 1$, then*

$$\mathbb{P}_{x_0} \{\tau_b(c) < \infty\} = \frac{\sqrt{2} \exp \left\{ -\frac{c \sigma^2 (\rho - 1)^2}{8} \right\} - \sigma (\rho - 1) \sqrt{\pi c} \Phi \left(-\frac{\sigma (\rho - 1) \sqrt{c}}{2} \right)}{\sqrt{2} \exp \left\{ -\frac{c \sigma^2 (\rho - 1)^2}{8} \right\} + \sigma (\rho - 1) \sqrt{\pi c} \Phi \left(\frac{\sigma (\rho - 1) \sqrt{c}}{2} \right)} \left(\frac{b}{x_0} \right)^{\rho - 1}.$$

Formula (3.7) with $b = 0$ retrieves equation (33) of Dassios and Wu (2008).

4 Numerical examples

As mentioned before, the only implicit part in formula (3.4) is the auxiliary quantity $A(a, b, c)$. By (3.2), $A(a, b, c)$ can be computed numerically via a PDE. In this section, we use the Crank–Nicolson method (see, e.g. Thomas (1995)) to solve $A(a, b, c)$. Usually, the Crank–Nicolson scheme is very accurate for small time steps. This is a second-order

implicit finite difference method, which is unconditionally convergent and stable. The local error is of order $O(\Delta x^2) + O(\Delta t^2)$, implying that the error for $A(a, b, c)$ is of order $O(\Delta x) + O(\Delta t^2)$.

The numerical experiments are carried out in a computer with an Intel Core i7-2600 CPU 3.40GHz 8GB RAM, and the software programs are run under Matlab R2011b.

4.1 For the Brownian motion case

First, we assume that the firm value follows a linear Brownian motion and that the capital structure remains unchanged during bankruptcy; that is,

$$dX_t = \mu dt + \sigma dW_t, \quad t > 0,$$

where $X_0 = x_0 \geq b$ and $\mu, \sigma > 0$. In risk theory, the arithmetic Brownian motion model can be considered as the limiting case of the classical Cramér-Lundberg insurance surplus model with exponential jumps; see Page 117–118 of Asmussen (2000). But the purpose of this section is for illustration only as the arithmetic Brownian motion structure is not very common for firm value models in general.

By Theorem 3.1, the probability of liquidation is given by

$$q(x_0) = \frac{A(a, b, c)}{A(a, b, c) + \rho} e^{-\rho(x_0 - b)} \quad (4.1)$$

with $\rho = 2\mu/\sigma^2$. The parameters are set to $\mu = 0.1$, $\sigma = 0.25$, $a = 0.1$, $b = 0.2$, and $c = 1$.

Table 4.1 The probability of liquidation for a linear Brownian motion without capital restructuring during bankruptcy

mesh	$A(a, b, c)$	$q(x_0)$	elapsed time (s)
$\Delta x = \Delta t = 0.005$	8.5534038	$1.3801420 \times e^{-3.2x_0}$	0.096548
$\Delta x = \Delta t = 0.001$	8.4987776	$1.3777311 \times e^{-3.2x_0}$	3.940972
$\Delta x = \Delta t = 0.0005$	8.4919795	$1.3774294 \times e^{-3.2x_0}$	32.536658
$\Delta x = \Delta t = 0.00025$	8.4885830	$1.3772786 \times e^{-3.2x_0}$	267.074039

By relation (2.8) and Corollary 3.1(i), the two traditional probabilities of bankruptcy and the Parisian ruin probability are equal to, respectively,

$$\begin{aligned} \mathbb{P}_{x_0} \{T_a < \infty\} &= 1.3771278 \times e^{-3.2x_0}, \\ \mathbb{P}_{x_0} \{T_b < \infty\} &= 1.8964809 \times e^{-3.2x_0}, \\ \mathbb{P}_{x_0} \{\tau_b(c) < \infty\} &= 0.6932042 \times e^{-3.2x_0}. \end{aligned}$$

The numerical results confirm the following facts:

$$\mathbb{P}_{x_0} \{T_a < \infty\} \vee \mathbb{P}_{x_0} \{\tau_b(c) < \infty\} < q(x_0) < \mathbb{P}_{x_0} \{T_b < \infty\}. \quad (4.2)$$

Evidently, $q(x_0)$ is very close to $\mathbb{P}_{x_0} \{T_a < \infty\}$, which means that, in the current situation, the liquidation happens mainly due to the firm value down-crossing level a rather than constantly staying below level b . In other words, $c = 1$ has been set relatively large. Conversely, for a relatively small $c > 0$, it is anticipated that $q(x_0)$ will be close to $\mathbb{P}_{x_0} \{T_b < \infty\}$. These are consistent with our analysis in (2.5) and (2.6).

Next, we propose a reorganization plan during bankruptcy. Assume that

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad t > 0,$$

where

$$\begin{aligned} \mu(x) &= \mu 1_{\{x > b\}} + \mu \left(1 - \frac{b-x}{2(b-a)}\right) 1_{\{a \leq x \leq b\}}, \\ \sigma(x) &= \sigma 1_{\{x > b\}} + \sigma \left(1 - \frac{b-x}{2(b-a)}\right) 1_{\{a \leq x \leq b\}}. \end{aligned}$$

This reorganization plan concerns the priority of the debt holder over the shareholders during bankruptcy by reducing $\mu(\cdot)$ and $\sigma(\cdot)$ over the range $[a, b]$. However, the Sharpe ratio $\mu(\cdot)/\sigma(\cdot)$ remains constant. Thus, by Theorem 3.1, formula (4.1) is still valid for $q(x_0)$ whereas the auxiliary quantity $A(a, b, c)$ needs to be recalculated according to the values of $\mu(\cdot)$ and $\sigma(\cdot)$ over the range $[a, b]$.

Table 4.2 The probability of liquidation for a linear Brownian motion with capital restructuring during bankruptcy

mesh	$A(a, b, c)$	$q(x_0)$	elapsed time (s)
$\Delta x = \Delta t = 0.005$	8.2173101	$1.3649425 \times e^{-3.2x_0}$	0.556764
$\Delta x = \Delta t = 0.001$	8.1639584	$1.3624470 \times e^{-3.2x_0}$	4.422223
$\Delta x = \Delta t = 0.0005$	8.1574008	$1.3621387 \times e^{-3.2x_0}$	34.587561
$\Delta x = \Delta t = 0.00025$	8.1541313	$1.3619848 \times e^{-3.2x_0}$	267.168596

Comparing Tables 4.1 and 4.2, we notice that this reorganization plan slightly reduces $q(x_0)$. However, we cannot draw a conclusion that capital restructuring according to the priority of the debt holder over the shareholders during bankruptcy always reduces the probability of liquidation. The following example illustrates the contrary.

4.2 For the geometric Brownian motion case

First, we assume that the firm value follows a geometric Brownian motion and that the capital structure remains unchanged during bankruptcy; that is,

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad t > 0,$$

where $X_0 = x_0 \geq b > 0$ and $\rho = 2\mu/\sigma^2 > 1$. By Theorem 3.1, the probability of liquidation is given by

$$q(x_0) = \frac{A(a, b, c)}{A(a, b, c) + (\rho - 1)/b} \left(\frac{b}{x_0}\right)^{\rho-1}. \quad (4.3)$$

The parameters are still set to $\mu = 0.1$, $\sigma = 0.25$, $a = 0.1$, $b = 0.2$, and $c = 1$.

Table 4.3 The probability of liquidation for a geometric Brownian motion without capital restructuring during bankruptcy

mesh	$A(a, b, c)$	$q(x_0)$	elapsed time (s)
$\Delta x = \Delta t = 0.005$	11.495846	$0.014815100 \times x_0^{-2.2}$	0.460361
$\Delta x = \Delta t = 0.001$	11.145638	$0.014590922 \times x_0^{-2.2}$	4.362864
$\Delta x = \Delta t = 0.0005$	11.101517	$0.014562175 \times x_0^{-2.2}$	31.490827
$\Delta x = \Delta t = 0.00025$	11.079429	$0.014547740 \times x_0^{-2.2}$	276.483609

For this case, the two traditional probabilities of bankruptcy and the Parisian ruin probability are equal to, respectively,

$$\begin{aligned} \mathbb{P}_{x_0} \{T_a < \infty\} &= 0.0063095734 \times x_0^{-2.2}, \\ \mathbb{P}_{x_0} \{T_b < \infty\} &= 0.028991187 \times x_0^{-2.2}, \\ \mathbb{P}_{x_0} \{\tau_b(c) < \infty\} &= 0.014533261 \times x_0^{-2.2}. \end{aligned}$$

Again, the numerical results confirm the facts in (4.2). However, $q(x_0)$ is now very close to $\mathbb{P}_{x_0} \{\tau_b(c) < \infty\}$, which means that, in the current situation, bankruptcy happens mainly due to the firm value constantly staying below level b rather than down-crossing level a .

Next, we propose a reorganization plan during bankruptcy. Assume that

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad t > 0,$$

where

$$\begin{aligned} \mu(x) &= \mu x 1_{\{x > b\}} + \mu x \left(1 - \frac{b-x}{2(b-a)}\right) 1_{\{a \leq x \leq b\}}, \\ \sigma(x) &= \sigma x 1_{\{x > b\}} + \sigma x \left(1 - \frac{b-x}{2(b-a)}\right) 1_{\{a \leq x \leq b\}}. \end{aligned}$$

This reorganization plan also concerns the priority of the debt holder over the shareholders during bankruptcy while the Sharpe ratio $\mu(\cdot)/\sigma(\cdot)$ remains constant. Thus, formula (4.3) is still valid for $q(x_0)$ whereas the auxiliary quantity $A(a, b, c)$ needs to be recalculated.

Table 4.4 The probability of liquidation for a geometric Brownian motion with capital restructuring during bankruptcy

mesh	$A(a, b, c)$	$q(x_0)$	elapsed time (s)
$\Delta x = \Delta t = 0.005$	12.348142	$0.015332581 \times x_0^{-2.2}$	0.567849
$\Delta x = \Delta t = 0.001$	11.970123	$0.015107802 \times x_0^{-2.2}$	4.432082
$\Delta x = \Delta t = 0.0005$	11.922681	$0.015079068 \times x_0^{-2.2}$	32.458421
$\Delta x = \Delta t = 0.00025$	11.898945	$0.015064648 \times x_0^{-2.2}$	258.392974

Comparing Tables 4.3 and 4.4, we notice that this reorganization plan slightly increases $q(x_0)$. The numerical results in Sections 4.1 and 4.2 show that the effect of capital restructuring during bankruptcy on the probability of liquidation is ambiguous but intriguing.

5 Proofs

To make our presentation self-contained, we briefly introduce some norms of Hölder spaces that will be used in the proof of Proposition 3.1. The reader is referred to Chapter 4 of Lieberman (1996) for general discussions.

Define $U_2 = (0, 1) \times (0, 2]$ and denote by $\bar{U}_2 = [0, 1] \times [0, 2]$ its closure. Note that the closure is taken in the parabolic topology generated by the parabolic distance $d(p_1, p_2) = (|x_1 - x_2|^2 + |t_1 - t_2|)^{1/2}$ between any two points $p_1 = (x_1, t_1)$ and $p_2 = (x_2, t_2)$. The parabolic boundary of U_2 is denoted by $\mathcal{P}U_2 = \bar{U}_2 \setminus U_2$. Furthermore, define a closed set $Q = [0, 1] \times [1/2, 3/2]$. For a nonnegative integer k and $\alpha \in (0, 1]$, define the norm

$$\begin{aligned} \|u\|_{C^{k+\alpha, (k+\alpha)/2}(Q)} &= \sum_{\beta+2j \leq k} \sup_{(x,t) \in Q} |D_x^\beta D_t^j u| + \sum_{\beta+2j=k} \sup_{p_1, p_2 \in Q, p_1 \neq p_2} \frac{|D_x^\beta D_t^j u(p_1) - D_x^\beta D_t^j u(p_2)|}{(d(p_1, p_2))^\alpha} \\ &\quad + \sum_{\beta+2j=k-1} \sup_{(x,t_1), (x,t_2) \in Q} \frac{|D_x^\beta D_t^j u(x, t_1) - D_x^\beta D_t^j u(x, t_2)|}{|t_1 - t_2|^{(1+\alpha)/2}}, \end{aligned}$$

where D_x^β is the β -th derivative operator with respect to x and D_t^j is the j -th derivative operator with respect to t for some nonnegative integers β and j . A function u is said to belong to some Hölder space if the corresponding norm is finite.

Proof of Proposition 3.1. Without loss of generality we assume that 0 and 1 are possible values of X , and for simplicity we set $a = 0$, $b = 1$, and $c = 1$. For $\varepsilon \geq 0$, we denote

$$v^\varepsilon(x, t) = \mathbb{P}_x \{T_{1+\varepsilon} > T_0 \wedge t\}, \quad 0 \leq x \leq 1 + \varepsilon \text{ and } t \geq 0.$$

It is well known that $v^\varepsilon(x, t)$ solves

$$\begin{cases} v_t^\varepsilon(x, t) = \frac{\sigma^2(x)}{2} v_{xx}^\varepsilon(x, t) + \mu(x) v_x^\varepsilon(x, t), & 0 < x < 1 + \varepsilon \text{ and } t > 0, \\ v^\varepsilon(0, t) = 1, & t \geq 0, \\ v^\varepsilon(1 + \varepsilon, t) = 0, & t \geq 0, \\ v^\varepsilon(x, 0) = 1, & 0 < x < 1 + \varepsilon. \end{cases}$$

Since the domain of $v^\varepsilon(\cdot, \cdot)$ varies in $\varepsilon \geq 0$, we use a scaling transform to define $u^\varepsilon(x, t) = v^\varepsilon(x + \varepsilon x, t)$ for $0 \leq x \leq 1$ and $t \geq 0$. It follows that $u^\varepsilon(x, t)$ solves

$$\begin{cases} u_t^\varepsilon(x, t) = \frac{\sigma^2(x+\varepsilon x)}{2(1+\varepsilon)^2} u_{xx}^\varepsilon(x, t) + \frac{\mu(x+\varepsilon x)}{1+\varepsilon} u_x^\varepsilon(x, t), & 0 < x < 1 \text{ and } t > 0, \\ u^\varepsilon(0, t) = 1, & t \geq 0, \\ u^\varepsilon(1, t) = 0, & t \geq 0, \\ u^\varepsilon(x, 0) = 1, & 0 < x < 1. \end{cases} \quad (5.1)$$

By (2.4) and the Schauder estimates (see e.g., Theorem 4.22 of Lieberman (1996)), it holds for any $\varepsilon \geq 0$, any $\alpha \in (0, 1]$, and some absolute constant $K > 0$ that

$$\|u^\varepsilon\|_{C^{2+\alpha, 1+\alpha/2}(\mathbb{Q})} \leq K \sup_{(x,t) \in \mathcal{P}U_2} |u^\varepsilon| = K. \quad (5.2)$$

We slightly abuse the notation K to denote an absolute constant which may vary from place to place. Relation (5.2) immediately implies the existence of the limit $\lim_{\varepsilon \downarrow 0} \frac{u^0(1-\varepsilon, 1)}{\varepsilon}$.

To prove that $\lim_{\varepsilon \downarrow 0} \frac{u^\varepsilon((1+\varepsilon)^{-1}, 1)}{\varepsilon}$ also exists and is equal to $\lim_{\varepsilon \downarrow 0} \frac{u^0(1-\varepsilon, 1)}{\varepsilon}$, it suffices to show that

$$|u^\varepsilon((1+\varepsilon)^{-1}, 1) - u^0(1-\varepsilon, 1)| = o(\varepsilon). \quad (5.3)$$

Actually, by (5.2) we have

$$|u^\varepsilon((1+\varepsilon)^{-1}, 1) - u^\varepsilon(1-\varepsilon, 1)| \leq K((1+\varepsilon)^{-1} - (1-\varepsilon)) \leq K\varepsilon^2. \quad (5.4)$$

Write $w(x, t) = u^\varepsilon(x, t) - u^0(x, t)$ for $0 \leq x \leq 1$ and $t \geq 0$. By (5.1), $w(x, t)$ solves

$$\begin{cases} w_t(x, t) = \frac{\sigma^2(x)}{2} w_{xx}(x, t) + \mu(x) w_x(x, t) + f^\varepsilon(x, t), & 0 < x < 1 \text{ and } t > 0, \\ w(0, t) = 0, & t \geq 0, \\ w(1, t) = 0, & t \geq 0, \\ w(x, 0) = 0, & 0 < x < 1, \end{cases}$$

where

$$f^\varepsilon(x, t) = \left(\frac{\sigma^2(x+\varepsilon x)}{2(1+\varepsilon)^2} - \frac{\sigma^2(x)}{2} \right) u_{xx}^\varepsilon(x, t) + \left(\frac{\mu(x+\varepsilon x)}{1+\varepsilon} - \mu(x) \right) u_x^\varepsilon(x, t).$$

By (2.4) and (5.2), it is straightforward to verify that,

$$\sup_{(x,t) \in \mathbb{Q}} \|f^\varepsilon\| \leq K\varepsilon.$$

By $W^{2,p}$ estimates (e.g. Theorem 7.32 of Lieberman (1996)) and then Sobolev embedding theorem (e.g. part II of Theorem 4.12 of Adams (1975)), for any $0 < \alpha < 1$, we have

$$\|w\|_{C^{1+\alpha, (1+\alpha)/2}(\mathbb{Q})} \leq K \sup_{(x,t) \in \mathbb{Q}} \|f^\varepsilon\| = K\varepsilon.$$

It follows that

$$|w(1 - \varepsilon, 1)| = |w(1 - \varepsilon, 1) - w(1, 1)| \leq \varepsilon \sup_{x \in [0,1]} |w_x(x, 1)| = K\varepsilon^2. \quad (5.5)$$

Relations (5.4) and (5.5) imply the desired result (5.3), and this ends the proof. ■

Proof of Proposition 3.2. We construct two approximations for the Parisian stopping time $\tau_b(c)$. Denote by θ the Markov shift operator such that $X_t \circ \theta_s = X_{s+t}$. For arbitrarily small $\varepsilon > 0$ and $i = 1, 2, \dots$, since $X_0 = b$, we define two sequences of exit times as follows:

$$\eta_1^+ = 0, v_1^+ = T_{b+\varepsilon}, \dots, \eta_{i+1}^+ = v_i^+ + T_b \circ \theta_{v_i^+}, v_{i+1}^+ = \eta_{i+1}^+ + T_{b+\varepsilon} \circ \theta_{\eta_{i+1}^+}$$

and

$$\eta_1^- = 0, v_1^- = T_{b-\varepsilon}, \dots, \eta_{i+1}^- = v_i^- + T_b \circ \theta_{v_i^-}, v_{i+1}^- = \eta_{i+1}^- + T_{b-\varepsilon} \circ \theta_{\eta_{i+1}^-}.$$

Further, we define two stopping times

$$\tau_b^{\varepsilon+}(c) = \inf \{t \in (\eta_i^+, v_i^+] : t - \eta_i^+ \geq c \text{ for some } i = 1, 2, \dots\}$$

and

$$\tau_b^{\varepsilon-}(c) = \inf \{t \in (v_i^-, \eta_{i+1}^-] : t - v_i^- \geq c \text{ for some } i = 1, 2, \dots\}.$$

From their definitions, it is easy to see $\tau_b^{\varepsilon+}(c)$ is monotone decreasing in ε and $\tau_b^{\varepsilon-}(c)$ is monotone increasing in ε . Moreover, we also have the inclusive relation

$$\tau_b^{\varepsilon+}(c) \leq \tau_b(c) \leq \tau_b^{\varepsilon-}(c), \quad (5.6)$$

almost surely for any $\varepsilon \geq 0$. Correspondingly, we define the probabilities

$$p_{\pm}^{\varepsilon}(x) = \mathbb{P}_x \{T_a \wedge \tau_b^{\varepsilon\pm}(c) = \infty\}, \quad \text{for } x > a. \quad (5.7)$$

By (5.6), it follows that

$$p_+^{\varepsilon}(x) \leq p(x) \leq p_-(x), \quad \text{for } x > a. \quad (5.8)$$

By the strong Markov property of X , we obtain

$$\begin{aligned} p_+^{\varepsilon}(b) &= \mathbb{P}_b \{T_{b+\varepsilon} < T_a \wedge c\} \mathbb{P}_{b+\varepsilon} \{T_a \wedge \tau_b^{\varepsilon}(c) = \infty\} \\ &= \mathbb{P}_b \{T_{b+\varepsilon} < T_a \wedge c\} p_+^{\varepsilon}(b + \varepsilon). \end{aligned} \quad (5.9)$$

Further, by (2.8),

$$\begin{aligned} p_+^{\varepsilon}(b + \varepsilon) &= \mathbb{P}_{b+\varepsilon} \{T_b = \infty\} + \mathbb{P}_{b+\varepsilon} \{T_b < \infty\} p_+^{\varepsilon}(b) \\ &= \frac{\int_b^{b+\varepsilon} G(y) dy}{\int_b^{\infty} G(y) dy} + \frac{\int_{b+\varepsilon}^{\infty} G(y) dy}{\int_b^{\infty} G(y) dy} p_+^{\varepsilon}(b). \end{aligned} \quad (5.10)$$

Substituting (5.10) to (5.9) and solving for $p_+^\varepsilon(b)$, we have

$$\begin{aligned} p_+^\varepsilon(b) &= \frac{\mathbb{P}_b \{T_{b+\varepsilon} < T_a \wedge c\} \int_b^{b+\varepsilon} G(y)dy}{\int_b^\infty G(y)dy - \mathbb{P}_b \{T_{b+\varepsilon} < T_a \wedge c\} \int_{b+\varepsilon}^\infty G(y)dy} \\ &= \frac{\mathbb{P}_b \{T_{b+\varepsilon} < T_a \wedge c\} \int_b^{b+\varepsilon} G(y)dy}{\int_b^{b+\varepsilon} G(y)dy + \mathbb{P}_b \{T_{b+\varepsilon} > T_a \wedge c\} \int_{b+\varepsilon}^\infty G(y)dy}. \end{aligned}$$

Dividing ε on the numerator and denominator above and letting $\varepsilon \downarrow 0$, by (5.8), it follows that

$$p(b) \geq \lim_{\varepsilon \downarrow 0} p_+^\varepsilon(b) = \frac{G(b)}{G(b) + A(a, b, c) \int_b^\infty G(y)dy}.$$

Similarly, by (5.7), we can show that

$$p_-^\varepsilon(b) = \frac{G(b)}{G(b) + \frac{\mathbb{P}_{b-\varepsilon} \{T_b > T_a \wedge c\}}{\varepsilon} \int_b^\infty G(y)dy}.$$

By Proposition 3.1 and (5.8), it follows that

$$p(b) \leq \lim_{\varepsilon \downarrow 0} p_-^\varepsilon(b) = \frac{G(b)}{G(b) + A(a, b, c) \int_b^\infty G(y)dy}.$$

This completes the proof. ■

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