Bivariate Regular Variation among Randomly Weighted Sums in General Insurance

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Abstract

The tail behavior of randomly weighted sums has become an increasingly interesting topic in applied probability and this study has played an important role in a few problems in insurance, finance, and risk management. In this paper, we extend the study to the case of non-standard bivariate regular variation and, as applications, we interpret the study in terms of bivariate processes of aggregate claims (without interest rate, with a constant force of interest, or with stochastic investment returns).

Keywords: bivariate regular variation; randomly weighted sums; asymptotics; limit measure; aggregate claims

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1 Introduction

The tail behavior of randomly weighted sums has become an increasingly interesting topic in applied probability and this study has played an important role in a few problems in insurance, finance, and risk management. See Resnick and Willekens (1991), Tang and Tsitsiashvili (2003b), Goovaerts et al. (2005), Hult and Samorodnitsky (2008), Zhang et al. (2009), Fougeres and Mercadier (2012), Olvera-Cravioto (2012), and Tang and Yuan (2014), to name a few that are closely related to the current study.

Let $\{X_i; i \in \mathbb{N}\}$ be a sequence of independent, identically distributed (i.i.d.), and nonnegative random variables with generic random variable X, and let $\{\xi_i; i \in \mathbb{N}\}$ be another sequence of nonnegative dependent random variables, independent of the former random variables. This study focuses on the tail behavior of

$$S_\infty = \sum_{i=1}^{\infty} \xi_i X_i, \tag{1.1}$$

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usually under a heavy-tailed condition on X and suitable moment conditions on $\{\xi_i; i \in \mathbb{N}\}$. In (1.1), if each X_i is interpreted as the amount of a benefit or a claim during period i and each ξ_i as the stochastic present value factor over the first i periods, then S_∞ is interpreted as the aggregate stochastic present value (time 0 equivalent value) of future benefits (namely, a perpetuity) or of future claims, which is often a key quantity to analyze in risk theory.

A well-known result in this study, which corresponds to the univariate case of Theorem 2.1 of Resnick and Willekens (1991), is the following:

Proposition 1.1 *Assume that* $X \in \mathrm{RV}_{-\alpha}$ *for some* $\alpha > 0$ (*see Section 2 for its definition*) *and that* $\{\xi_i; i \in \mathbb{N}\}$ *satisfy the following moment conditions (which we denote as* $\mathbb{M}_\alpha$ *for convenience to quote later):*

- *in case* $\alpha \in (0,1)$, $\sum_{i=1}^{\infty} E\left[\max\left\{\xi_i^{\alpha-\varepsilon}, \xi_i^{\alpha+\varepsilon}\right\}\right] < \infty$ *for some small* $\varepsilon > 0$;
- *in case* $\alpha \in [1,\infty)$, $\sum_{i=1}^{\infty} \left(E\left[\max\left\{\xi_i^{\alpha-\varepsilon}, \xi_i^{\alpha+\varepsilon}\right\}\right]\right)^{\frac{1}{\alpha+\varepsilon}} < \infty$ *for some small* $\varepsilon > 0$.

Then

$$\lim_{x\to\infty} \frac{P\left(S_\infty > x\right)}{P(X > x)} = \sum_{i=1}^{\infty} E\left[\xi_i^{\alpha}\right]. \tag{1.2}$$

Such an asymptotic result has immediate implications for a few problems in insurance, finance, and risk management. For example, in risk theory, for various cases with heavy-tailed insurance claims and with/without risky investments, the probability of ruin usually has the same asymptotic behavior as $P\left(S_\infty > x\right)$, and thus relation (1.2) readily shows an approximation $P(X > x)\sum_{i=1}^{\infty} E\left[\xi_i^{\alpha}\right]$ for the probability of ruin. See, for example, Tang (2005) for such a treatment; see also Paulsen (2008) and Asmussen and Albrecher (2010) for reviews of some early works on this study.

For another example, following the aforementioned interpretation of S_∞ as the aggregate stochastic present value of future insurance claims, the randomly weighted sum in (1.1) serves as an effective platform for the interplay of two fundamental risk sources faced by nowadays insurance business, namely, insurance risk caused by insurance claims and financial risk due to risky investments. In this way, S_∞ becomes a risk management tool. The study of its tail behavior lends important insights on the control of both risks to an insurance risk manager who manages its insurance and investment portfolios for maximizing gains. Originating from the time-honored subject of portfolio optimization in finance, it has developed to be an active research topic in insurance, so called portfolio optimization under solvency constraints; see Tang and Yuan (2012), Asanga et al. (2014), and Asimit et al. (2014) for recent discussions on this topic in the insurance context. Under the help of the approximation for $P\left(S_\infty > x\right)$ as given in (1.2), those tail-related risk measures such as Value-at-Risk and Tail-Value-at-Risk of S_∞, used to describe various solvency constraints, are readily computable with precision at an acceptable level.

In this paper, we will extend the study to a bivariate case in which the primary random variables form a sequence of i.i.d. nonnegative random pairs possessing a non-standard bivariate regular variation structure, and the random weights are independent of the primary random variables and satisfy certain moment conditions similar to the one in Proposition 1.1. As applications, we will interpret the study in various situations in general insurance. The pursued extension is of vital meaning for academics, practitioners, and regulators, as in practice every insurance company runs multiple insurance business lines and it needs to closely monitor and analyze the risk of the multiple lines and allocate risk capital among them. To keep the paper short, we only consider the bivariate case, but we would like to point out that the extension to multivariate cases is straightforward. After establishing two general theoretical results showing the bivariate regular variation structure for two associated randomly weighted sums, we interpret the results in terms of a bivariate process of aggregate claims, a bivariate aggregate stochastic present value process (according to either a constant force of interest or stochastic investment returns) of future claims, and so on.

The rest of the paper is organized as follows. Section 2 introduces the concepts of univariate and bivariate regular variation, Section 3 establishes as main results the bivariate regular variation structure for a two associated randomly weighted sums, Section 4 applies the study to various situations in general insurance and conducts numerical studies to examine the accuracy of the obtained formulas, and finally Section 5 collects all proofs of the main results, their corollaries, as well as two needed lemmas.

2 Regular variation

Throughout the paper, all limit relations are according to $x \to \infty$ unless otherwise stated. For two positive functions f_1 and f_2, write $f_1 \sim f_2$ if $\lim f_1/f_2 = 1$. For any $p, q \in \mathbb{R}$, write $p \vee q = \max\{p, q\}$. For an event A, denote its indicator function by 1_A. Denote by F_X the distribution of a random variable X, by $\overline{F_X} = 1 - F_X$ the tail of X, and so on.

2.1 Univariate regular variation

A positive function f on $\mathbb{R}_+ = [0, \infty)$ is said to be regularly varying at ∞ with index $\alpha \in \mathbb{R}$, written as $f \in \mathrm{RV}_\alpha$, if

$$\lim_{x\to\infty} \frac{f(xy)}{f(x)} = y^\alpha, \qquad y > 0.$$

When $\alpha = 0$, this defines a slowly varying function at ∞. See Bingham et al. (1987) and Resnick (1987) for textbook treatments of regular variation.

For simplicity, for a random variable X with distribution F_X, write

$$U_X(x) = \left(\frac{1}{\overline{F_X}}\right)^{\leftarrow}(x) = \inf\left\{y \in \mathbb{R} : \frac{1}{\overline{F_X}(y)} \geq x\right\}, \qquad x > 0,$$

where $\inf \emptyset = \infty$ by convention. We are interested in random variables X with a regularly tail $\overline{F_X} \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 0$, for which case we simply write $X \in \mathrm{RV}_{-\alpha}$. This regular variation can be restated as follows. There exists a limit measure ν non-degenerate (i.e., not identically 0) on $\mathbb{R}_+$ such that the limit relation

$$\lim_{x\to\infty} xP\left(\frac{X}{U_X(x)} \in B\right) = \nu(B)$$

holds for every Borel set $B \subset \mathbb{R}_+$ that is bounded away from 0 and ν-continuous. This measure ν is actually given by $\nu(s,\infty) = s^{-\alpha}$ for $s > 0$ and hence is homogeneous in the sense that the relation

$$\nu(\lambda^{\frac{1}{\alpha}} B) = \lambda^{-1}\nu(B) \tag{2.1}$$

holds for any $\lambda > 0$ and any Borel set $B \subset \mathbb{R}_+$ that is away from 0.

The well-known Breiman's theorem states that for two independent nonnegative random variables X and ξ, if $X \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 0$ and $E\left[\xi^{\alpha+\varepsilon}\right] < \infty$ for some $\varepsilon > 0$, then

$$\lim_{x\to\infty} \frac{P\left(\xi X > x\right)}{\overline{F_X}(x)} = E\left[\xi^{\alpha}\right].$$

See Breiman (1965) for the original version of the result and see Cline and Samorodnitsky (1994), Denisov and Zwart (2007), and Fougeres and Mercadier (2012) for related discussions and extensions. To be consistent with the main context of this paper, we rewrite Breiman's theorem in terms of U_X as

$$\lim_{x\to\infty} xP\left(\frac{\xi X}{U_X(x)} > 1\right) = E\left[\xi^{\alpha}\right]. \tag{2.2}$$

2.2 Non-standard bivariate regular variation

A nonnegative random pair (X, Y) is said to follow a non-standard bivariate regular variation (BRV) structure if there exists a limit measure ν non-degenerate on $\mathbb{R}_+^2$ such that the limit relation

$$\lim_{x\to\infty} xP\left(\left(\frac{X}{U_X(x)}, \frac{Y}{U_Y(x)}\right) \in B\right) = \nu(B) \tag{2.3}$$

holds for every Borel set $B \subset \mathbb{R}_+^2$ that is bounded away from $\mathbf{0}$ and ν-continuous. Necessarily, both $\overline{F_X}$ and $\overline{F_Y}$ are regularly varying. We assume $X \in \mathrm{RV}_{-\alpha}$ and $Y \in \mathrm{RV}_{-\beta}$ for some $\alpha, \beta > 0$, for which case we write $(X, Y) \in \mathrm{BRV}_{-\alpha,-\beta}(\nu)$. If F_X and F_Y are identical, so are U_X and U_Y, then relation (2.3) reduces to a standard BRV structure: with U denoting $U_X = U_Y$,

$$\lim_{x\to\infty} xP\left(\frac{(X,Y)}{U(x)} \in B\right) = \nu(B). \tag{2.4}$$

Throughout the paper, unless otherwise stated we always simply write BRV but we mean a non-standard, hence more general, BRV structure. Remarkably, the information of dependence of (X,Y) in the upper-right tail is contained in the limit measure ν. It allows a variety of tail dependence structures for (X,Y) ranging from asymptotic independence, asymptotic dependence at a varying degree, to asymptotic full dependence. The reader is referred to de Haan and Resnick (1981) and Resnick (1987, 2007) for the introduction and comprehensive treatments of multivariate regular variation.

Very recently, Tang and Xiao (2018) show that, similar to (2.1) in the univariate case, the limit measure ν in (2.3) possesses a homogeneity property in the sense that, for any $\lambda > 0$ and any Borel set $B \subset \mathbb{R}_+^2 \backslash \{\mathbf{0}\}$,

$$\nu(B^\lambda) = \lambda^{-1}\nu(B), \tag{2.5}$$

where $B^\lambda = \{(\lambda^{\frac{1}{\alpha}} p, \lambda^{\frac{1}{\beta}} q) : (p,q) \in B\}$. This is also consistent with the homogeneity property known for the standard case; for the latter, see, e.g., page 178 in Resnick (2007) and Theorem 3.1 of Lindskog et al. (2014).

3 BRV of randomly weighted sums

Let $\{(X_i, Y_i); i \in \mathbb{N}\}$ be a sequence of i.i.d. nonnegative random pairs with generic pair (X,Y), interpreted as primary random pairs, and let $\{\xi_i, \eta_i; i \in \mathbb{N}\}$ be another sequence of nonnegative dependent random variables, interpreted as random weights, independent of $\{(X_i, Y_i); i \in \mathbb{N}\}$. Write

$$S_n = \sum_{i=1}^{n} \xi_i X_i \qquad \text{and} \qquad T_n = \sum_{i=1}^{n} \eta_i Y_i, \qquad n \in \mathbb{N} \cup \{\infty\}. \tag{3.1}$$

The dependence between S_n and T_n comes from both the intricate dependence structure between each pair (X_i, Y_i) and the arbitrary dependence structure among the random weights. We have restricted the pairs (X_i, Y_i) to be nonnegative, but we would like to point out that the whole work, subject to some minor and obvious adjustments, is valid for real-valued random pairs (X_i, Y_i).

Here comes our first main result:

Theorem 3.1 *Assume that $(X,Y) \in \mathrm{BRV}_{-\alpha,-\beta}(\nu)$ for some $\alpha, \beta > 0$ and that $E\left[\xi_i^{\alpha+\varepsilon}\right] + E\left[\eta_i^{\beta+\varepsilon}\right] < \infty$ for some $\varepsilon > 0$. Then, for each $n \in \mathbb{N}$, the random pair (S_n, T_n) still* *possesses* $\mathrm{BRV}_{-\alpha,-\beta}(\tilde{\nu})$ *with the limit measure* $\tilde{\nu}$ *characterized by, for* $(p,q) > \mathbf{0}$*,*

$$\tilde{\nu}[\mathbf{0},(p,q)]^c = \frac{1}{p^\alpha}\sum_{i=1}^{n} E\left[\xi_i^\alpha\right] + \frac{1}{q^\beta}\sum_{i=1}^{n} E\left[\eta_i^\beta\right] - \sum_{i=1}^{n} E\left[\nu\left(\left(\frac{p}{\xi_i},\frac{q}{\eta_i}\right),\boldsymbol{\infty}\right]\right]. \tag{3.2}$$

By Lemma 5.2, each weighted pair $(\xi_i X_i, \eta_i Y_i)$ possesses $\mathrm{BRV}_{-\alpha,-\beta}(\tilde{\nu}_i)$ with the limit measure $\tilde{\nu}_i$ characterized by, for $(p,q) > \mathbf{0}$,

$$\tilde{\nu}_i[\mathbf{0},(p,q)]^c = \frac{1}{p^\alpha} E\left[\xi_i^\alpha\right] + \frac{1}{q^\beta} E\left[\eta_i^\beta\right] - E\left[\nu\left(\left(\frac{p}{\xi_i},\frac{q}{\eta_i}\right),\boldsymbol{\infty}\right]\right].$$

Thus, the limit measure $\tilde{\nu}$ of the BRV structure of (S_n, T_n) satisfies

$$\tilde{\nu} = \sum_{i=1}^{n} \tilde{\nu}_i.$$

In the next result we extend Theorem 3.1 to $n = \infty$, which is crucially important for our applications.

Theorem 3.2 *Assume that* $(X,Y) \in \mathrm{BRV}_{-\alpha,-\beta}(\nu)$ *for some* $\alpha,\beta > 0$ *and that* $\{\xi_i; i \in \mathbb{N}\}$ *and* $\{\eta_i; i \in \mathbb{N}\}$ *satisfy the moment conditions* $\mathbb{M}_\alpha$ *and* $\mathbb{M}_\beta$ *introduced in Proposition 1.1, respectively. Then the random pair* (S_∞, T_∞) *still possesses* $\mathrm{BRV}_{-\alpha,-\beta}(\tilde{\nu})$ *with the limit measure* $\tilde{\nu}$ *characterized by, for* $(p,q) > \mathbf{0}$*,*

$$\tilde{\nu}[\mathbf{0},(p,q)]^c = \frac{1}{p^\alpha}\sum_{i=1}^{\infty} E\left[\xi_i^\alpha\right] + \frac{1}{q^\beta}\sum_{i=1}^{\infty} E\left[\eta_i^\beta\right] - \sum_{i=1}^{\infty} E\left[\nu\left(\left(\frac{p}{\xi_i},\frac{q}{\eta_i}\right),\boldsymbol{\infty}\right]\right]. \tag{3.3}$$

A recent study on the topic is Li (2018). As is easily seen, our main results are related to but different from the corresponding results in Li (2018) in several ways. An essential difference exists in that the latter, although considering slightly more general distribution classes than the regular variation class, assumes that X and Y are so-called strongly asymptotically independent, while our study is concentrated on the asymptotic dependence case.

4 Applications to general insurance

4.1 A bivariate risk model

In this section we show various situations in general insurance where the BRV structure is preserved under randomly weighted sums. Consider a general insurance business consisting of two business lines, each generating a flow of claims to be described below.

Suppose that accidents under the insurance coverage occur successively at epochs $0 \le \tau_1 \le \tau_2 \le \cdots$, constituting a general counting process

$$N_t = \sup\{n \in \mathbb{N} : \tau_n \le t\}, \qquad t \ge 0,$$

where $\sup \emptyset = 0$ by convention. Clearly, its mean function is given by

$$E[N_t] = \sum_{i=1}^{\infty} P\left(N_t \geq i\right) = \sum_{i=1}^{\infty} P\left(\tau_i \leq t\right).$$

Among the simplest are homogeneous, nonhomogeneous, and mixed Poisson processes (the latter also known as a Cox process), for which cases the mean function takes an explicit expression. Certain renewal counting processes with i.i.d. interarrival times having a phase-type distribution (of which the Erlang distribution is a special case) also admit an explicit expression for the mean function $E[N_t]$; see, e.g., Asmussen (2003).

Suppose that each accident i simultaneously causes two types of claims, X_i and Y_i. As we will use nonnegative random variables to model these claims, in case only one type of claim is caused, the other type of claim is recorded as zero. Another explanation is that accident i causes one loss that is however split into two parts, X_i and Y_i, to be covered separately by the two lines of the business.

Such a bivariate risk model has been extensively explored in risk theory; see Chan et al. (2003), Cai and Li (2005, 2007), Li et al. (2007), Chen et al. (2011), Hu et al. (2013), Yang and Li (2014) and Foss et al. (2017), to name a few.

Throughout this section, assume that $\{(X_i, Y_i); i \in \mathbb{N}\}$ is a sequence of i.i.d. nonnegative random pairs, with generic pair (X, Y), independent of the general counting process $\{N_t; t \geq 0\}$.

4.2 In a world without economic factors

Consider the bivariate risk model introduced above in a world without economic factors. Then the aggregate amounts of claims of the two types by time t are, respectively,

$$S_{N_t} = \sum_{i=1}^{N_t} X_i \qquad \text{and} \qquad T_{N_t} = \sum_{i=1}^{N_t} Y_i, \qquad t \geq 0. \tag{4.1}$$

The dependence between S_{N_t} and T_{N_t} comes from both the common counting process N_t shared by the two lines and the intricate dependence structure between each pair (X_i, Y_i).

Corollary 4.1 *Assume $(X, Y) \in \mathrm{BRV}_{-\alpha,-\beta}(\nu)$ for some $\alpha, \beta > 0$ and $0 < E[N_t^{\gamma}] < \infty$ for some $\gamma > \alpha \vee \beta \vee 1$. Then the random pair (S_{N_t}, T_{N_t}) still possesses $\mathrm{BRV}_{-\alpha,-\beta}(\tilde{\nu})$ with the limit measure $\tilde{\nu}$ characterized by, for $(p, q) > \mathbf{0}$,*

$$\tilde{\nu}[\mathbf{0}, (p, q)]^c = E[N_t] \left(\frac{1}{p^{\alpha}} + \frac{1}{q^{\beta}} - \nu\left((p, q), \boldsymbol{\infty}\right] \right). \tag{4.2}$$

For an insurer who operates two business lines, given that one line runs into financial stress, the insurer should be keen to whether or not the other line will run into financial stress too. Formally, denote by P^* such a conditional probability that line two runs into financial stress given that line one has already. If the random pair (S_{N_t}, T_{N_t}) possesses $\mathrm{BRV}_{-\alpha,-\beta}(\tilde{\nu})$, then it holds for any fixed $(p,q) > \mathbf{0}$ that

$$\begin{aligned} P^* &= \lim_{x\to\infty} P\left(T_{N_t} > qU_Y(x) | S_{N_t} > pU_X(x)\right) \\ &= \lim_{x\to\infty} \frac{xP\left(S_{N_t} > pU_X(x), T_{N_t} > qU_Y(x)\right)}{xP\left(S_{N_t} > pU_X(x)\right)} \\ &= \frac{\tilde{\nu}\left((p,q), \boldsymbol{\infty}\right]}{\tilde{\nu}\left((p,0), \boldsymbol{\infty}\right]}. \end{aligned} \tag{4.3}$$

Under the conditions of Corollary 4.1, plugging into (4.3) the limit measure $\tilde{\nu}$ characterized by (4.2) yields

$$P^* = \frac{E[N_t]\nu\left((p,q), \boldsymbol{\infty}\right]}{E[N_t]p^{-\alpha}} = p^{\alpha}\nu\left((p,q), \boldsymbol{\infty}\right].$$

4.3 In the presence of a constant force of interest

We continue to consider the bivariate risk model introduced in Subsection 4.1, and we now introduce a constant force of interest r to the model; that is, one dollar invested now becomes e^{rt} dollars at time $t \geq 0$, or one dollar at time $t \geq 0$ is equivalent to e^{-rt} dollar at time 0. Then the aggregate stochastic present values of claims of the two types by time t are, respectively,

$$S_{N_t} = \sum_{i=1}^{N_t} e^{-r\tau_i} X_i \qquad \text{and} \qquad T_{N_t} = \sum_{i=1}^{N_t} e^{-r\tau_i} Y_i, \qquad t \geq 0. \tag{4.4}$$

Corollary 4.2 *Under the conditions of Corollary 4.1, the random pair* (S_{N_t}, T_{N_t}) *still possesses* $\mathrm{BRV}_{-\alpha,-\beta}(\tilde{\nu})$ *with the limit measure* $\tilde{\nu}$ *characterized by, for* $(p,q) > \mathbf{0}$*,*

$$\tilde{\nu}[\mathbf{0}, (p,q)]^c = \int_{0-}^{t} \left(\frac{1}{p^{\alpha} e^{\alpha rs}} + \frac{1}{q^{\beta} e^{\beta rs}} - \nu\left((pe^{rs}, qe^{rs}), \boldsymbol{\infty}\right] \right) dE[N_s]. \tag{4.5}$$

If (X,Y) follows the standard BRV structure (2.4) so that $\alpha = \beta$, the homogeneity property (2.1) of ν implies that $\nu\left((pe^{rs}, qe^{rs}), \boldsymbol{\infty}\right] = e^{-\alpha rs}\nu\left((p,q), \boldsymbol{\infty}\right]$. Thus, the limit measure given by (4.5) is simplified to

$$\tilde{\nu}[\mathbf{0}, (p,q)]^c = \left(\frac{1}{p^{\alpha}} + \frac{1}{q^{\alpha}} - \nu\left((p,q), \boldsymbol{\infty}\right] \right) \int_{0-}^{t} e^{-\alpha rs} dE[N_s].$$

Still consider the conditional probability P^* in (4.3). In the current situation of Corollary 4.2, plugging into (4.3) the limit measure $\tilde{\nu}$ characterized by (4.5) yields

$$P^* = \frac{\int_{0-}^{t} \nu\left((pe^{rs}, qe^{rs}), \boldsymbol{\infty}\right] dE[N_s]}{\int_{0-}^{t} p^{-\alpha} e^{-\alpha rs} dE[N_s]}.$$

Similarly to the above, under the standard BRV structure (2.4) for (X,Y), this can be simplified to

$$P^* = p^{\alpha}\nu\left((p,q), \boldsymbol{\infty}\right].$$

4.4 In the presence of stochastic returns on risky investments

Now suppose that the insurer makes investments into some risk-free and risky assets and earns stochastic returns. Denote by A_i^X and A_i^Y the stochastic accumulation factors of the two lines based on respective overall returns over each period i. Note that both A_i^X and A_i^Y are positive random variables. Then

$$\xi_i = \left(\prod_{j=1}^{i} A_j^X\right)^{-1} \qquad \text{and} \qquad \eta_i = \left(\prod_{j=1}^{i} A_j^Y\right)^{-1}, \qquad i \in \mathbb{N}, \tag{4.6}$$

represent the stochastic present value factors of the two lines, respectively, over the first i periods. Furthermore, the randomly weighted sums

$$S_n = \sum_{i=1}^{n} \xi_i X_i \qquad \text{and} \qquad T_n = \sum_{i=1}^{n} \eta_i Y_i, \qquad n \in \mathbb{N} \cup \{\infty\},$$

represent the aggregate stochastic present values of the claims along the two lines, respectively, within the first n periods. This discrete-time risk model has been extensively explored in risk theory after the work of Tang and Tsitsiashvili (2003a), but most of the study focuses on the one-dimensional case only.

Considering that the two business lines coexist in the same external economic environment, it now becomes crucially important to allow ξ_i and η_i for $i \in \mathbb{N}$ defined in (4.6) to be arbitrarily dependent on each other, of which we have taken care throughout this study. In conclusion, Theorems 3.1–3.2 are readily applicable to this situation and yield the BRV structure of (S_n, T_n) for $n \in \mathbb{N} \cup \{\infty\}$.

The expressions (3.2)–(3.3) for the limit measure given by Theorems 3.1–3.2, especially the last terms, are rather involved for the general case. Now we show a special case which admits a substantial simplification. Assume that the accumulation factors A_i^X and A_i^Y for both lines are identical, so are the stochastic present value factors ξ_i and η_i. This assumption would be acceptable if the insurer collects gains from both lines and makes investments all together. Further assume that (X, Y) follows the standard BRV structure (2.4) so that $\alpha = \beta$, for which case the homogeneity property (2.1) of ν implies that

$$E\left[\nu\left(\left(\left(\frac{p}{\xi_i}, \frac{q}{\xi_i}\right), \boldsymbol{\infty}\right]\right)\right] = E\left[\xi_i^\alpha\right] \nu\left(\left(p, q\right), \boldsymbol{\infty}\right], \qquad i \in \mathbb{N}.$$

Thus, the expressions (3.2)–(3.3) for the limit measure $\tilde{\nu}$ are simplified to

$$\tilde{\nu}[\mathbf{0}, (p, q)]^c = \left(\frac{1}{p^\alpha} + \frac{1}{q^\alpha} - \nu\left(\left(p, q\right), \boldsymbol{\infty}\right]\right) \sum_{i=1}^{n} E\left[\xi_i^\alpha\right], \qquad n \in \mathbb{N} \cup \{\infty\}.$$

In this idealized situation, the conditional probability P^* as that in (4.3) again takes the form

$$P^* = \lim_{x \to \infty} P\left(T_n > qU_Y(x) | S_n > pU_X(x)\right) = \frac{\tilde{\nu}\left((p, q), \boldsymbol{\infty}\right]}{\tilde{\nu}\left((p, 0), \boldsymbol{\infty}\right]} = p^\alpha \nu\left((p, q), \boldsymbol{\infty}\right].$$

4.5 Numerical studies

In this subsection we conduct numerical studies to examine the accuracy of the asymptotic expressions for P^* defined in (4.3) in Subsections 4.2–4.3. For this purpose, we compare the conditional tail probability $P\left(T_{N_t} > qU_Y(x) | S_{N_t} > pU_X(x)\right)$ obtained by crude Monte Carlo (CMC) simulation with the limit $P^* = p^{\alpha}\nu\left((p,q),\boldsymbol{\infty}\right]$ obtained by the asymptotic formula.

As can be easily verified, a random pair (X,Y) possessing a Gumbel copula and regularly varying marginal tails follows the BRV structure (2.3); see also Subsection 5.1 of Tang and Yuan (2003) and Section 4 of Tang and Yang (2018). Recall that a Gumbel copula is of the form

$$C(u,v) = \exp\left\{-\left((-\log u)^{\rho} + (-\log v)^{\rho}\right)^{\frac{1}{\rho}}\right\}, \qquad (u,v) \in [\mathbf{0},\mathbf{1}],$$

where the parameter $\rho \geq 1$ controls the strength of the tail dependence. Let X and Y follow Pareto distributions

$$F_X(x) = 1 - \left(\frac{\theta_1}{x+\theta_1}\right)^{\alpha}, \qquad F_Y(x) = 1 - \left(\frac{\theta_2}{x+\theta_2}\right)^{\beta}, \qquad x > 0,$$

with parameters $\alpha, \beta, \theta_1, \theta_2 > 0$, so that $\overline{F_X} \in \mathrm{RV}_{-\alpha}$ and $\overline{F_Y} \in \mathrm{RV}_{-\beta}$. Then $(X,Y) \in \mathrm{BRV}_{-\alpha,-\beta}(\nu)$ with the limit measure ν satisfying

$$\nu[\mathbf{0},(p,q)]^c = \left(p^{-\alpha\rho} + q^{-\beta\rho}\right)^{\frac{1}{\rho}}, \qquad (p,q) > \mathbf{0};$$

see Tang and Yuan (2003) or Tang and Yang (2018).

Furthermore, the counting process $\{N_t; t \geq 0\}$ is specified to a Poisson process with intensity $\lambda > 0$.

We first consider the risk model without economic factors in Subsection 4.2. The various parameters are set to:

- $\rho = 9$;
- $\alpha = 1.6$, $\beta = 1.3$, $\theta_1 = \theta_2 = 10$;
- $\lambda = 100$, $t = 10$;
- $(p,q) = (0.2, 0.3)$.

The asymptotic estimate $P^* = p^{\alpha}\nu\left((p,q),\boldsymbol{\infty}\right]$ is computed by numerical integration and calculated to be 0.36. For the CMC estimation, we first generate the value of N_t and then generate m samples $\{(X_i^{(k)}, Y_i^{(k)}); i = 1,\ldots,N_t\}$ for $k = 1,\ldots,m$. The conditional tail probability $P\left(T_{N_t} > qU_Y(x) | S_{N_t} > pU_X(x)\right)$ is estimated by

$$\frac{\sum_{k=1}^{m} 1_{\left(S_{N_t}^{(k)} > pU_X(x),\, T_{N_t}^{(k)} > qU_Y(x)\right)}}{\sum_{k=1}^{m} 1_{\left(S_{N_t}^{(k)} > pU_X(x)\right)}},$$

where the random pairs $(S_{N_t}^{(k)}, T_{N_t}^{(k)})$ for $k = 1, \ldots, m$ are i.i.d. copies of (S_{N_t}, T_{N_t}) defined by (4.1).

In Figure 4.1, we compare the CMC estimate for $P(T_{N_t} > qU_Y(x)|S_{N_t} > pU_X(x))$ with the asymptotic estimate $P^* = p^\alpha \nu((p,q), \boldsymbol{\infty}]$ given by (4.3) on the left and show their ratio on the right. The CMC simulation is conducted with a sample of size $m = 10^8$. From the figure, we see that as x increases from $10^{7.5}$ to $10^{9.5}$ the conditional tail probability gradually becomes close to the asymptotic estimate. The fluctuation when x is large is due to the poor performance of the CMC method.

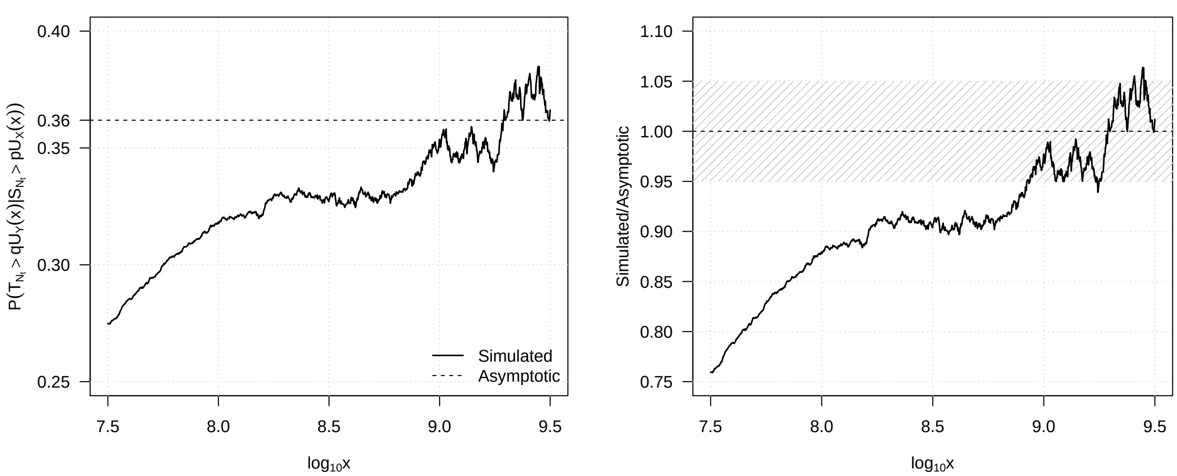

Figure 4.1: Comparison of the simulated conditional tail probability with the asymptotic estimate in Subsection 4.2

We next consider the risk model with a constant force of interest r in Subsection 4.3. The various parameters are set to:

- $\rho = 8$;
- $\alpha = \beta = 1.4$, $\theta_1 = \theta_2 = 10$;
- $\lambda = 100$, $t = 30$;
- $r = 0.05$;
- $(p, q) = (0.2, 0.3)$.

The asymptotic estimate $P^* = p^\alpha \nu((p,q), \boldsymbol{\infty}]$ is computed by numerical integration and calculated to be 0.52. The procedure of the simulated estimation with (S_{N_t}, T_{N_t}) defined by (4.4) is similar to the previous case.

In Figure 4.2, we compare the CMC estimate for $P(T_{N_t} > qU_Y(x)|S_{N_t} > pU_X(x))$ with the asymptotic estimate $P^* = p^\alpha \nu((p,q), \boldsymbol{\infty}]$ on the left and show their ratio on the right. The simulated estimate is obtained with a sample of size $m = 10^8$ and x from 10^7 to 10^{10}.

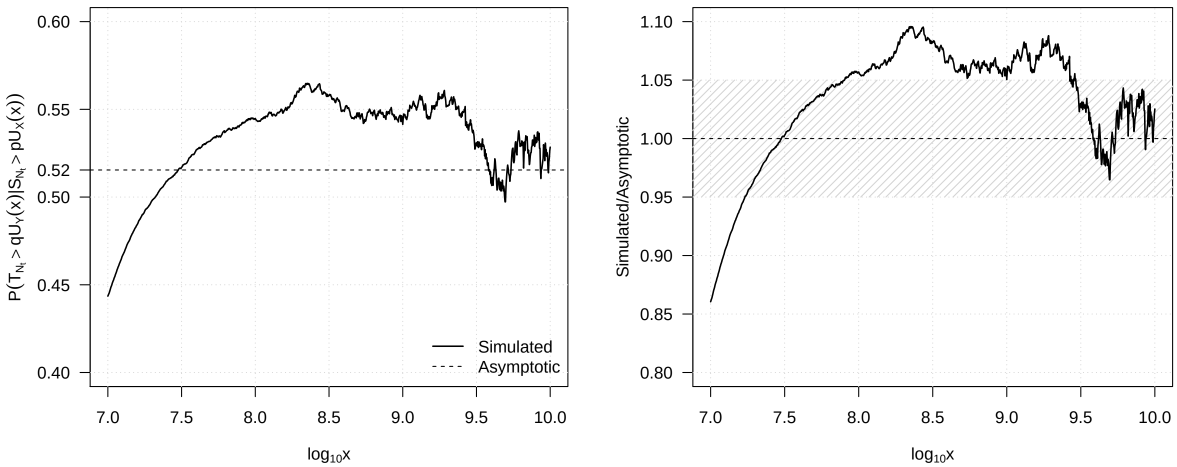

Figure 4.2: Comparison of the simulated conditional tail probability with the asymptotic estimate in Subsection 4.3

5 Proofs

5.1 Lemmas

The following lemma can be easily proven by applying Breiman's theorem as quoted in (2.2); see also Lemma 7 of Tang and Yuan (2014) for a slightly extended version:

Lemma 5.1 *Let $X \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 0$, let ξ be a nonnegative random variable with $E[\xi^{\alpha+\varepsilon}] < \infty$ for some $\varepsilon > 0$, let $\{\Delta_x; x \in \mathbb{R}_+\}$ be a set of random events satisfying $\lim_{x\to\infty} P(\Delta_x) = 0$, and let $\{\xi, \{\Delta_x; x \in \mathbb{R}_+\}\}$ be independent of X. Then*

$$\lim_{x\to\infty} xP\left(\frac{\xi X}{U_X(x)} > 1, \Delta_x\right) = 0.$$

The following lemma shows that the BRV structure can be preserved when being randomly weighted. The lemma extends Proposition A.1 of Basrak et al. (2002) and Theorem 1 of Fougeres and Mercadier (2012) to the non-standard case.

Lemma 5.2 *Let two nonnegative random pairs (X, Y) and (ξ, η) be independent of each other. If $(X, Y) \in \mathrm{BRV}_{-\alpha,-\beta}(\nu)$ for some $\alpha, \beta > 0$ and $E[\xi^{\alpha+\varepsilon}] + E[\eta^{\beta+\varepsilon}] < \infty$ for some $\varepsilon > 0$, then the random pair $(\xi X, \eta Y)$ still possesses $\mathrm{BRV}_{-\alpha,-\beta}(\tilde{\nu})$ with the limit measure $\tilde{\nu}$ characterized by, for $(p, q) > \mathbf{0}$,*

$$\tilde{\nu}[\mathbf{0}, (p, q)]^c = \frac{E[\xi^\alpha]}{p^\alpha} + \frac{E[\eta^\beta]}{q^\beta} - E\left[\nu\left(\left(\frac{p}{\xi}, \frac{q}{\eta}\right), \boldsymbol{\infty}\right]\right].$$

Proof. It follows straightforwardly from Breiman's theorem as quoted in (2.2) that

$$\lim_{x\to\infty} xP\left(\frac{\xi X}{U_X(x)} > p\right) = \frac{E[\xi^\alpha]}{p^\alpha}, \qquad \lim_{x\to\infty} xP\left(\frac{\eta Y}{U_Y(x)} > q\right) = \frac{E[\eta^\beta]}{q^\beta}. \tag{5.1}$$

Moreover, for arbitrarily fixed $M > 0$, we do the split

$$\begin{aligned}
& xP\left(\frac{\xi X}{U_X(x)} > p, \frac{\eta Y}{U_Y(x)} > q\right) \\
= & xP\left(\frac{\xi X}{U_X(x)} > p, \frac{\eta Y}{U_Y(x)} > q, (\xi \leq M) \cap (\eta \leq M)\right) \\
& + xP\left(\frac{\xi X}{U_X(x)} > p, \frac{\eta Y}{U_Y(x)} > q, (\xi > M) \cup (\eta > M)\right) \\
= & I_1 + I_2.
\end{aligned}$$

For I_1, by the dominated convergence theorem,

$$\begin{aligned}
\lim_{x\to\infty} I_1 &= \lim_{x\to\infty} \iint_{0<y,z\leq M} xP\left(\frac{yX}{U_X(x)} > p, \frac{zY}{U_Y(x)} > q\right) P\left(\xi \in dy, \eta \in dz\right) \\
&= \iint_{0<y,z\leq M} \lim_{x\to\infty} xP\left(\frac{yX}{U_X(x)} > p, \frac{zY}{U_Y(x)} > q\right) P\left(\xi \in dy, \eta \in dz\right) \\
&= \iint_{0<y,z\leq M} \nu\left(\left(\frac{p}{y}, \frac{q}{z}\right), \boldsymbol{\infty}\right] P\left(\xi \in dy, \eta \in dz\right) \\
&\to E\left[\nu\left(\left(\frac{p}{\xi}, \frac{q}{\eta}\right), \boldsymbol{\infty}\right]\right], \qquad M \uparrow \infty.
\end{aligned}$$

For I_2, by Breiman's theorem,

$$\begin{aligned}
I_2 &\leq xP\left(\frac{\xi X}{U_X(x)} > p, \xi > M\right) + xP\left(\frac{\eta Y}{U_Y(x)} > q, \eta > M\right) \\
&\to \frac{1}{p^\alpha} E\left[\xi^\alpha 1_{(\xi>M)}\right] + \frac{1}{q^\beta} E\left[\eta^\beta 1_{(\eta>M)}\right] \\
&\to 0, \qquad M \uparrow \infty.
\end{aligned}$$

It follows that

$$\lim_{x\to\infty} xP\left(\frac{\xi X}{U_X(x)} > p, \frac{\eta Y}{U_Y(x)} > q\right) = E\left[\nu\left(\left(\frac{p}{\xi}, \frac{q}{\eta}\right), \boldsymbol{\infty}\right]\right].$$

Combining this with the two relations in (5.1), we obtain

$$\lim_{x\to\infty} xP\left(\left(\frac{\xi X}{U_X(x)}, \frac{\eta Y}{U_Y(x)}\right) \in [\mathbf{0}, (p,q)]^c\right) = \frac{E\left[\xi^\alpha\right]}{p^\alpha} + \frac{E\left[\eta^\beta\right]}{q^\beta} - E\left[\nu\left(\left(\frac{p}{\xi}, \frac{q}{\eta}\right), \boldsymbol{\infty}\right]\right].$$

This concludes the proof. ■

5.2 Proof of Theorem 3.1

By some well-known results in the study of randomly weighted sums, we have

$$\lim_{x\to\infty} xP\left(\frac{S_n}{U_X(x)} > p\right) = \frac{1}{p^\alpha} \sum_{i=1}^n E\left[\xi_i^\alpha\right], \qquad \lim_{x\to\infty} xP\left(\frac{T_n}{U_Y(x)} > q\right) = \frac{1}{q^\beta} \sum_{i=1}^n E\left[\eta_i^\beta\right];$$

see, e.g., Theorem 3 of Tang and Yuan (2014). Now we are going to prove that

$$\lim_{x\to\infty} xP\left(\frac{S_n}{U_X(x)} > p, \frac{T_n}{U_Y(x)} > q\right) = \sum_{i=1}^{n} E\left[\nu\left(\left(\frac{p}{\xi_i}, \frac{q}{\eta_i}\right), \boldsymbol{\infty}\right]\right]. \tag{5.2}$$

To derive the upper bound for (5.2), for arbitrarily fixed small $\delta > 0$ write

$$A = \left(\bigvee_{i=1}^{n} \frac{\xi_i X_i}{U_X(x)} > (1-\delta)p\right), \qquad B = \left(\bigvee_{i=1}^{n} \frac{\eta_i Y_i}{U_Y(x)} > (1-\delta)q\right).$$

In terms of these sets, we do the split

$$\begin{aligned} xP\left(\frac{S_n}{U_X(x)} > p, \frac{T_n}{U_Y(x)} > q\right) &\leq xP\left(\frac{S_n}{U_X(x)} > p, \frac{T_n}{U_Y(x)} > q, A\cap B\right) \\ &\quad + xP\left(\frac{S_n}{U_X(x)} > p, \frac{T_n}{U_Y(x)} > q, A^c\right) \\ &\quad + xP\left(\frac{S_n}{U_X(x)} > p, \frac{T_n}{U_Y(x)} > q, B^c\right) \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{5.3}$$

For I_1, we have

$$\begin{aligned} I_1 &\leq xP\left(A\cap B\right) \\ &\leq \sum_{i=1}^{n}\sum_{j=1}^{n} xP\left(\frac{\xi_i X_i}{U_X(x)} > (1-\delta)p, \frac{\eta_j Y_j}{U_Y(x)} > (1-\delta)q\right) \\ &= \sum_{1\leq i=j\leq n} + \sum_{1\leq i\neq j\leq n}. \end{aligned} \tag{5.4}$$

By Lemma 5.2, each term in the sum $\sum_{1\leq i=j\leq n}$ satisfies

$$\lim_{x\to\infty} xP\left(\frac{\xi_i X_i}{U_X(x)} > (1-\delta)p, \frac{\eta_i Y_i}{U_Y(x)} > (1-\delta)q\right) = E\left[\nu\left(\left(\frac{(1-\delta)p}{\xi_i}, \frac{(1-\delta)q}{\eta_i}\right), \boldsymbol{\infty}\right]\right].$$

Note that for each term in the sum $\sum_{1\leq i\neq j\leq n}$ the product $\eta_j Y_j$ is independent of X_i. By Lemma 5.1, we have

$$\lim_{x\to\infty} xP\left(\frac{\xi_i X_i}{U_X(x)} > (1-\delta)p, \frac{\eta_j Y_j}{U_Y(x)} > (1-\delta)q\right) = 0.$$

Plugging these two estimates into (5.4) yields

$$\begin{aligned} \limsup_{x\to\infty} I_1 &\leq \sum_{i=1}^{n} E\left[\nu\left(\left(\frac{(1-\delta)p}{\xi_i}, \frac{(1-\delta)q}{\eta_i}\right), \boldsymbol{\infty}\right]\right] \\ &\leq \sum_{i=1}^{n} E\left[\nu\left(\left(\frac{(1-\delta)^{\frac{\alpha\vee\beta}{\alpha}}p}{\xi_i}, \frac{(1-\delta)^{\frac{\alpha\vee\beta}{\beta}}q}{\eta_i}\right), \boldsymbol{\infty}\right]\right] \end{aligned}$$

$$
\begin{aligned}
&= (1-\delta)^{-(\alpha\vee\beta)}\sum_{i=1}^{n} E\left[\nu\left(\left(\frac{p}{\xi_i},\frac{q}{\eta_i}\right),\boldsymbol{\infty}\right]\right] \\
&\to \sum_{i=1}^{n} E\left[\nu\left(\left(\frac{p}{\xi_i},\frac{q}{\eta_i}\right),\boldsymbol{\infty}\right]\right], \qquad \delta\downarrow 0, \qquad (5.5)
\end{aligned}
$$

where the second step is due to the fact that the set $\left(\left(\frac{sp}{\xi_i},\frac{sq}{\eta_i}\right),\boldsymbol{\infty}\right]$ increases as $s>0$ decreases, and the third step due to the homogeneity property (2.5) of ν. For I_2, it holds that

$$
\begin{aligned}
I_2 &\le xP\left(\frac{S_n}{U_X(x)}>p, A^c\right) \\
&= xP\left(\frac{S_n}{U_X(x)}>p, \bigvee_{i=1}^{n}\frac{\xi_iX_i}{U_X(x)}>\frac{p}{n}, \bigvee_{j=1}^{n}\frac{\xi_jX_j}{U_X(x)}\le(1-\delta)p\right) \\
&\le \sum_{i=1}^{n} xP\left(\frac{\xi_iX_i}{U_X(x)}>\frac{p}{n}, \sum_{k=1,k\ne i}^{n}\frac{\xi_kX_k}{U_X(x)}>\delta p\right) \\
&\to 0, \qquad (5.6)
\end{aligned}
$$

where the last step is due to Lemma 5.1 again. In the same way,

$$
\lim_{x\to\infty} I_3 = 0. \qquad (5.7)
$$

Plugging the estimates of (5.5)–(5.7) into (5.3) yields the upper bound for (5.2).

Next we derive the lower bound for (5.2). Write

$$
C_i = \left(\frac{\xi_iX_i}{U_X(x)}>p, \frac{\eta_iY_i}{U_Y(x)}>q\right), \qquad i=1,\dots,n.
$$

Since $\{(X_i,Y_i); i\in\mathbb{N}\}$ are nonnegative, applying Bonferroni's inequality leads to

$$
\begin{aligned}
xP\left(\frac{S_n}{U_X(x)}>p, \frac{T_n}{U_Y(x)}>q\right) &\ge xP\left(\bigcup_{i=1}^{n} C_i\right) \\
&\ge \sum_{i=1}^{n} xP(C_i) - \sum_{1\le i<j\le n} xP(C_i\cap C_j) \\
&= J_1 - J_2. \qquad (5.8)
\end{aligned}
$$

For J_1, applying Lemma 5.2 we have

$$
\lim_{x\to\infty} J_1 = \sum_{i=1}^{n} E\left[\nu\left(\left(\frac{p}{\xi_i},\frac{q}{\eta_i}\right),\boldsymbol{\infty}\right]\right]. \qquad (5.9)
$$

For J_2 in (5.8), similarly to (5.6), each term in it is no more than

$$
xP\left(\frac{\xi_iX_i}{U_X(x)}>p, \frac{\eta_jY_j}{U_Y(x)}>q\right) \to 0
$$

by Lemma 5.1. Hence,

$$\lim_{x\to\infty} J_2 = 0. \tag{5.10}$$

Plugging the estimations of (5.9)–(5.10) into (5.8) yields the lower bound for (5.2).

5.3 Proof of Theorem 3.2

Applying Proposition 1.1 to both sums S_∞ and T_∞, we have, respectively,

$$\begin{aligned}
\lim_{x\to\infty} xP\left(\frac{S_\infty}{U_X(x)} > p\right) &= \frac{1}{p^\alpha}\sum_{i=1}^{\infty} E\left[\xi_i^\alpha\right], \\
\lim_{x\to\infty} xP\left(\frac{T_\infty}{U_Y(x)} > q\right) &= \frac{1}{q^\beta}\sum_{i=1}^{\infty} E\left[\eta_i^\beta\right].
\end{aligned}$$

Thus, it remains to prove that

$$\lim_{x\to\infty} xP\left(\frac{S_\infty}{U_X(x)} > p, \frac{T_\infty}{U_Y(x)} > q\right) = \sum_{i=1}^{\infty} E\left[\nu\left(\left(\frac{p}{\xi_i}, \frac{q}{\eta_i}\right), \boldsymbol{\infty}\right]\right]. \tag{5.11}$$

The finiteness of the sum in (5.11) can be verified as follows:

$$\begin{aligned}
\sum_{i=1}^{\infty} E\left[\nu\left(\left(\frac{p}{\xi_i}, \frac{q}{\eta_i}\right), \boldsymbol{\infty}\right]\right] &\le \sum_{i=1}^{\infty} E\left[\nu\left(\left(\frac{p}{\xi_i}, 0\right), \boldsymbol{\infty}\right]\right] \\
&= p^{-\alpha}\nu\left((1,0), \boldsymbol{\infty}\right]\sum_{i=1}^{\infty} E\left[\xi_i^\alpha\right] \\
&< \infty.
\end{aligned}$$

By relation (5.2), it holds for arbitrarily fixed $n \in \mathbb{N}$ that

$$\begin{aligned}
xP\left(\frac{S_\infty}{U_X(x)} > p, \frac{T_\infty}{U_Y(x)} > q\right) &\ge xP\left(\frac{S_n}{U_X(x)} > p, \frac{T_n}{U_Y(x)} > q\right) \\
&\to \sum_{i=1}^{n} E\left[\nu\left(\left(\frac{p}{\xi_i}, \frac{q}{\eta_i}\right), \boldsymbol{\infty}\right]\right] \\
&\to \sum_{i=1}^{\infty} E\left[\nu\left(\left(\frac{p}{\xi_i}, \frac{q}{\eta_i}\right), \boldsymbol{\infty}\right]\right], \qquad n\uparrow\infty,
\end{aligned}$$

giving the lower bound for (5.11). Now we aim at the corresponding upper bound. For arbitrarily fixed $n \in \mathbb{N}$ and $\varepsilon \in (0,1)$, we derive

$$\begin{aligned}
xP\left(\frac{S_\infty}{U_X(x)} > p, \frac{T_\infty}{U_Y(x)} > q\right) &\le xP\left(\frac{S_n}{U_X(x)} > (1-\varepsilon)p, \frac{T_n}{U_Y(x)} > (1-\varepsilon)q\right) \\
&\quad + xP\left(\frac{\sum_{i=n+1}^{\infty}\xi_i X_i}{U_X(x)} > \varepsilon p\right)
\end{aligned}$$

$$
\begin{aligned}
& +xP\left(\frac{\sum_{i=n+1}^{\infty}\eta_i Y_i}{U_Y(x)} > \varepsilon q\right) \\
= \;& K_1 + K_2 + K_3.
\end{aligned}
$$

Applying Theorem 3.1 and then following (5.5), we have

$$
\begin{aligned}
\lim_{x\to\infty} K_1 &= \sum_{i=1}^{n} E\left[\nu\left(\left(\frac{(1-\varepsilon)p}{\xi_i}, \frac{(1-\varepsilon)q}{\eta_i}\right), \boldsymbol{\infty}\right]\right] \\
&\leq (1-\varepsilon)^{-(\alpha\vee\beta)}\sum_{i=1}^{n} E\left[\nu\left(\left(\frac{p}{\xi_i}, \frac{q}{\eta_i}\right), \boldsymbol{\infty}\right]\right] \\
&\to \sum_{i=1}^{n} E\left[\nu\left(\left(\frac{p}{\xi_i}, \frac{(q}{\eta_i}\right), \boldsymbol{\infty}\right]\right], \qquad \varepsilon \downarrow 0.
\end{aligned}
$$

By Proposition 1.1,

$$
\begin{aligned}
\lim_{x\to\infty} K_2 &= \frac{1}{p^{\alpha}}\sum_{i=n+1}^{\infty} E\left[\xi_i^{\alpha}\right] \to 0, \qquad n \uparrow \infty; \\
\lim_{x\to\infty} K_3 &= \frac{1}{q^{\beta}}\sum_{i=n+1}^{\infty} E\left[\eta_i^{\beta}\right] \to 0, \qquad n \uparrow \infty.
\end{aligned}
$$

Putting these together, we obtain

$$
\limsup_{x\to\infty} xP\left(\frac{S_\infty}{U_X(x)} > p, \frac{T_\infty}{U_Y(x)} > q\right) \leq \sum_{i=1}^{\infty} E\left[\nu\left(\left(\frac{p}{\xi_i}, \frac{(q}{\eta_i}\right), \boldsymbol{\infty}\right]\right],
$$

as desired.

5.4 Proof of Corollary 4.1

Rewrite the random pair (S_{N_t}, T_{N_t}) as

$$
(S_{N_t}, T_{N_t}) = \left(\sum_{i=1}^{\infty} 1_{(N_t\geq i)}X_i, \sum_{i=1}^{\infty} 1_{(N_t\geq i)}Y_i\right),
$$

which becomes a pair of randomly weighted sums (3.1) with random weights $\xi_i = \eta_i = 1_{(N_t\geq i)}$ for $i \in \mathbb{N}$. Thus, the current proof is merely a validation of Theorem 3.2. The limit measure given by (3.2) becomes

$$
\begin{aligned}
\tilde{\nu}[\mathbf{0}, (p,q)]^c &= \frac{1}{p^{\alpha}}\sum_{i=1}^{\infty} E\left[\xi_i^{\alpha}\right] + \frac{1}{q^{\beta}}\sum_{i=1}^{\infty} E\left[\eta_i^{\beta}\right] - \sum_{i=1}^{\infty} E\left[\nu\left(\left(\frac{p}{\xi_i}, \frac{q}{\eta_i}\right), \boldsymbol{\infty}\right]\right] \\
&= \frac{1}{p^{\alpha}}\sum_{i=1}^{\infty} E\left[1_{(N_t\geq i)}\right] + \frac{1}{q^{\beta}}\sum_{i=1}^{\infty} E\left[1_{(N_t\geq i)}\right] - \sum_{i=1}^{\infty} E\left[\nu\left((p,q), \boldsymbol{\infty}\right] 1_{(N_t\geq i)}\right]
\end{aligned}
$$

$$= \quad E[N_t]\left(\frac{1}{p^\alpha}+\frac{1}{q^\beta}-\nu\left((p,q),\boldsymbol{\infty}\right]\right).$$

It remains to verify the moment conditions $\mathbb{M}_\alpha$ and $\mathbb{M}_\beta$ (introduced in Proposition 1.1) on the random weights. In case $\alpha\in(0,1)$, for any small $\varepsilon>0$,

$$\sum_{i=1}^\infty E\left[\xi_i^{\alpha-\varepsilon}\vee\xi_i^{\alpha+\varepsilon}\right]=\sum_{i=1}^\infty E\left[1_{(N_t\geq i)}\right]=E[N_t]<\infty.$$

In case $\alpha\in[1,\infty)$, for any small $\varepsilon>0$ such that $\gamma>\alpha+\varepsilon$,

$$\begin{aligned}\sum_{i=1}^\infty\left(E\left[\xi_i^{\alpha-\varepsilon}\vee\xi_i^{\alpha+\varepsilon}\right]\right)^{\frac{1}{\alpha+\varepsilon}} &= \sum_{i=1}^\infty\left(P\left(N_t\geq i\right)\right)^{\frac{1}{\alpha+\varepsilon}}\\ &\leq \sum_{i=1}^\infty\left(\frac{E[N_t^\gamma]}{i^\gamma}\right)^{\frac{1}{\alpha+\varepsilon}}\\ &= \left(E[N_t^\gamma]\right)^{\frac{1}{\alpha+\varepsilon}}\sum_{i=1}^\infty i^{-\frac{\gamma}{\alpha+\varepsilon}}<\infty,\end{aligned}$$

where the second step applies Markov's inequality. This verifies $\mathbb{M}_\alpha$. The other condition $\mathbb{M}_\beta$ can be verified in the same way.

5.5 Proof of Corollary 4.2

Rewrite the random pair (S_{N_t},T_{N_t}) as

$$(S_{N_t},T_{N_t})=\left(\sum_{i=1}^\infty e^{-r\tau_i}1_{(\tau_i\leq t)}X_i,\sum_{i=1}^\infty e^{-r\tau_i}1_{(\tau_i\leq t)}Y_i\right),$$

which becomes a pair of randomly weighted sums (3.1) with random weights $\xi_i=\eta_i=e^{-r\tau_i}1_{(\tau_i\leq t)}$ for $i\in\mathbb{N}$. Thus, the current proof is merely another validation of Theorem 3.2. In the limit measure given by (3.2), the first term becomes

$$\begin{aligned}\frac{1}{p^\alpha}\sum_{i=1}^\infty E\left[\xi_i^\alpha\right] &= \frac{1}{p^\alpha}\sum_{i=1}^\infty E\left[e^{-\alpha r\tau_i}1_{(\tau_i\leq t)}\right]\\ &= \frac{1}{p^\alpha}\sum_{i=1}^\infty\int_{0-}^t e^{-\alpha rs}P\left(\tau_i\in ds\right)\\ &= \frac{1}{p^\alpha}\int_{0-}^t e^{-\alpha rs}\sum_{i=1}^\infty P\left(\tau_i\in ds\right)\\ &= \frac{1}{p^\alpha}\int_{0-}^t e^{-\alpha rs}dE[N_s].\end{aligned}$$

In the same way, the second term becomes

$$\frac{1}{q^\beta}\sum_{i=1}^\infty E\left[\eta_i^\beta\right]=\frac{1}{q^\beta}\sum_{i=1}^\infty E\left[e^{-\beta r\tau_i}1_{(\tau_i\leq t)}\right]=\frac{1}{q^\beta}\int_{0-}^t e^{-\beta rs}dE[N_s],$$

and the last term becomes

$$\begin{aligned}\sum_{i=1}^{\infty} E\left[\nu\left(\left(\frac{p}{\xi_i}, \frac{q}{\eta_i}\right), \infty\right]\right] &= \sum_{i=1}^{\infty} E\left[\nu\left(\left(pe^{r\tau_i}, qe^{r\tau_i}\right), \infty\right] 1_{(\tau_i \le t)}\right] \\ &= \int_{0-}^{t} \nu\left(\left(pe^{rs}, qe^{rs}\right), \infty\right] dE[N_s].\end{aligned}$$

Putting these results together we obtain the limit measure as given. The verifications of the moment conditions on the random weights are similar to the previous proof and therefore are omitted.

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