Valuing equity-linked death benefits and other contingent options: a discounted density approach

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Abstract

Motivated by the Guaranteed Minimum Death Benefits in various deferred annuities, we investigate the calculation of the expected discounted value of a payment at the time of death. The payment depends on the price of a stock at that time and possibly also on the history of the stock price. If the payment turns out to be the payoff of an option, we call the contract for the payment a (life) contingent option. Because each time-until-death distribution can be approximated by a combination of exponential distributions, the analysis is made for the case where the time until death is exponentially distributed,

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i.e., under the assumption of a constant force of mortality. The time-until-death random variable is assumed to be independent of the stock price process which is a geometric Brownian motion. Our key tool is a discounted joint density function. A substantial series of closed-form formulas is obtained, for the contingent call and put options, for lookback options, for barrier options, for dynamic fund protection, and for dynamic withdrawal benefits. In a section on several stocks, the method of Esscher transforms proves to be useful for finding among others an explicit result for valuing contingent Margrabe options or exchange options. For the case where the contracts have a finite expiry date, closed-form formulas are found for the contingent call and put options. From these, results for De Moivre's law are obtained as limits. We also discuss equity-linked death benefit reserves and investment strategies for maintaining such reserves. The elasticity of the reserve with respect to the stock price plays an important role. Whereas in the most important applications the stopping time is the time of death, it could be different in other applications, for example, the time of the next catastrophe.

Key words: Equity-linked death benefits, variable annuities, minimum guaranteed death benefits, exponential stopping, option pricing, discounted density.

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1 Introduction

This paper is dedicated to the celebration of the 65th birthday of Professor Marc Goovaerts. Parts of it were presented on June 13, 2011, at the Memorable Actuarial Research Conference, Katholieke Universiteit Leuven.

A key motivation for this paper is the problem of valuing *Guaranteed Minimum Death Benefits* (GMDB) in various variable annuity and equity-indexed annuity contracts. Consider a customer age x paying a single premium for one unit of a mutual fund or stock fund. For $t \geq 0$, let $S(t)$ denote the value of one unit of the fund at time t. Consider a GMDB rider that guarantees the following payment to the customer's estate when the customer dies,

$$\max(S(T_x), K), \tag{1.1}$$

where T_x is the time-until-death random variable for a life age x, and K is the guaranteed amount. Because

$$\max(S(T_x), K) = S(T_x) + [K - S(T_x)]_+, \tag{1.2}$$

the problem of valuing the guarantee becomes the problem of valuing a K-strike *put* option that is exercised at time T_x. Since T_x is a random variable, the put option is of neither the European style nor the American style. It is a *life-contingent* put option.

Thus we are interested in evaluating the expectation,

$$\mathrm{E}[e^{-\delta T_x} b(S(T_x))], \tag{1.3}$$

where δ denotes a force of interest and $b(s)$ is an equity-indexed death benefit function. Let $f_{T_x}(t)$ denote the probability density function of T_x. Under the assumption that T_x is independent of the stock price process $S(t)$, the expectation (1.3) is

$$\int_0^\infty \mathrm{E}[b(S(t))] e^{-\delta t} f_{T_x}(t) \mathrm{d}t. \tag{1.4}$$

If the function $f_{T_x}(t)$ is a linear combination of some other probability density functions, i.e., if

$$f_{T_x}(t) = \sum_j c_j f_{T_j}(t), \tag{1.5}$$

then

$$\begin{aligned} \mathrm{E}[e^{-\delta T_x} b(S(T_x))] &= \sum_j c_j \int_0^\infty \mathrm{E}[b(S(t))] e^{-\delta t} f_{T_j}(t) \mathrm{d}t \\ &= \sum_j c_j \mathrm{E}[e^{-\delta T_j} b(S(T_j))]. \end{aligned} \tag{1.6}$$

Now, combinations of exponential distributions are (weakly) dense in the space of all probability distributions on the positive axis (Dufresne 2007a, b; Ko and Ng 2007); see also Section 3 in Shang et al. (2011). Thus, if we can find a formula for the expectation

$$\mathrm{E}[e^{-\delta \tau} b(S(\tau))], \tag{1.7}$$

where τ is an exponential random variable independent of the stock-price process $S(t)$, we have found a way to approximate the expectation (1.3). Indeed, there is such a formula, if the stock-price process is a geometric Brownian motion.

In fact, we can generalize the death-benefit function b in (1.7) to the case where it also depends on the *running maximum* of the stock price. The result is

$$\begin{aligned} &\mathrm{E}[e^{-\delta \tau} b(S(\tau), \max_{0 \le t \le \tau} S(t))] \\ &= \frac{2}{\mathrm{E}[\tau] \mathrm{Var}[\ln S(1)]} \int_0^\infty \left[\int_{-\infty}^y b(S(0)e^x, S(0)e^y) e^{-\alpha x} \mathrm{d}x \right] e^{-(\beta - \alpha) y} \mathrm{d}y, \end{aligned} \tag{1.8}$$

where $\alpha < 0$ and $\beta > 0$ are the roots of the quadratic equation (2.5) in the next section. This elementary calculus formula is an immediate consequence of formula (2.7).

In this paper, $X(t)$ denotes a (linear) Brownian motion, $M(t)$ its running maximum, and $m(t)$ its running minimum; τ denotes an exponential random variable independent of the Brownian motion. The time-t fund price is modeled as

$$S(t) = S(0)e^{X(t)}. \tag{1.9}$$

Thus, the left-hand side (LHS) of (1.8) is

$$\mathrm{E}[e^{-\delta\tau} b(S(0)e^{X(\tau)}, S(0)e^{M(\tau)})]. \tag{1.10}$$

This expectation can be evaluated by means of (2.7), the discounted joint density function of $X(\tau)$ and $M(\tau)$, which is derived in Section 3.

Many interesting consequences of (2.7) are given in Section 2. The random variable $X(\tau)$ has a two-sided exponential distribution. The random variables $M(\tau)$ and $[M(\tau) - X(\tau)]$ are independent and exponentially distributed with means $1/\beta$ and $-1/\alpha$, respectively (where $\alpha < 0$ and $\beta > 0$ are the solutions of equation (2.5) with $\delta = 0$); the same statement is true for the random variables $[X(\tau) - m(\tau)]$ and $-m(\tau)$. The random variables $M(\tau)$ and $X(\tau)$ have the same joint distribution as $[X(\tau) - m(\tau)]$ and $X(\tau)$; the random variables $-m(\tau)$ and $X(\tau)$ have the same joint distribution as $[M(\tau) - X(\tau)]$ and $X(\tau)$.

In Sections 4 to 6, we evaluate the expectation (1.10) for various forms of the equity-indexed death benefit function b. If b is the payoff function of an option, we use the term *contingent option*. In Section 4, we derive formulas for valuing contingent call and put options. The formulas are particularly simple when the options are out-of-the-money,

$$\begin{aligned}
&\mathrm{E}[e^{-\delta\tau}[S(\tau) - K]_+] \\
&= \frac{2}{\mathrm{E}[\tau]\mathrm{Var}[X(1)]} \frac{K}{\beta(\beta-1)(\beta-\alpha)} \left[\frac{S(0)}{K}\right]^{\beta}, \qquad S(0) \le K,
\end{aligned}$$

$$\begin{aligned}
&\mathrm{E}[e^{-\delta\tau}[K - S(\tau)]_+] \\
&= \frac{2}{\mathrm{E}[\tau]\mathrm{Var}[X(1)]} \frac{K}{-\alpha(1-\alpha)(\beta-\alpha)} \left[\frac{K}{S(0)}\right]^{-\alpha}, \qquad S(0) \ge K.
\end{aligned}$$

These two formulas are (4.19) and (4.25), respectively. The in-the-money formulas are obtained by put-call parity. In Section 5, we value contingent lookback options. In Section 6, we study the valuation of contingent barrier options. With the aid of the mathematical software *Mathematica*, we evaluate various versions of the iterated integral (1.8); the results are listed in the Appendix.

Section 7 values "dynamic fund protection" (Gerber and Pafumi 2000; Gerber and Shiu 2003b) when the guarantee is effective until time τ. Section 8 considers the dual concept of "dynamic withdrawal benefit" (Ko et al. 2010). The concepts of dynamic fund protection and dynamic withdrawal benefit can be generalized to the situation where the boundary is another geometric Brownian motion. Section 9 discusses such a generalization. It also evaluates the contingent Margrabe option, whose payoff is

$$[S_1(\tau) - S_2(\tau)]_+.$$

Some of the valuation formulas can be expressed as a factor times

$$\mathrm{E}[e^{-\delta\tau} S(\tau)]. \tag{1.11}$$

The expectation (1.11) can be interpreted as the time-0 value for obtaining one unit of the stock fund at time τ; formulas for it are (4.7) and (4.10). Table 1 presents a list of such formulas. For each payoff in the left column, the middle column gives the factor.

Table 1: Valuation formulas that can be expressed as a factor times (1.11)

Payoff at time τ	Factor multiplied to (1.11)	Equation number
$[S(\tau) - S(0)]_+$	$(1-\alpha)/[\beta(\beta-\alpha)]$	(4.21)
$[S(0) - S(\tau)]_+$	$(\beta-1)/[-\alpha(\beta-\alpha)]$	(4.27)
$S(0)e^{M(\tau)}$	$1 + (-\alpha)^{-1}$	(5.11)
$S(0)e^{m(\tau)}$	$1 - \beta^{-1}$	(5.26)
$[\gamma S(0)e^{M(\tau)} - S(\tau)]_+$ with $0 < \gamma \leq 1$	$\gamma^{1-\alpha}/(-\alpha)$	(5.17)
$[S(\tau) - \gamma S(0)e^{m(\tau)}]_+$ with $\gamma \geq 1$	$(1/\gamma)^{\beta-1}/\beta$	(5.30)
$S(\tau)\{[L/S(0)]e^{-m(\tau)} - 1\}_+$ with $0 \leq L \leq S(0)$	$[L/S(0)]^{1-\alpha}/(-\alpha)$	(7.7)
$S(\tau)\{1 - [L/S(0)]e^{-M(\tau)}\}_+$ with $L \geq S(0)$	$[S(0)/L]^{\beta-1}/\beta$	(8.7)

Most options and guarantees have a finite expiry date. Section 10 presents explicit formulas for evaluating

$$\mathrm{E}[e^{-\delta\tau}[K - S(\tau)]_+ I_{(\tau \leq T)}],$$

where $I_{(.)}$ denotes the indicator function and T is a fixed positive number. It turns out that by taking limits, the results can be used to evaluate options whose time of exercise is uniformly distributed between 0 and T (De Moivre's law). This is shown in Section 11.

Section 12 assumes that the actuarial reserve for a life-contingent option or equity-linked death benefit is calculated as an expected present value. It shows that the reserve satisfies a generalization of the celebrated *Thiele's differential equation*. It discusses investment strategies related to maintaining the value of the reserve through time. This section can be read independently of the others.

We should emphasize that results in this paper are not restricted to valuing death benefits. Instead of a time-until-death random variable, we can consider a time-until-catastrophe random variable, and so on. A key assumption is that such a random variable is independent of the geometric Brownian motion $S(t)$.

In the actuarial literature, we have found the papers Milevsky and Posner (2001) and Ulm (2006, 2008) containing results related to ours. We have verified numerically that their formulas are equivalent to ours. A recent paper on variable annuities is Bacinello et al. (2011).

2 Exponential stopping of Brownian motion

Let

$$X(t) = \mu t + \sigma W(t), \qquad t \geq 0, \tag{2.1}$$

where $W(t)$ is a standard Brownian motion (Wiener process), and μ and $\sigma > 0$ are constants. Let

$$M(t) = \max\{X(s); \quad 0 \leq s \leq t\} \tag{2.2}$$

denote the *running maximum* of the process. Let $f_{X(t),M(t)}(x, y)$, $y \geq \max(x, 0)$, denote the joint probability density function of $X(t)$ and $M(t)$. The process $X(t)$ is stopped at time τ, an independent exponential random variable with probability density function

$$f_\tau(t) = \lambda e^{-\lambda t}, \qquad t > 0. \tag{2.3}$$

For $\delta > -\lambda$, we define the function

$$f^\delta_{X(\tau),M(\tau)}(x, y) = \int_0^\infty e^{-\delta t} f_{X(t),M(t)}(x, y) f_\tau(t) \mathrm{d}t, \quad y \geq \max(x, 0). \tag{2.4}$$

We call such functions *discounted density functions* even in the case of negative δ, where the adjective *inflated* might be more appropriate. Unless stated otherwise, in this paper $\alpha < 0$ and $\beta > 0$ are the roots of the quadratic equation

$$D\xi^2 + \mu\xi - (\lambda + \delta) = 0, \tag{2.5}$$

where

$$D = \frac{1}{2}\sigma^2. \tag{2.6}$$

The following result is a key to a series of formulas that are useful in actuarial and financial applications:

$$f^{\delta}_{X(\tau),M(\tau)}(x, y) = \frac{\lambda}{D}e^{-\alpha x-(\beta-\alpha)y}, \qquad y \geq \max(x, 0). \tag{2.7}$$

A self-contained proof of this surprisingly simple formula will be given in Section 3. In the remainder of this section, we discuss easy consequences of (2.7).

Let

$$f^{\delta}_{X(\tau),M(\tau)-X(\tau)}(x, z) = \int_0^\infty e^{-\delta t} f_{X(t),M(t)-X(t)}(x, z) f_\tau(t)\mathrm{d}t \tag{2.8}$$

denote the discounted joint density function of $X(\tau)$ and $M(\tau) - X(\tau)$. It follows from (2.8),

$$f_{X(t),M(t)-X(t)}(x, z) = f_{X(t),M(t)}(x, x + z),$$

(2.4), and (2.7) that

$$f^{\delta}_{X(\tau),M(\tau)-X(\tau)}(x, z) = \frac{\lambda}{D}e^{-\beta x-(\beta-\alpha)z}, \qquad z \geq \max(-x, 0). \tag{2.9}$$

Similarly, let us consider

$$f^{\delta}_{M(\tau),M(\tau)-X(\tau)}(y, z) = \int_0^\infty e^{-\delta t} f_{M(t),M(t)-X(t)}(y, z) f_\tau(t)\mathrm{d}t, \tag{2.10}$$

the discounted joint density function of $M(\tau)$ and $M(\tau) - X(\tau)$. Then

$$f^{\delta}_{M(\tau),M(\tau)-X(\tau)}(y, z) = \frac{\lambda}{D}e^{-\beta y+\alpha z}, \qquad y \geq 0,\ z \geq 0. \tag{2.11}$$

Note that if $\delta = 0$, this shows that $M(\tau)$ and $[M(\tau) - X(\tau)]$ are independent random variables (even though $M(t)$ and $[M(t) - X(t)]$ are not independent).

Let $f^{\delta}_{X(\tau)}(x)$, $f^{\delta}_{M(\tau)}(y)$, and $f^{\delta}_{M(\tau)-X(\tau)}(z)$ denote the discounted density functions of $X(\tau)$, $M(\tau)$, and $M(\tau) - X(\tau)$, respectively. When we integrate (2.7) over y, we have to distinguish whether x is positive or negative. This way we find that

$$f^{\delta}_{X(\tau)}(x) = \begin{cases} \kappa e^{-\alpha x}, & \text{if } \ x \leq 0, \\ \kappa e^{-\beta x}, & \text{if } \ x \geq 0, \end{cases} \tag{2.12}$$

with the notation

$$\kappa = \frac{\lambda}{D(\beta - \alpha)} = \frac{\lambda}{\lambda + \delta} \frac{-\alpha\beta}{\beta - \alpha}. \tag{2.13}$$

Note that $D(\beta - \alpha)$ is the square root of the discriminant of the quadratic polynomial in (2.5) and that

$$\frac{\lambda}{\lambda + \delta} = \mathrm{E}[e^{-\delta\tau}]. \tag{2.14}$$

If we integrate (2.7) over x (from $-\infty$ to y), we obtain the formula

$$f^{\delta}_{M(\tau)}(y) = \frac{\lambda}{-\alpha D} e^{-\beta y} = \frac{\lambda}{\lambda + \delta} \beta e^{-\beta y}, \qquad y \geq 0. \tag{2.15}$$

Finally, we integrate (2.9) over x (from $-z$ to ∞) and obtain

$$f^{\delta}_{M(\tau)-X(\tau)}(z) = \frac{\lambda}{\beta D} e^{\alpha z} = \frac{\lambda}{\lambda + \delta}(-\alpha) e^{\alpha z}, \qquad z \geq 0. \tag{2.16}$$

Of course, (2.15) and (2.16) can be also obtained easily from (2.11).

For certain applications, we are interested in the *running minimum*

$$m(t) = \min\{X(s); \quad 0 \leq s \leq t\} \tag{2.17}$$

of the process $X(t)$. Because

$$m(t) = -\max\{-X(s); \quad 0 \leq s \leq t\}, \tag{2.18}$$

we can use the previous results with $M(t)$ replaced by $-m(t)$. We just keep in mind that μ is to be replaced by $-\mu$, if $X(s)$ is replaced by $-X(s)$. By (2.5), α is replaced by $-\beta$ and β by $-\alpha$. Hence, if we have the result

$$\mathrm{E}[e^{-\delta\tau}g(X(\tau), M(\tau))] = h(\alpha, \beta),$$

then we can translate it to

$$\mathrm{E}[e^{-\delta\tau}g(-X(\tau), -m(\tau))] = h(-\beta, -\alpha).$$

Thus the formulas (2.7), (2.9), (2.11), (2.15), and (2.16) are translated as

$$f^{\delta}_{X(\tau),m(\tau)}(x,y) = \frac{\lambda}{D}e^{-\beta x+(\beta-\alpha)y}, \qquad y \leq \min(x,0), \tag{2.19}$$

$$f^{\delta}_{X(\tau),X(\tau)-m(\tau)}(x,z) = \frac{\lambda}{D}e^{-\alpha x-(\beta-\alpha)z}, \qquad z \geq \max(x,0), \tag{2.20}$$

$$f^{\delta}_{m(\tau),X(\tau)-m(\tau)}(y,z) = \frac{\lambda}{D}e^{-\alpha y-\beta z}, \qquad y \leq 0, z \geq 0, \tag{2.21}$$

$$f^{\delta}_{m(\tau)}(y) = \frac{\lambda}{\beta D}e^{-\alpha y} = \frac{\lambda}{\lambda+\delta}(-\alpha)e^{-\alpha y}, \qquad y \leq 0, \tag{2.22}$$

$$f^{\delta}_{X(\tau)-m(\tau)}(z) = \frac{\lambda}{-\alpha D}e^{-\beta z} = \frac{\lambda}{\lambda+\delta}\beta e^{-\beta z}, \qquad z \geq 0. \tag{2.23}$$

Note that with $\delta = 0$ formula (2.21) shows that $m(\tau)$ and $[X(\tau) - m(\tau)]$ are independent random variables.

Remark 2.1: Let

$$M_Y(t) = \mathrm{E}[e^{tY}] \tag{2.24}$$

denote the moment-generating function of a random variable Y. Then, the quadratic equation (2.5) can be rewritten as

$$\ln[M_{X(1)}(\xi)] - (\lambda + \delta) = 0. \tag{2.25}$$

From this we see that α and β can also be characterized as the two values of ξ for which the process $e^{-(\lambda+\delta)t+\xi X(t)}$ is a martingale. Equivalently, they are values of ξ

such that the process $e^{-\delta t+\xi X(t)} I_{(\tau>t)}$ is a martingale. In this paper, I_A denotes the indicator function of an event A.

Remark 2.2: We present an independent proof of (2.15), which is equivalent to

$$\mathrm{E}[e^{-\delta\tau}(1-F_{M(\tau)}(y))] = \frac{\lambda}{\lambda+\delta} e^{-\beta y}, \quad y > 0. \tag{2.26}$$

Let $\mathcal{T}$ denote the first passage time of $X(t)$ at the level y. Then the LHS of (2.26) is

$$\mathrm{E}[e^{-\delta(\tau-\mathcal{T})} e^{-\delta\mathcal{T}} I_{(\mathcal{T}<\tau)}] = \frac{\lambda}{\lambda+\delta}\mathrm{E}[e^{-\delta\mathcal{T}} I_{(\mathcal{T}<\tau)}] \tag{2.27}$$

because of (2.14) and the fact that the conditional distribution of $\tau-\mathcal{T}$, given $\mathcal{T}<\tau$, is the same as the distribution of τ. By stopping the martingale $e^{-\delta t+\beta X(t)} I_{(t<\tau)}$ at time $\mathcal{T}$ and using the optional stopping theorem, we see that

$$\mathrm{E}[e^{-\delta\mathcal{T}} I_{(\mathcal{T}<\tau)}] = e^{-\beta y}. \tag{2.28}$$

This yields (2.26). This martingale proof can obviously be generalized to the case where $X(t)$ is a Lévy process that is skip-free upwards (spectrally negative Lévy process). See also Section 4 in Kyprianou and Palmowski (2005).

Remark 2.3: The following formulas can be found in books such as Borodin and Salminen (2002) and Jeanblanc et al. (2009). For each $t > 0$,

$$\begin{aligned}
f_{X(t)}(x) &= \frac{1}{2\sqrt{\pi D t}} e^{-\frac{(x-\mu t)^2}{4Dt}}, & -\infty < x < \infty, \quad &(2.29)\\
f_{M(t)}(y) &= \frac{1}{\sqrt{\pi D t}} e^{-\frac{(y-\mu t)^2}{4Dt}} - \frac{\mu}{D} e^{\frac{\mu y}{D}} \Phi\left(\frac{-y-\mu t}{\sqrt{2Dt}}\right), & y \geq 0, \quad &(2.30)\\
f_{X(t),M(t)}(x,y) &= \frac{2y-x}{2\sqrt{\pi D^3 t^3}} e^{(\mu x-\frac{1}{2}\mu^2 t-\frac{(2y-x)^2}{2t})/(2D)}, & y \geq \max(x,0). \quad &(2.31)
\end{aligned}$$

Note that the corresponding discounted density functions (2.12), (2.15) and (2.7) are much simpler.

Remark 2.4: Here is a sketch of a derivation for (2.31). If the drift μ of the Brownian motion $X(t)$ is zero, it follows from the *reflection principle* that

$$\Pr(X(t) \leq x, M(t) > y) = \Pr(X(t) \leq x - 2y), \qquad y \geq \max(x,0). \tag{2.32}$$

By changing the probability measure, we can change the drift. If the drift μ is an arbitrary constant, the identity (2.32) is generalized as

$$\Pr(X(t) \le x, M(t) > y) = e^{y\mu/D} \Pr(X(t) \le x - 2y), \qquad y \ge \max(x, 0). \quad (2.33)$$

The joint density function of $X(t)$ and $M(t)$ can then be obtained by differentiating (2.33),

$$\begin{aligned} f_{X(t),M(t)}(x,y) &= -\frac{\partial^2}{\partial y \partial x} \Pr(X(t) \le x, M(t) > y) \\ &= -\frac{\partial}{\partial y}[e^{y\mu/D} f_{X(t)}(x - 2y)], \qquad y \ge \max(x, 0). \end{aligned} \quad (2.34)$$

Apply (2.29).

Remark 2.5: Formulas (2.15) and (2.23) show that the random variables $M(\tau)$ and $[X(\tau) - m(\tau)]$ have the same discounted density function. This is expected because, for each $t > 0$, the random variables $M(t)$ and $[X(t) - m(t)]$ have the same distribution, as can be seen as follows. For $t > 0$,

$$M(t) = \max\{X(s), 0 \le s \le t\} = \max\{X(t - s), 0 \le s \le t\}$$

and

$$X(t) - m(t) = \max\{X(t) - X(s), 0 \le s \le t\}.$$

Because $X(t-s)$ and $[X(t)-X(s)]$ have the same distribution, the random variables $M(t)$ and $[X(t) - m(t)]$ have the same distribution. On the other hand, it seems unexpected that the random variables $M(\tau)$ and $X(\tau)$ have the same discounted joint density function as $[X(\tau) - m(\tau)]$ and $X(\tau)$; this fact is obtained by comparing (2.7) with (2.20). Similarly, by comparing (2.9) with (2.19) we see that $[M(\tau) - X(\tau)]$ and $X(\tau)$ have the same discounted joint density function as $-m(\tau)$ and $X(\tau)$.

Remark 2.6: The following identity of moment-generating functions is a version of the *Wiener-Hopf factorization*,

$$M_{X(\tau)}(z) = M_{M(\tau)} \times M_{m(\tau)}(z). \quad (2.35)$$

For more general results, see Section XI.4d of Asmussen and Albrecher (2010) and Chapter 6 of Kyprianou (2006).

Remark 2.7: If τ is an Erlang(n, λ) random variable independent of $X(t)$, it can be shown that

$$f^{\delta}_{X(\tau)}(x) = \begin{cases} \kappa^n e^{-\alpha x} \sum_{j=1}^{n} \frac{\binom{2n-j-1}{n-j}}{(j-1)!(\beta-\alpha)^{n-j}} (-x)^{j-1}, & \text{if } x \le 0, \\ \kappa^n e^{-\beta x} \sum_{j=1}^{n} \frac{\binom{2n-j-1}{n-j}}{(j-1)!(\beta-\alpha)^{n-j}} x^{j-1}, & \text{if } x \ge 0, \end{cases} \tag{2.36}$$

which is a generalization of (2.12).

3 Proof of (2.7)

One way to establish (2.7) is to evaluate the integral in (2.4) with $f_{X(t),M(t)}$ given by (2.31) and the following formula for the Laplace transform of the probability density function for the first passage time of a standard Brownian motion at the level a, $a > 0$,

$$\int_0^\infty e^{-\zeta t} \frac{a e^{-a^2/(2t)}}{\sqrt{2\pi t^3}} \mathrm{d}t = e^{-a\sqrt{2\zeta}}, \qquad \zeta \ge 0. \tag{3.1}$$

Here, we present a self-contained proof of (2.7) based on college calculus. Knowledge of (2.31) or (3.1) is not required.

Let $\pi(x, y)$, $-\infty < x \le y$ and $y \ge 0$, be an arbitrary bounded function differentiable with respect to y and satisfying $\pi(x, \infty) = 0$. We define

$$\chi(x, y) = \mathrm{E}[e^{-\delta\tau} \pi(x + X(\tau), \max(x + M(\tau), y))]. \tag{3.2}$$

If π is interpreted as a reward, χ is the expected discounted reward at time τ. Because

$$\chi(0, 0) = \int_0^\infty \int_{-\infty}^{y} \pi(x, y) f^{\delta}_{X(\tau),M(\tau)}(x, y) \mathrm{d}x \mathrm{d}y, \tag{3.3}$$

our strategy to derive $f^{\delta}_{X(\tau),M(\tau)}(x, y)$ is to determine $\chi(0, 0)$.

Let y be a positive number. As a function of x, $\chi(x,y)$ satisfies the differential equation

$$D\chi_{xx}(x,y)+\mu\chi_x(x,y)-(\lambda+\delta)\chi(x,y)+\lambda\pi(x,y)=0, \quad x\in(-\infty,y), \tag{3.4}$$

where the subscripts denote partial derivatives. The general solution of the corresponding homogeneous equation is a linear combination of $e^{\alpha x}$ and $e^{\beta x}$, where $\alpha<0$ and $\beta>0$ are the roots of the characteristic equation (2.5). To obtain a particular solution $\chi^p(x,y)$ of (3.4), we apply the method of *variation of parameters* (or *variation of constants*) and find that

$$\chi^p(u,y)=[\kappa\int_0^u\pi(x,y)e^{-\alpha x}\mathrm{d}x]e^{\alpha u}+[-\kappa\int_0^u\pi(x,y)e^{-\beta x}\mathrm{d}x]e^{\beta u}, \tag{3.5}$$

where κ is defined by (2.13). The reader who is not familiar with the method of variation of parameters can substitute (3.5) in (3.4) to check that it is a particular solution. Hence, the general solution of (3.4) is of the form

$$\begin{aligned}\chi(u,y) &= A(y)e^{\alpha u}+B(y)e^{\beta u}+\chi^p(u,y),\\ &= [A(y)+\kappa\int_0^u\pi(x,y)e^{-\alpha x}\mathrm{d}x]e^{\alpha u}\\ &\quad +[B(y)-\kappa\int_0^u\pi(x,y)e^{-\beta x}\mathrm{d}x]e^{\beta u}, \qquad u\le y.\end{aligned} \tag{3.6}$$

Note that $\chi^p(0,0)=0$. Hence, the LHS of (3.3) is $A(0)+B(0)$.

Because $\chi(u,y)$ is bounded for $u\to-\infty$, and because α is negative, it follows from (3.6) that

$$A(y)=\kappa\int_{-\infty}^0\pi(x,y)e^{-\alpha x}\mathrm{d}x. \tag{3.7}$$

Applying (3.7) to (3.6) yields

$$\chi(u,y)=\kappa e^{\alpha u}\int_{-\infty}^u\pi(x,y)e^{-\alpha x}\mathrm{d}x+e^{\beta u}[B(y)-\kappa\int_0^u\pi(x,y)e^{-\beta x}\mathrm{d}x]. \tag{3.8}$$

For $y\to\infty$, $\chi(y,y)$ is bounded. It follows from this, (3.8) and β being positive that

$$B(\infty)=\kappa\int_0^\infty\pi(x,\infty)e^{-\beta x}\mathrm{d}x=0, \tag{3.9}$$

because we made the assumption that $\pi(x,\infty)=0$.

If x is close to y, we can be "almost sure" that the process will attain the value y (and hence the maximum will increase) before the contingent event (governed by τ) happens. Thus, if x is close to y, the value of χ is insensitive to small changes in y, that is,

$$\chi_y(y,y)=0. \tag{3.10}$$

For further discussion, see Goldman, Sosin and Gatto (1979). Some authors use the term *normal reflection condition* to describe (3.10). Differentiating (3.8) with respect to y, applying (3.10), and rearranging, we obtain

$$B'(y)=\kappa\int_0^y \pi_y(x,y)e^{-\beta x}\mathrm{d}x-\kappa e^{-(\beta-\alpha)y}\int_{-\infty}^y \pi_y(x,y)e^{-\alpha x}\mathrm{d}x. \tag{3.11}$$

We use this and (3.9) to see that

$$B(0)=-\int_0^\infty B'(y)\mathrm{d}y=I_1+I_2+I_3, \tag{3.12}$$

with

$$\begin{aligned}
I_1 &= -\kappa\int_0^\infty\int_0^y \pi_y(x,y)e^{-\beta x}\mathrm{d}x\mathrm{d}y,\\
I_2 &= \kappa\int_0^\infty e^{-(\beta-\alpha)y}\int_{-\infty}^0 \pi_y(x,y)e^{-\alpha x}\mathrm{d}x\mathrm{d}y,\\
I_3 &= \kappa\int_0^\infty e^{-(\beta-\alpha)y}\int_0^y \pi_y(x,y)e^{-\alpha x}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

To evaluate I_1, we change the order of integration and find that

$$I_1 = \kappa\int_0^\infty \pi(x,x)e^{-\beta x}\mathrm{d}x.$$

To evaluate I_2 and I_3, we change the order of integration and integrate by parts. This way we find that

$$\begin{aligned}
I_2 &= -\kappa\int_{-\infty}^0 \pi(x,0)e^{-\alpha x}\mathrm{d}x+(\beta-\alpha)\kappa\int_{-\infty}^0 e^{-\alpha x}\int_0^\infty \pi(x,y)e^{-(\beta-\alpha)y}\mathrm{d}y\mathrm{d}x,\\
I_3 &= -\kappa\int_0^\infty \pi(x,x)e^{-\beta x}\mathrm{d}x+(\beta-\alpha)\kappa\int_0^\infty e^{-\alpha x}\int_x^\infty \pi(x,y)e^{-(\beta-\alpha)y}\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

By (2.13), $(\beta-\alpha)\kappa = \frac{\lambda}{D}$. Thus

$$\begin{aligned}
\chi(0,0) &= A(0) + I_1 + I_2 + I_3 \\
&= \frac{\lambda}{D}\int_{-\infty}^{0} e^{-\alpha x}\int_{0}^{\infty}\pi(x,y)e^{-(\beta-\alpha)y}\mathrm{d}y\mathrm{d}x \\
&\quad +\frac{\lambda}{D}\int_{0}^{\infty} e^{-\alpha x}\int_{x}^{\infty}\pi(x,y)e^{-(\beta-\alpha)y}\mathrm{d}y\mathrm{d}x \\
&= \frac{\lambda}{D}\int_{0}^{\infty} e^{-(\beta-\alpha)y}\int_{-\infty}^{y}\pi(x,y)e^{-\alpha x}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

By comparing the last expression with the right-hand side (RHS) of (3.3), we obtain formula (2.7).

Remark 3.1: The differential equation (3.4) can be obtained from basic principles. Let $x < y$. Interpret $\chi(x,y)$ as the value of an investment which provides a single payment of $\pi(x+X(\tau), \max(x+M(\tau),y))$ at time τ and consider a time interval of length $\mathrm{d}t$. Then the instantaneous interest due on the investment must equal the expected change of value within $\mathrm{d}t$, that is

$$\chi(x,y)\delta\mathrm{d}t = [D\chi_{xx}(x,y) + \mu\chi_x(x,y)]\mathrm{d}t + \lambda\mathrm{d}t[\pi(x,y) - \chi(x,y)]. \tag{3.13}$$

From this, (3.4) follows.

Remark 3.2: For readers who are familiar with two-sided Laplace transforms, here is an alternative derivation for (2.12). The two-sided Laplace transform of $f^{\delta}_{X(\tau)}(x)$ with respect to the parameter ζ is

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\zeta x} f^{\delta}_{X(\tau)}(x)\mathrm{d}x &= \mathrm{E}[e^{-\delta\tau-\zeta X(\tau)}] = \mathrm{E}[\mathrm{E}[e^{-\delta\tau-\zeta X(\tau)}|\tau]] = \mathrm{E}[e^{-(\delta+\mu\zeta-D\zeta^2)\tau}] \\
&= \frac{\lambda}{\lambda+\delta+\mu\zeta-D\zeta^2} = \frac{\lambda}{-D(\zeta+\beta)(\zeta+\alpha)} \\
&= \kappa\left(\frac{1}{\zeta+\beta} - \frac{1}{\zeta+\alpha}\right).
\end{aligned} \tag{3.14}$$

For the Laplace transform to exist, there is the condition that the real part of ζ is between $-\beta$ and $-\alpha$. With this condition, (2.12) is the inversion of (3.14). For more applications of the method, see Albrecher et al. (2012).

Remark 3.3: Because (2.34) is valid for all $t > 0$, we can replace t by τ,

$$f_{X(\tau),M(\tau)}(x,y) = -\frac{\partial}{\partial y}[e^{y\mu/D} f_{X(\tau)}(x-2y)], \qquad y \geq \max(x,0). \tag{3.15}$$

This gives rise to a derivation of (2.7) for the case $\delta = 0$. By (2.12),

$$f_{X(\tau)}(x-2y) = \kappa e^{-\alpha(x-2y)}. \tag{3.16}$$

Here, α and β are for the case $\delta = 0$. Because $(\mu/D)+2\alpha = -(\beta-\alpha)$, formula (3.15) becomes

$$f_{X(\tau),M(\tau)}(x,y) = \kappa(\beta-\alpha)e^{-\alpha x-(\beta-\alpha)y}. \tag{3.17}$$

If τ is an Erlang(n,λ) random variable, we use (2.36) in place of (2.12).

Remark 3.4: If $\delta = 0$, (2.7) can be obtained by differentiating formula (2.1.1.6) on page 251 of Borodin and Salminen (2002).

4 Valuation of basic options

In the rest of this paper, $S(t)$ denotes the time-t price of a share of a stock or a unit of a mutual fund. We assume

$$S(t) = S(0)e^{X(t)}, \qquad t \geq 0, \tag{4.1}$$

where $X(t)$ is the linear Brownian motion defined by (2.1). We note that

$$\mathrm{E}[S(t)] = S(0)e^{\vartheta t}, \qquad t \geq 0,$$

where

$$\vartheta = \mu + D. \tag{4.2}$$

In this section we evaluate the expected discounted value of the payoff $b(S(\tau))$,

$$\mathrm{E}[e^{-\delta\tau}b(S(\tau))], \tag{4.3}$$

for various payoff or benefit functions $b(s)$. Under the assumption that the random variable τ is independent of the process $X(t)$, the expectation (4.3) is the double integral, with respect to x and t, of

$$e^{-\delta t} b(S(0)e^x) f_{X(t)}(x) f_\tau(t). \tag{4.4}$$

Integrating out the t variable (from 0 to ∞) yields

$$\mathrm{E}[e^{-\delta\tau} b(S(\tau))] = \int_{-\infty}^{\infty} b(S(0)e^x) f^{\delta}_{X(\tau)}(x) \mathrm{d}x. \tag{4.5}$$

Because τ is exponential, we can use (2.12) to see that

$$\mathrm{E}[e^{-\delta\tau} b(S(\tau))] = \kappa \int_{-\infty}^{0} b(S(0)e^x) e^{-\alpha x} \mathrm{d}x + \kappa \int_{0}^{\infty} b(S(0)e^x) e^{-\beta x} \mathrm{d}x. \tag{4.6}$$

With this formula, determining the expected discounted value of a payoff $b(S(\tau))$ becomes a first-year calculus exercise.

In the special case where $b(s) = s$, equation (4.6) yields

$$\begin{aligned} \mathrm{E}[e^{-\delta\tau} S(\tau)] &= \frac{\kappa(\beta-\alpha)}{(1-\alpha)(\beta-1)} S(0) \\ &= \frac{\lambda}{\lambda+\delta} \frac{-\alpha\beta}{(1-\alpha)(\beta-1)} S(0). \end{aligned} \tag{4.7}$$

Because α and β are the roots of (2.5), we have

$$\alpha + \beta = -\frac{\mu}{D}, \quad -\alpha\beta = \frac{\lambda+\delta}{D}, \tag{4.8}$$

and it follows that

$$\frac{1}{\lambda+\delta} \frac{-\alpha\beta}{(1-\alpha)(\beta-1)} = \frac{1}{\lambda+\delta-\vartheta}. \tag{4.9}$$

With this, (4.7) becomes

$$\mathrm{E}[e^{-\delta\tau} S(\tau)] = \frac{\lambda}{\lambda+\delta-\vartheta} S(0). \tag{4.10}$$

It is interesting to compare this formula with

$$e^{-\delta t} \mathrm{E}[S(t)] = e^{-(\delta-\vartheta)t} S(0), \tag{4.11}$$

which is for each positive t. We note that (4.11) can be used to confirm (4.10) and with that, (4.7). In the special case

$$\vartheta = \delta \tag{4.12}$$

the coefficients of $S(0)$ in (4.10) and (4.11) are 1. Condition (4.12) arises in the case where the stock pays no dividends, δ is the risk-free force of interest and the probability measure is risk-neutral.

Out-of-the-money all-or-nothing call option

The payoff function is

$$b(s) = s^n I_{(s>K)}. \tag{4.13}$$

Here, n is a real number; $n = 0$ and $n = 1$ are two cases of particular interest. The constant K is greater than $S(0)$; the term "out-of-the-money" means that the option, if exercised now, is worth nothing. Let

$$k = \ln[K/S(0)], \tag{4.14}$$

which is positive because $K > S(0)$. Then (4.6) is

$$\begin{aligned} \mathrm{E}[e^{-\delta\tau}[S(\tau)]^n I_{(S(\tau)>K)}|S(0) < K] &= \kappa \int_k^\infty [S(0)e^x]^n e^{-\beta x}\mathrm{d}x \\ = \kappa[S(0)]^n \frac{e^{-(\beta-n)k}}{\beta - n} &= \frac{\kappa K^n}{\beta - n}\left[\frac{S(0)}{K}\right]^\beta. \end{aligned} \tag{4.15}$$

The convergence of the integral requires the condition $\beta > n$.

At-the-money all-or-nothing call option

By setting $S(0) = K$ in (4.15), we have

$$\mathrm{E}[e^{-\delta\tau}[S(\tau)]^n I_{(S(\tau)>K)}|S(0) = K] = \frac{\kappa K^n}{\beta - n}. \tag{4.16}$$

Remark 4.1: With (4.16), we can now give an interpretation for (4.15). For the out-of-the-money option to have any value, the stock price must first reach the level K. At that time, the option becomes an at-the-money option; this explains the factor $\kappa K^n/(\beta - n)$ in (4.15). To understand the remaining factor in (4.15), we let $\mathcal{T}$ be the first time when the stock price process $S(t)$ rises to level K and apply (2.28) with $y = \ln[K/S(0)]$. Thus we have

$$\mathrm{E}[e^{-\delta\mathcal{T}} I_{(\tau>\mathcal{T})}] = \left[\frac{S(0)}{K}\right]^{\beta}. \tag{4.17}$$

The LHS is the expected discounted value of a contingent payment of 1 payable at the first time when the stock price rises to level K, if τ has not yet occurred. It may remind actuaries the concept of a single premium for a $\mathcal{T}$-year *pure endowment* in life insurance mathematics, where $\mathcal{T}$, however, is fixed and certain.

Out-of-the-money call option

The payoff function is

$$b(s) = (s - K)_+ = sI_{(s>K)} - KI_{(s>K)}. \tag{4.18}$$

Here, $K > S(0)$ because the option is out-of-the-money. By applying (4.15) with $n = 1$ and $n = 0$, we have

$$\begin{aligned} \mathrm{E}[e^{-\delta\tau}[S(\tau) - K]_+ | S(0) < K] &= \kappa[S(0)]^{\beta}\frac{K^{-(\beta-1)}}{\beta - 1} - K\kappa[S(0)]^{\beta}\frac{K^{-\beta}}{\beta} \\ &= \frac{\kappa K}{\beta(\beta-1)}\left[\frac{S(0)}{K}\right]^{\beta}. \end{aligned} \tag{4.19}$$

Remark 4.2: As a check, we differentiate both sides of (4.19) with respect to K using the formula

$$\frac{\mathrm{d}}{\mathrm{d}K}(s - K)_+ = -I_{(s>K)}, \qquad K \neq s.$$

The result is the negative of (4.13) for $n = 0$.

At-the-money call option

The payoff function is

$$b(s) = [s - S(0)]_+, \tag{4.20}$$

which is (4.18) with $K = S(0)$. Thus, it follows from (4.19) that

$$\begin{aligned} \mathrm{E}[e^{-\delta\tau}[S(\tau) - S(0)]_+] &= \frac{\kappa S(0)}{\beta(\beta-1)} \\ &= \frac{1-\alpha}{\beta(\beta-\alpha)}\mathrm{E}[e^{-\delta\tau}S(\tau)] \end{aligned} \tag{4.21}$$

by the first equality in (4.7).

Remark 4.3: The two factors in the last expression of (4.19) can be explained by (4.17) and the first equality in (4.21).

Out-of-the-money all-or-nothing put option

The payoff function is

$$b(s) = s^n I_{(s<K)}. \tag{4.22}$$

Here, n is a real number, and $K < S(0)$ because the option is out-of-the-money. Since $k = \ln[K/S(0)] < 0$, it follows from (4.6) that

$$\begin{aligned} &\mathrm{E}[e^{-\delta\tau}[S(\tau)]^n I_{(S(\tau)<K)}|S(0) > K] = \kappa \int_{-\infty}^{k} [S(0)e^x]^n e^{-\alpha x}\mathrm{d}x \\ &= \kappa[S(0)]^n \frac{e^{-(\alpha-n)k}}{-(\alpha-n)} = \frac{\kappa K^n}{n-\alpha}\left[\frac{K}{S(0)}\right]^{-\alpha}. \end{aligned} \tag{4.23}$$

The convergence of the integral requires the condition $\alpha < n$.

Remark 4.4: The factor $[K/S(0)]^{-\alpha}$ in (4.23) is the expected discounted value of a contingent payment of 1 payable at the first time when the stock price drops to level K, if τ has not yet occurred.

Out-of-the-money put option

The payoff function is

$$b(s) = (K - s)_+ = KI_{(s<K)} - sI_{(s<K)}. \tag{4.24}$$

By applying (4.23) with $n = 0$ and $n = 1$, we have

$$\mathrm{E}[e^{-\delta\tau}[K - S(\tau)]_+ | S(0) > K] \;=\; \frac{\kappa K}{-\alpha(1-\alpha)}\left[\frac{K}{S(0)}\right]^{-\alpha}. \tag{4.25}$$

At-the-money put option

The payoff function is

$$b(s) = [S(0) - s]_+, \tag{4.26}$$

which is (4.24) with $K = S(0)$. Thus, it follows from (4.25) that

$$\begin{aligned} \mathrm{E}[e^{-\delta\tau}[S(0) - S(\tau)]_+] &= \frac{\kappa S(0)}{-\alpha(1-\alpha)} \\ &= \frac{\beta - 1}{-\alpha(\beta - \alpha)}\mathrm{E}[e^{-\delta\tau}S(\tau)] \end{aligned} \tag{4.27}$$

by the first equality in (4.7).

In-the-money put and call options

To evaluate in-the-money put and call options, we can use *put-call parity.* To derive the put-call parity relationship, we start with the identity

$$[K - S(\tau)]_+ - [S(\tau) - K]_+ = K - S(\tau). \tag{4.28}$$

Multiplying (4.28) with $e^{-\delta\tau}$, taking expectations, and applying (2.14) yields

$$\begin{aligned} &\mathrm{E}[e^{-\delta\tau}[K - S(\tau)]_+] - \mathrm{E}[e^{-\delta\tau}[S(\tau) - K]_+] \\ &= \frac{\lambda}{\lambda + \delta}K - \mathrm{E}[e^{-\delta\tau}S(\tau)]. \end{aligned} \tag{4.29}$$

From this, we obtain the in-the-money formulas:

$$\begin{aligned}
&\mathrm{E}[e^{-\delta\tau}[K - S(\tau)]_+|S(0) < K] \\
&= \frac{\kappa K}{\beta(\beta-1)}\left[\frac{S(0)}{K}\right]^{\beta} + \frac{\lambda}{\lambda+\delta}K - \mathrm{E}[e^{-\delta\tau}S(\tau)]
\end{aligned} \tag{4.30}$$

by (4.19), and

$$\begin{aligned}
&\mathrm{E}[e^{-\delta\tau}[S(\tau) - K]_+|S(0) > K] \\
&= \frac{\kappa K}{-\alpha(1-\alpha)}\left[\frac{K}{S(0)}\right]^{-\alpha} - \frac{\lambda}{\lambda+\delta}K + \mathrm{E}[e^{-\delta\tau}S(\tau)]
\end{aligned} \tag{4.31}$$

by (4.25).

Remark 4.5: Apply (4.10) to the last term in (4.31). Then, equating the RHS of (4.19) at $S(0) = K$ with that of (4.31) at $S(0) = K$ yields the identity

$$\frac{\kappa}{\beta(\beta-1)} = \frac{\kappa}{-\alpha(1-\alpha)} - \frac{\lambda}{\lambda+\delta} + \frac{\lambda}{\lambda+\delta-\vartheta}, \tag{4.32}$$

which we shall use in Section 11. Equating the derivative of the RHS of (4.19) with respect to $S(0)$ at $S(0) = K$ with that of (4.31) at $S(0) = K$ yields a simpler identity

$$\frac{\kappa}{\beta-1} = -\frac{\kappa}{1-\alpha} + \frac{\lambda}{\lambda+\delta-\vartheta}. \tag{4.33}$$

As a check, we replace κ in (4.33) by the last expression in (2.13) and retrieve (4.9) after simplification.

Remark 4.6: Results corresponding to the expected discounted value of the put option payoff,

$$b(S(\tau)) = [K - S(\tau)]_+,$$

can be found in the literature. For a "rollup" GMDB in a variable annuity, one would consider a more general payoff,

$$b(\tau, S(\tau)) = [Ke^{p\tau} - S(\tau)]_+. \tag{4.34}$$

Here, we follow Ulm (2006, 2008) to use the letter p to denote the "roll-up" rate. Because (4.34) can be rewritten as

$$e^{p\tau}[K - e^{-p\tau}S(\tau)]_+, \tag{4.35}$$

its expected discounted value can be determined using formulas in this section with δ changed to $\delta - p$ and μ changed to $\mu - p$. With this substitution, (4.25) and (4.30) should be compared to formula (24) in Ulm (2008). For the special case of $\mu = \delta - D$ (which is equivalent to (4.12)), $p = \delta$ and $K = S(0)$, we have

$$\begin{aligned} \mathrm{E}[e^{-\delta\tau}[Ke^{p\tau} - S(\tau)]_+] &= S(0)\mathrm{E}[[1 - e^{-D\tau+\sigma W(\tau)}]_+] \\ &= S(0)\frac{1}{\sqrt{1+\frac{4\lambda}{D}}}. \end{aligned} \tag{4.36}$$

Formula (4.36) can be obtained using (4.27) or the second formula on page 20 of Profeta et al. (2010).

Remark 4.7: We shall derive in Section 9 the expected discounted value of a Margrabe option or exchange option, whose payoff is

$$[S_1(\tau) - S_2(\tau)]_+. \tag{4.37}$$

It is obvious that (4.34) is a special case of (4.37).

Remark 4.8: In the context of variable annuities, Ulm (2006, 2008) has considered the possibility of lapses or policy surrenders. To make the problem tractable, we follow Ulm in assuming that lapses are independent of mortality and the stock price process. We model the possibility of a lapse by means of a nonincreasing function $\eta(t)$: Given $T_x = t$, $\eta(t)$ is the probability that the policy has not lapsed by time t. Here, T_x is the time-until-death random variable for a life age x. For a payoff such as (4.34), there is zero cash value or surrender value. Thus, the problem is to evaluate

$$\mathrm{E}[e^{-\delta T_x}\eta(T_x)b(T_x, S(T_x))]. \tag{4.38}$$

If $\eta(t) = e^{-\nu t}$, $t > 0$, i.e., if the force of surrender is a positive constant ν, then (4.38) is

$$\mathrm{E}[e^{-(\delta+\nu)T_x} b(T_x, S(T_x))],$$

which means that we use a higher force of interest. (Our ν is Ulm's λ.) By approximating the density function of T_x with a linear combination of exponential densities and the lapse function $\eta(t)$ with a linear combination of exponential functions, we can evaluate (4.38) with sufficient accuracy.

5 Lookback options

Many equity-indexed annuities credit interest using a high water mark method or a low water mark method (Streiff and DiBiase 1999, Chapter 4; Tiong 2000; Lee 2003). These methods are forms of lookback options. In this section we value lookback options exercised at time τ. The corresponding time-T formulas, where T is a fixed time, can be found in Gerber and Shiu (2003a). The readers will find that the formulas in this section are much simpler.

Fixed strike lookback call option

The payoff at time τ is

$$[\max(H, \max_{0 \le t \le \tau} S(t)) - K]_+ = [\max(H, S(0)e^{M(\tau)}) - K]_+. \tag{5.1}$$

Here, H is a positive constant with $H \ge S(0)$; it can be interpreted as the maximum level of the stock's historical ($t \le 0$) prices. To value this payoff, we need to distinguish whether the strike price K is higher or lower than the historical maximum price H, that is, we need to distinguish whether the option is out-of-the money or in-the-money.

Out-of-the-money fixed strike lookback call option

For $K > H$, the payoff (5.1) simplifies as

$$[S(0)e^{M(\tau)} - K]_+, \tag{5.2}$$

whose time-0 value, because of formula (2.15), is

$$\begin{aligned} \int_k^\infty [S(0)e^y - K] f^\delta_{M(\tau)}(y) \mathrm{d}y &= \frac{\lambda}{\lambda+\delta}[S(0)\frac{\beta e^{-(\beta-1)k}}{\beta-1} - Ke^{-\beta k}] \\ &= \frac{\lambda}{\lambda+\delta}\frac{K}{\beta-1}\left[\frac{S(0)}{K}\right]^\beta. \end{aligned} \tag{5.3}$$

The lower limit of the integral is $k = \ln[K/S(0)]$. It is positive because $K > H \geq S(0)$. See (4.17) in Remark 4.1 for an interpretation of the factor $[S(0)/K]^\beta$. Also, by the second half of (4.8), another expression for the option value is

$$\frac{\lambda}{D}\frac{K}{-\alpha\beta(\beta-1)}\left[\frac{S(0)}{K}\right]^\beta. \tag{5.4}$$

In-the-money fixed strike lookback call option

For $K < H$, the payoff (5.1) is

$$\max(H, S(0)e^{M(\tau)}) - K. \tag{5.5}$$

By rewriting (5.5) as

$$H - K + [S(0)e^{M(\tau)} - H]_+ \tag{5.6}$$

and using (5.3) with K replaced by H, we find that the time-0 value of (5.5) is

$$\frac{\lambda}{\lambda+\delta}\left\{H - K + \frac{H}{\beta-1}\left[\frac{S(0)}{H}\right]^\beta\right\}. \tag{5.7}$$

Floating strike lookback put option

The payoff at time τ is

$$\max(H, \max_{0\le t\le\tau} S(t)) - S(\tau), \tag{5.8}$$

where $H \ge S(0)$. By comparing (5.8) with (5.5), we see that its time-0 value is (5.7) but with $\frac{\lambda}{\lambda+\delta}K$, which is $\mathrm{E}[e^{-\delta\tau}K]$, replaced by $\mathrm{E}[e^{-\delta\tau}S(\tau)]$. The result is

$$\frac{\lambda}{\lambda+\delta}\left\{H + \frac{H}{\beta-1}\left[\frac{S(0)}{H}\right]^{\beta}\right\} - \mathrm{E}[e^{-\delta\tau}S(\tau)]. \tag{5.9}$$

In the special case where $H = S(0)$, the time-0 value (5.9) simplifies as

$$\begin{aligned}\frac{\lambda}{\lambda+\delta}\frac{\beta}{\beta-1}S(0) - \mathrm{E}[e^{-\delta\tau}S(\tau)] &= \frac{1-\alpha}{-\alpha}\mathrm{E}[e^{-\delta\tau}S(\tau)] - \mathrm{E}[e^{-\delta\tau}S(\tau)] \\ &= \frac{1}{-\alpha}\mathrm{E}[e^{-\delta\tau}S(\tau)]. \end{aligned} \tag{5.10}$$

This result can be reformulated as

$$\mathrm{E}[e^{-\delta\tau}\max_{0\le t\le\tau} S(t)] = \left(\frac{1}{-\alpha}+1\right)\mathrm{E}[e^{-\delta\tau}S(\tau)]. \tag{5.11}$$

Fractional floating strike lookback put option

For a given $\gamma \in (0,1]$, we consider the time-τ payoff

$$[\gamma \max_{0\le t\le\tau} S(t) - S(\tau)]_+ = S(0)[\gamma e^{M(\tau)} - e^{X(\tau)}]_+. \tag{5.12}$$

We want to determine its expected discounted value,

$$S(0)\mathrm{E}[e^{-\delta\tau}[\gamma e^{M(\tau)} - e^{X(\tau)}]_+]. \tag{5.13}$$

One way is to use formula (2.7),

$$\mathrm{E}[e^{-\delta\tau}[\gamma e^{M(\tau)} - e^{X(\tau)}]_+] = \int_0^\infty \left[\int_{-\infty}^{y+\ln\gamma} (\gamma e^y - e^x) f^{\delta}_{X(\tau),M(\tau)}(x,y)dx\right]\mathrm{d}y. \tag{5.14}$$

Another way is to notice

$$[\gamma e^{M(\tau)} - e^{X(\tau)}]_+ = e^{M(\tau)}[\gamma - e^{X(\tau)-M(\tau)}]_+$$

for which formula (2.11) can be applied,

$$\begin{aligned}
\mathrm{E}[e^{-\delta\tau}e^{M(\tau)}[\gamma - e^{X(\tau)-M(\tau)}]_+] &= \int_0^\infty \int_0^\infty e^y[\gamma - e^{-z}]_+ f^\delta_{M(\tau),M(\tau)-X(\tau)}(y,z)\mathrm{d}y\mathrm{d}z \\
&= \frac{\lambda}{D}\left[\int_0^\infty e^y e^{-\beta y}\mathrm{d}y\right]\left[\int_0^\infty [\gamma - e^{-z}]_+ e^{\alpha z}\mathrm{d}z\right] \\
&= \frac{\lambda}{D}\frac{1}{\beta-1}\frac{\gamma^{1-\alpha}}{-\alpha(1-\alpha)} \\
&= \gamma^{1-\alpha}\frac{\lambda}{\lambda+\delta}\frac{\beta}{(1-\alpha)(\beta-1)} \\
&= \gamma^{1-\alpha}\frac{1}{-\alpha}\mathrm{E}[e^{-\delta\tau}e^{X(\tau)}]. \qquad (5.15)
\end{aligned}$$

In view of (5.10), formula (5.15) can be rewritten in the following intriguing way,

$$\mathrm{E}[e^{-\delta\tau}[\gamma e^{M(\tau)} - e^{X(\tau)}]_+] = \gamma^{1-\alpha}\mathrm{E}[e^{-\delta\tau}(e^{M(\tau)} - e^{X(\tau)})]. \qquad (5.16)$$

Finally, we multiply (5.15) by $S(0)$ to obtain

$$\mathrm{E}[e^{-\delta\tau}[\gamma \max_{0\le t\le\tau} S(t) - S(\tau)]_+] = \frac{\gamma^{1-\alpha}}{-\alpha}\mathrm{E}[e^{-\delta\tau}S(\tau)], \qquad (5.17)$$

which generalizes (5.11). The surprising formulas (5.16) and (5.17) do not seem to have probabilistic interpretations.

Fixed strike lookback put option

The payoff at time τ is

$$[K - \min(H, \min_{0\le t\le\tau} S(t))]_+ = [K - \min(H, S(0)e^{m(\tau)})]_+. \qquad (5.18)$$

Here, H is a positive constant, with $H \le S(0)$; it can be interpreted as the minimum level of the stock's historical ($t < 0$) prices. To value this payoff, we need to distinguish whether the strike price K is lower or higher than the historical minimum price H, that is, we need to distinguish whether the option is out-of-the money or in-the-money.

Out-of-the-money fixed strike lookback put option

For $K < H$, the payoff (5.18) simplifies as

$$[K - S(0)e^{m(\tau)}]_+, \tag{5.19}$$

whose time-0 value, because of formula (2.22), is

$$\int_{-\infty}^{k} [K - S(0)e^{y}] f^{\delta}_{m(\tau)}(y)\mathrm{d}y = \frac{\lambda}{\lambda+\delta}\frac{K}{1-\alpha}\left[\frac{K}{S(0)}\right]^{-\alpha}. \tag{5.20}$$

The upper limit of the integral is $k = \ln[K/S(0)]$. It is negative because $K < H \leq S(0)$.

In-the-money fixed strike lookback put option

For $K > H$, the payoff (5.18) is

$$K - \min(H, S(0)e^{m(\tau)}) = K - H + [H - S(0)e^{m(\tau)}]_+, \tag{5.21}$$

whose time-0 value is

$$\frac{\lambda}{\lambda+\delta}\left\{K - H + \frac{H}{1-\alpha}\left[\frac{H}{S(0)}\right]^{-\alpha}\right\}. \tag{5.22}$$

Floating strike lookback call option

The payoff at time τ is

$$S(\tau) - \min(H, \min_{0\leq t\leq\tau} S(t)), \tag{5.23}$$

where $0 < H \leq S(0)$. Its time-0 value is (5.22) with $\frac{\lambda}{\lambda+\delta}K$ replaced by $\mathrm{E}[e^{-\delta\tau}S(\tau)]$, namely,

$$\mathrm{E}[e^{-\delta\tau}S(\tau)] + \frac{\lambda}{\lambda+\delta}\left\{-H + \frac{H}{1-\alpha}\left[\frac{H}{S(0)}\right]^{-\alpha}\right\}. \tag{5.24}$$

In the special case where $H = S(0)$, the time-0 value (5.24) simplifies as

$$\begin{aligned}
&\mathrm{E}[e^{-\delta\tau}S(\tau)] - \frac{\lambda}{\lambda+\delta}\frac{-\alpha}{1-\alpha}S(0) \\
&= \mathrm{E}[e^{-\delta\tau}S(\tau)] - \frac{\beta-1}{\beta}\mathrm{E}[e^{-\delta\tau}S(\tau)] \\
&= \frac{1}{\beta}\mathrm{E}[e^{-\delta\tau}S(\tau)]. \qquad (5.25)
\end{aligned}$$

Analogous to (5.11), this result can be reformulated as

$$\mathrm{E}[e^{-\delta\tau}\min_{0\le t\le\tau} S(t)] = \left(1-\frac{1}{\beta}\right)\mathrm{E}[e^{-\delta\tau}S(\tau)]. \tag{5.26}$$

Fractional floating strike lookback call option

For $\gamma \ge 1$, we consider the time-τ payoff

$$\begin{aligned}
[S(\tau)-\gamma\min_{0\le t\le\tau} S(t)]_+ &= S(0)[e^{X(\tau)}-\gamma e^{m(\tau)}]_+. \\
&= S(0)e^{m(\tau)}[e^{X(\tau)-m(\tau)}-\gamma]_+ \qquad (5.27)
\end{aligned}$$

Its expected discounted value is $S(0)$ times the following expectation, evaluated with formula (2.21),

$$\begin{aligned}
\mathrm{E}[e^{-\delta\tau}e^{m(\tau)}[e^{X(\tau)-m(\tau)}-\gamma]_+] &= \frac{\lambda}{D}\left[\int_{-\infty}^{0} e^{y}e^{-\alpha y}\mathrm{d}y\right]\left[\int_{0}^{\infty}[e^{z}-\gamma]_+e^{-\beta z}\mathrm{d}z\right] \\
&= \frac{\lambda}{D}\frac{1}{1-\alpha}\frac{\gamma^{1-\beta}}{\beta(\beta-1)} \\
&= \frac{1}{\gamma^{\beta-1}}\frac{\lambda}{\lambda+\delta}\frac{-\alpha}{(1-\alpha)(\beta-1)} \\
&= \frac{1}{\gamma^{\beta-1}}\frac{1}{\beta}\mathrm{E}[e^{-\delta\tau}e^{X(\tau)}]. \qquad (5.28)
\end{aligned}$$

In view of (5.25), formula (5.28) can be rewritten as

$$\mathrm{E}[e^{-\delta\tau}[e^{X(\tau)}-\gamma e^{m(\tau)}]_+] = \gamma^{-(\beta-1)}\mathrm{E}[e^{-\delta\tau}(e^{X(\tau)}-e^{m(\tau)})]. \tag{5.29}$$

Similar to (5.17), we have

$$\mathrm{E}[e^{-\delta\tau}[S(\tau)-\gamma\min_{0\le t\le\tau} S(t)]_+] = \frac{1}{\beta\gamma^{\beta-1}}\mathrm{E}[e^{-\delta\tau}S(\tau)]. \tag{5.30}$$

High-low option

The high-low option is also called the length-of-range option. Its payoff at time τ is

$$\max(\overline{H}, \max_{0\le t\le\tau} S(t)) - \min(\underline{H}, \min_{0\le t\le\tau} S(t)), \tag{5.31}$$

where $0 < \underline{H} \leq S(0) \leq \overline{H}$. The parameters $\underline{H}$ and $\overline{H}$ can be interpreted as the past stock-price minimum and maximum, respectively. We note that the payoff (5.31) is the sum of (5.8) with $H = \overline{H}$ and (5.23) with $H = \underline{H}$. Hence it follows from (5.9) and (5.24) that the time-0 value of the high-low option is

$$\frac{\lambda}{\lambda+\delta}\left\{\overline{H} + \frac{\overline{H}}{\beta-1}\left[\frac{S(0)}{\overline{H}}\right]^{\beta} - \underline{H} + \frac{\underline{H}}{1-\alpha}\left[\frac{\underline{H}}{S(0)}\right]^{-\alpha}\right\}. \tag{5.32}$$

In the special case where $\underline{H} = S(0) = \overline{H}$, the time-0 value (5.32) simplifies as

$$S(0)\frac{\lambda}{\lambda+\delta}\frac{\beta-\alpha}{(\beta-1)(1-\alpha)} = \frac{\beta-\alpha}{-\alpha\beta}\mathrm{E}[e^{-\delta\tau}S(\tau)]. \tag{5.33}$$

By rewriting (5.33) as

$$\left(\frac{1}{-\alpha} + \frac{1}{\beta}\right)\mathrm{E}[e^{-\delta\tau}S(\tau)], \tag{5.34}$$

we can check this result using (5.11) and (5.26).

Remark 5.1: Milevsky and Posner (2001) have evaluated (5.8) with a risk-neutral stock price process and $H = S(0)$. They also assume that the stock pays dividends continuously at a rate proportional to its price. With l denoting the dividend yield rate, $\delta = r$, and $\mu = r - D - l$, the RHS of (5.10) is

$$\frac{2D}{(r-D-l) + \sqrt{(r-D-l)^2 + 4D(\lambda+r)}} \times S(0)\frac{\lambda}{\lambda+l}. \tag{5.35}$$

Although it seems rather different from formula (38) on page 117 of Milevsky and Posner (2001), both formulas produce the same values.

Remark 5.2: Multiplying (5.11) with (5.26) and then applying (4.7) and (2.14), we obtain the identity,

$$\mathrm{E}[e^{-\delta\tau}\max_{0\leq t\leq\tau} S(t)]\mathrm{E}[e^{-\delta\tau}\min_{0\leq t\leq\tau} S(t)] = \mathrm{E}[e^{-\delta\tau}S(\tau)]\mathrm{E}[e^{-\delta\tau}]S(0). \tag{5.36}$$

The quantity $\mathrm{E}[e^{-\delta\tau}]$ can be interpreted as the time-0 value of a contingent zero-coupon bond that pays 1 at time τ. By considering $aX(t)$ instead of $X(t)$, where a

is an arbitrary real number, (5.36) can be generalized as

$$\mathrm{E}[e^{-\delta\tau} \max_{0\le t\le\tau} S(t)^a]\mathrm{E}[e^{-\delta\tau} \min_{0\le t\le\tau} S(t)^a] = \mathrm{E}[e^{-\delta\tau} S(\tau)^a]\mathrm{E}[e^{-\delta\tau}]S(0)^a, \tag{5.37}$$

which is the Wiener-Hopf factorization (2.35) when $\delta = 0$ and $S(0) = 1$.

6 Barrier options

A *barrier option* is an option whose payoff depends on whether or not the price of the underlying asset has reached a predetermined level or *barrier*. *Knock-out options* are those which go out of existence if the asset price reaches the barrier, and *knock-in options* are those which come into existence if the barrier is reached. We have the following parity relation:

$$\text{Knock-out option} + \text{Knock-in option} = \text{Ordinary option}. \tag{6.1}$$

As in the previous two sections, we let $S(t)$ denote the price of one unit of the underlying asset at time t. Let L denote the barrier and $\ell = \ln[L/S(0)]$. The option is exercised at time τ, an exponential random variable independent of the asset price process.

If $L > S(0)$ ($\ell > 0$), we are dealing with *up-and-out* and *up-and-in* options, whose payoffs are

$$I_{([\max_{0\le t\le\tau} S(t)]<L)} b(S(\tau)) = I_{(M(\tau)<\ell)} b(S(0)e^{X(\tau)}) \tag{6.2}$$

and

$$I_{([\max_{0\le t\le\tau} S(t)]\ge L)} b(S(\tau)) = I_{(M(\tau)\ge\ell)} b(S(0)e^{X(\tau)}), \tag{6.3}$$

respectively. The expected discounted value of (6.2) is

$$\begin{aligned}&\int_0^\infty \left[\int_{-\infty}^y I_{(y<\ell)} b(S(0)e^x) f^\delta_{X(\tau),M(\tau)}(x,y)\mathrm{d}x\right]\mathrm{d}y\\ &= \frac{\lambda}{D}\int_0^\ell \left[\int_{-\infty}^y b(S(0)e^x)e^{-\alpha x}\mathrm{d}x\right] e^{-(\beta-\alpha)y}\mathrm{d}y \end{aligned}\tag{6.4}$$

because of (2.7). Similarly, the expected discounted value of (6.3) is

$$\frac{\lambda}{D}\int_{\ell}^{\infty}\left[\int_{-\infty}^{y} b(S(0)e^{x})e^{-\alpha x}\mathrm{d}x\right]e^{-(\beta-\alpha)y}\mathrm{d}y; \tag{6.5}$$

it can also be determined from the parity relationship (6.1), that is, it is the difference between (4.6) and (6.4).

If $0 < L < S(0)$ ($\ell < 0$), we are dealing with *down-and-out* and *down-and-in* options, whose payoffs are

$$I_{([\min_{0\le t\le\tau} S(t)]>L)}b(S(\tau)) = I_{(m(\tau)>\ell)}b(S(0)e^{X(\tau)}) \tag{6.6}$$

and

$$I_{([\min_{0\le t\le\tau} S(t)]\le L)}b(S(\tau)) = I_{(m(\tau)\le\ell)}b(S(0)e^{X(\tau)}), \tag{6.7}$$

respectively. Their expected discounted values are

$$\frac{\lambda}{D}\int_{\ell}^{0}\left[\int_{y}^{\infty} b(S(0)e^{x})e^{-\beta x}\mathrm{d}x\right]e^{(\beta-\alpha)y}\mathrm{d}y \tag{6.8}$$

and

$$\frac{\lambda}{D}\int_{-\infty}^{\ell}\left[\int_{y}^{\infty} b(S(0)e^{x})e^{-\beta x}\mathrm{d}x\right]e^{(\beta-\alpha)y}\mathrm{d}y, \tag{6.9}$$

respectively.

By evaluating (6.4), (6.5), (6.8) or (6.9) for various payoff functions $b(.)$, we obtain valuation formulas for various barrier options. We present the results in an Appendix.

7 Dynamic fund protection

Let $S(t)$ denote the value of one unit of a mutual fund at time t. Consider an investor purchasing one unit of the fund at time 0, together with the following "dynamic fund protection" guarantee effective until time τ. His account value will never drop below a fixed level $L,\ 0 < L \leq S(0)$. As soon as his account value drops below the guaranteed level L, his account will be credited with a sufficient number of fund units to restore the account value to the guaranteed level L. For $t \geq 0$, let $n(t)$ denote the number of units of the mutual fund in the investor's account. The following three conditions must be satisfied:

(i) $n(0) = 1$;

(ii) $n(t)$ is a nondecreasing function of t;

(iii) $n(t)S(t) \geq L, \quad t \geq 0.$

Condition (i) merely states that, at time 0, the investor has one unit of the mutual fund. Condition (ii) means that additional units can be credited to the investor's account, but they can never be taken away afterwards. Condition (iii) is the guarantee. From conditions (ii) and (iii), it follows that

$$n(t) \geq n(s) \geq \frac{L}{S(s)} \quad \text{for } 0 \leq s \leq t;$$

hence

$$n(t) \geq \max_{0 \leq s \leq t} \frac{L}{S(s)} = \frac{L}{S(0)} e^{-m(t)},$$

where $m(t)$ is defined by (2.17). Because of (i), we have

$$n(t) \geq \max\{1, \frac{L}{S(0)} e^{-m(t)}\}. \tag{7.1}$$

Thus, by replacing the inequality sign in (7.1) by an equal sign, we obtain the number-of-units function for providing the guarantee with the least cost,

$$n(t) = \max\{1, \frac{L}{S(0)} e^{-m(t)}\}, \tag{7.2}$$

which is formula (1.5) in Gerber and Pafumi (2000).

The expected discounted value of the contract is

$$\mathrm{E}[e^{-\delta\tau}n(\tau)S(\tau)], \tag{7.3}$$

which is the sum of two expected discounted values, (1.11) and

$$\mathrm{E}[e^{-\delta\tau}[n(\tau)-1]S(\tau)]. \tag{7.4}$$

The quantity (7.4) can be interpreted as the cost for providing the guarantee.

To evaluate (7.4), note that

$$\begin{aligned}
[n(\tau)-1]S(\tau) &= [\frac{L}{S(0)}e^{-m(\tau)}-1]_{+}S(\tau) \\
&= [L-S(0)e^{m(\tau)}]_{+}e^{X(\tau)-m(\tau)}
\end{aligned} \tag{7.5}$$

and apply (2.21). Then, the expectation (7.4) is

$$\begin{aligned}
&\frac{\lambda}{D}\left[\int_{-\infty}^{0}[L-S(0)e^{y}]_{+}e^{-\alpha y}\mathrm{d}y\right]\left[\int_{0}^{\infty}e^{z}e^{-\beta z}\mathrm{d}z\right] \\
&= \frac{\lambda}{D}\frac{L}{-\alpha(1-\alpha)}\left[\frac{L}{S(0)}\right]^{-\alpha}\frac{1}{\beta-1} \\
&= \left[\frac{L}{S(0)}\right]^{-\alpha}L\frac{\lambda}{\lambda+\delta}\frac{\beta}{(1-\alpha)(\beta-1)},
\end{aligned} \tag{7.6}$$

which, in view of (4.7), can be expressed as

$$\left[\frac{L}{S(0)}\right]^{1-\alpha}\frac{1}{-\alpha}\mathrm{E}[e^{-\delta\tau}S(\tau)]. \tag{7.7}$$

An alternative way to derive (7.7) is to use Remark 2.5 that the random variables $X(\tau)-m(\tau)$ and $X(\tau)$ have the same discounted joint density function as $M(\tau)$ and $X(\tau)$. Hence,

$$\begin{aligned}
\mathrm{E}[e^{-\delta\tau}[n(\tau)-1]S(\tau)] &= \mathrm{E}[e^{-\delta\tau}[Le^{X(\tau)-m(\tau)}-S(\tau)]_{+}] \\
&= \mathrm{E}[e^{-\delta\tau}[Le^{M(\tau)}-S(\tau)]_{+}],
\end{aligned} \tag{7.8}$$

which yields (7.7) because of (5.15) with $\gamma = L/S(0)$. Also, by (5.16), we have

$$\mathrm{E}[e^{-\delta\tau}[n(\tau)-1]S(\tau)] = \left[\frac{L}{S(0)}\right]^{1-\alpha}\mathrm{E}[e^{-\delta\tau}[\max_{0\le t\le\tau}S(t)-S(\tau)]]. \tag{7.9}$$

The last expectation is the time-0 value of an at-the-money contingent floating strike lookback put option.

8 Dynamic withdrawals

In the last section, we considered an investor who does not want the value of his investments to ever drop below a predetermined level. In this section, we consider an investor who does not want the value of his investments to ever go *above* a predetermined level. If his investments ever go above that level, he wants the excess be immediately paid back to him as "dividends." Ko et al. (2010) use the term *dynamic withdrawal benefit* to describe such a payoff feature. The motivation of the problem comes from "living benefits" in variable annuities.

As before, $S(t)$ denotes the value of one unit of a mutual fund or stock fund at time t. Let L denote the level of the "dividend barrier." Here, $L \geq S(0)$, which is a condition opposite to that in the last section. At time 0, the investor has one unit of the mutual fund. If, between time 0 and time τ, the investor's account value ever exceeds the level L, just enough units of the mutual fund are sold so that the account value stays at level L, and the proceeds are paid back to the investor. Let $n(t)$ denote the number of units in the investor's account at time t. Then, the three conditions for the $n(t)$ function in this section are:

(i) $n(0) = 1$;

(ii) $n(t)$ is a nonincreasing function of t;

(iii) $n(t)S(t) \leq L, \quad t \geq 0$.

In place of (7.2), here we have

$$n(t) = \min\{1, \min_{0\le s\le t} \frac{L}{S(s)}\} = \min\{1, \frac{L}{S(0)} e^{-M(t)}\}, \tag{8.1}$$

which is formula (1.1) in Ko et al. (2010).

If no "dividends" are paid, the investor's account value at time τ is $S(\tau)$. With "dividends" paid, the account value at time τ is $n(\tau)S(\tau)$. Hence, the expected discounted value of all dividends paid between time 0 and time τ is

$$\mathrm{E}[e^{-\delta\tau}[1 - n(\tau)]S(\tau)]. \tag{8.2}$$

If we consider

$$[1 - n(\tau)]S(\tau) = e^{X(\tau)}[S(0) - Le^{-M(\tau)}]_+, \tag{8.3}$$

we can use the discounted joint density formula (2.7) to evaluate (8.2). A more efficient way is to consider

$$[1 - n(\tau)]S(\tau) = e^{X(\tau)-M(\tau)}[S(0)e^{M(\tau)} - L]_+, \tag{8.4}$$

and to use (2.11). A third derivation is to consider

$$[1 - n(\tau)]S(\tau) = [S(\tau) - Le^{X(\tau)-M(\tau)}]_+, \tag{8.5}$$

which, because of the last sentence in Remark 2.5, has the same distribution as

$$[S(\tau) - Le^{m(\tau)}]_+ = [S(\tau) - \gamma \min_{0\le t\le\tau} S(t)]_+ \tag{8.6}$$

with $\gamma = L/S(0)$. Thus, we can use the fractional floating strike lookback call formula (5.30) to obtain

$$\mathrm{E}[e^{\delta\tau}[1 - n(\tau)]S(\tau)] = \left[\frac{S(0)}{L}\right]^{\beta-1} \frac{1}{\beta} \mathrm{E}[e^{-\delta\tau} S(\tau)], \tag{8.7}$$

which is the counterpart of (7.7). Furthermore, from (8.7) and (5.25), we see that

$$\mathrm{E}[e^{\delta\tau}[1 - n(\tau)]S(\tau)] = \left[\frac{S(0)}{L}\right]^{\beta-1} \mathrm{E}[e^{-\delta\tau}[S(\tau) - \min_{0\le t\le\tau} S(t)]]. \tag{8.8}$$

This formula corresponds to (7.9).

Suppose that, in addition to the "dividends," an amount of at least K is required at time τ, where K is a positive constant less than L. That is, at time τ, there is to be a payoff of amount

$$\max(K, n(\tau)S(\tau)) = n(\tau)S(\tau) + [K - n(\tau)S(\tau)]_+. \tag{8.9}$$

The expected discounted value of the payoff in excess to the account value is

$$\mathrm{E}[e^{-\delta\tau}[K - n(\tau)S(\tau)]_+], \tag{8.10}$$

which will be determined in the remainder of this section.

One way to determine (8.10) is to evaluate the integral

$$S(0)\int_0^\infty \left[\int_{-\infty}^y [e^k - \min(1, e^{\ell-y})e^x]_+ f^\delta_{X(\tau),M(\tau)}(x,y)\mathrm{d}x\right]\mathrm{d}y,$$

where $k = \ln[K/S(0)]$ and $\ell = \ln[L/S(0)]$. An easier way is to note that by (8.1),

$$\begin{aligned}&[K - n(\tau)S(\tau)]_+\\ &= I_{(M(\tau)\geq\ell)}[K - e^{\ell-M(\tau)}S(\tau)]_+ + I_{(M(\tau)<\ell)}[K - S(\tau)]_+,\end{aligned} \tag{8.11}$$

or

$$\begin{aligned}&[K - n(\tau)S(\tau)]_+ - [K - S(\tau)]_+\\ &= I_{(M(\tau)\geq\ell)}[K - e^{\ell-M(\tau)}S(\tau)]_+ - I_{(M(\tau)\geq\ell)}[K - S(\tau)]_+.\end{aligned} \tag{8.12}$$

The first term on the RHS of (8.12) can be rewritten as

$$I_{(M(\tau)\geq\ell)}[K - Le^{-[M(\tau)-X(\tau)]}]_+,$$

whose expected discounted value can be readily determined by using formula (2.11),

$$\frac{\lambda}{D}\left[\int_\ell^\infty e^{-\beta y}\mathrm{d}y\right]\left[\int_0^\infty [K - Le^{-z}]_+ e^{\alpha z}\mathrm{d}z\right] = \frac{\lambda}{D}\frac{[S(0)/L]^\beta}{\beta}\frac{L^\alpha K^{1-\alpha}}{-\alpha(1-\alpha)}. \tag{8.13}$$

The second term on the RHS of (8.12) is the time-τ payoff of an up-and-in put option with $K < L$ (or $k < \ell$). Its expected discounted value can be found in (A.9).

The expected discounted value of the RHS of (8.12) is the difference between (8.13) and the second expression in (A.9); it is

$$\frac{\kappa K^{1-\alpha} L^{\alpha}}{(1-\alpha)\beta} \left[\frac{S(0)}{L}\right]^{\beta}. \tag{8.14}$$

For $S(0) > K$, the expected discounted value (8.10) is the sum of (4.25) and (8.14),

$$\frac{\kappa K}{1-\alpha} \left[\frac{K}{S(0)}\right]^{-\alpha} \left\{\frac{1}{-\alpha} + \left[\frac{S(0)}{L}\right]^{\beta-\alpha} \frac{1}{\beta}\right\}. \tag{8.15}$$

For $S(0) \leq K$, the expected discounted value (8.10) is the sum of (4.30) and (8.14), resulting in a formula somewhat more complicated than (8.15).

Remark 8.1: The second term on the RHS of (8.11) is the time-τ payoff of an up-and-out put option with $K < L$. Its expected discounted value has two expressions, depending on whether $S(0) > K$ or $S(0) \leq K$. They are the second and third expressions in (A.5). Thus the expected discounted value (8.10) can also be obtained by adding (8.13) to (A.5).

Remark 8.2: If τ is changed to a fixed time T, $S(0) = 1$, $\delta = r$, the fund pays no dividends, and the expectation is taken with respect to the risk-neutral probability measure, then a formula for the expectation (8.10) can be found in (3.8) of Ko et al. (2010).

9 Several stocks

Certain results in previous sections can be extended to the case of two or more stocks (or stock funds). Let

$$\mathbf{X}(t) = (X_1(t), X_2(t), \cdots, X_n(t))' \tag{9.1}$$

be an n-dimensional Brownian motion. Let $\boldsymbol{\mu}$ denote the mean vector and $\mathbf{C}$ the covariance matrix of $\mathbf{X}(1)$. Let

$$g_t(\mathbf{X}) \tag{9.2}$$

denote a real-valued *functional* of the process up to time t. For $n = 1$, examples of (9.2) are functions of $X(t)$, $M(t)$ and $m(t)$, where $M(t)$ and $m(t)$ are defined by (2.2) and (2.17), respectively. The process is stopped at time τ, an independent positive random variable (in this paragraph, τ does not have to be exponential). Let $\mathbf{h}$ be an n-dimensional vector of real numbers; by the method of *Esscher transforms* (Gerber and Shiu 1994, 1996), we have

$$\mathrm{E}[e^{-\delta\tau}e^{\mathbf{h}'\mathbf{X}(\tau)}g_\tau(\mathbf{X})] = \mathrm{E}[e^{-\delta(\mathbf{h})\tau}g_\tau(\mathbf{X}); \mathbf{h}], \tag{9.3}$$

where the second expectation is taken with respect to the changed probability measure indexed by the parameter vector $\mathbf{h}$ and

$$\begin{aligned} \delta(\mathbf{h}) &= \delta - \ln[M_{\mathbf{X}(1)}(\mathbf{h})] \\ &= \delta - \mathbf{h}'\boldsymbol{\mu} - \tfrac{1}{2}\mathbf{h}'\mathbf{C}\mathbf{h}. \end{aligned} \tag{9.4}$$

To derive (9.3), we condition on $\tau = t$; then the LHS of (9.3) is

$$\int_0^\infty e^{-\delta t}\mathrm{E}[e^{\mathbf{h}'\mathbf{X}(t)}g_t(\mathbf{X})]f_\tau(t)\mathrm{d}t. \tag{9.5}$$

By the *factorization formula* in the method of Esscher transforms, the expectation inside the integrand of (9.5) can be written as the product of two expectations,

$$\begin{aligned} &\mathrm{E}[e^{\mathbf{h}'\mathbf{X}(t)}] \times \mathrm{E}[g_t(\mathbf{X}); \mathbf{h}] \\ &= [M_{\mathbf{X}(1)}(\mathbf{h})]^t \times \mathrm{E}[g_t(\mathbf{X}); \mathbf{h}]. \end{aligned} \tag{9.6}$$

Thus the integral (9.5) is

$$\int_0^\infty e^{-\delta(\mathbf{h})t}\mathrm{E}[g_t(\mathbf{X}); \mathbf{h}]f_\tau(t)\mathrm{d}t,$$

which is the RHS of (9.3).

Let $\mathbf{k}$ be an n-dimensional vector of real numbers. Consider the one-dimensional Brownian motion $\mathbf{k}'\mathbf{X}(t)$. Let $q_t(\mathbf{k}'\mathbf{X})$ denote a real-valued functional of the process up to time t. The process is stopped at time τ, an independent exponential random variable with mean $1/\lambda$. Then, it follows from (9.3) that

$$\mathrm{E}[e^{-\delta\tau}e^{\mathbf{h}'\mathbf{X}(\tau)}q_\tau(\mathbf{k}'\mathbf{X})] = \mathrm{E}[e^{-\delta(\mathbf{h})\tau}q_\tau(\mathbf{k}'\mathbf{X}); \mathbf{h}]. \tag{9.7}$$

By considering $\mathbf{k}'\mathbf{X}(t)$ as $X(t)$, we can thus use results in the earlier sections. To this end, we need the two zeros of the quadratic polynomial corresponding to the one on the LHS of (2.5). The polynomial is

$$\begin{aligned}&\tfrac{1}{2}\mathrm{Var}[\mathbf{k}'\mathbf{X}(1); \mathbf{h}]\xi^2 + \mathrm{E}[\mathbf{k}'\mathbf{X}(1); \mathbf{h}]\xi - [\lambda + \delta(\mathbf{h})] \\ &= \tfrac{1}{2}\mathbf{k}'\mathbf{C}\mathbf{k}\xi^2 + \mathbf{k}'(\boldsymbol{\mu} + \mathbf{C}\mathbf{h})\xi - (\lambda + \delta - \mathbf{h}'\boldsymbol{\mu} - \tfrac{1}{2}\mathbf{h}'\mathbf{C}\mathbf{h})\end{aligned} \tag{9.8}$$

by Section 7 of Gerber and Shiu (1994). We can also find the polynomial by considering (2.25). The moment-generating function of $\mathbf{k}'\mathbf{X}(1)$ with respect to the transformed probability measure is

$$\mathrm{E}\left[e^{\xi\mathbf{k}'\mathbf{X}(1)}\frac{e^{\mathbf{h}'\mathbf{X}(1)}}{\mathrm{E}[e^{\mathbf{h}'\mathbf{X}(1)}]}\right].$$

Hence, the polynomial corresponding to (2.25) is

$$\begin{aligned}&\ln\{\mathrm{E}[e^{(\xi\mathbf{k}+\mathbf{h})'\mathbf{X}(1)}]/\mathrm{E}[e^{\mathbf{h}'\mathbf{X}(1)}]\} - [\lambda + \delta(\mathbf{h})] \\ &= \ln[M_{\mathbf{X}(1)}(\xi\mathbf{k} + \mathbf{h})] - (\lambda + \delta).\end{aligned} \tag{9.9}$$

Under the condition

$$\lambda + \delta(\mathbf{h}) > 0, \tag{9.10}$$

the two zeros of the polynomial have opposite signs.

In the remainder of this section, we consider $n = 2$, and let $S_1(t) = S_1(0)e^{X_1(t)}$ and $S_2(t) = S_2(0)e^{X_2(t)}$ be the prices of two stocks or stock funds at time t. Here

$$\boldsymbol{\mu} = (\mu_1, \mu_2)' \tag{9.11}$$

and

$$\mathbf{C} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \tag{9.12}$$

The payoff of a contingent *Margrabe option* (*exchange option*) is

$$[S_1(\tau) - S_2(\tau)]_+. \tag{9.13}$$

Special cases of it are the contingent call and put options in Section 4. If we rewrite (9.13) as

$$e^{X_2(\tau)}[S_1(0)e^{X_1(\tau)-X_2(\tau)} - S_2(0)]_+,$$

we can find its expected discounted value by using formula (9.7) with $\mathbf{h} = (0 \quad 1)'$ and $\mathbf{k} = (1 \quad -1)'$, and formula (4.19) or (4.31). For simplicity, we only consider the out-of-the-money case, $S_1(0) < S_2(0)$. By (9.7) and (4.19),

$$\mathrm{E}[e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+ | S_1(0) < S_2(0)] = \frac{\kappa^* S_2(0)}{\beta^*(\beta^* - 1)} \left[\frac{S_1(0)}{S_2(0)}\right]^{\beta^*}. \tag{9.14}$$

Here,

$$\begin{aligned} \kappa^* &= \frac{\lambda}{D^*(\beta^* - \alpha^*)}, \\ D^* &= \tfrac{1}{2}\mathrm{Var}[X_1(1) - X_2(1)] = \tfrac{1}{2}(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2), \end{aligned} \tag{9.15}$$

and $\alpha^* < 0$ and $\beta^* > 0$ are the zeros of the quadratic polynomial (9.8), which is

$$D^*\xi^2 + (\mu_1 - \mu_2 + \rho\sigma_1\sigma_2 - \sigma_2^2)\xi - (\lambda + \delta - \mu_2 - \tfrac{1}{2}\sigma_2^2). \tag{9.16}$$

For the two zeros to have opposite signs, we need the constant term of the polynomial to be negative (this is inequality (9.10)). Under risk-neutral valuation and if stock 2 pays no dividends, the constant term simplifies to $-\lambda$, which is negative.

On the other hand, if we write (9.13) as

$$e^{X_1(\tau)}[S_1(0) - S_2(0)e^{X_2(\tau)-X_1(\tau)}]_+,$$

we would use formula (9.7) with $\mathbf{h} = (1 \;\; 0)'$ and $\mathbf{k} = (-1 \;\; 1)'$ and the put option formula (4.25) to evaluate (9.13) for the out-of-money case. Then,

$$\begin{aligned} &\mathrm{E}[e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+ | S_1(0) < S_2(0)] \\ &= \frac{\kappa^{**} S_1(0)}{-\alpha^{**}(1-\alpha^{**})} \left[\frac{S_1(0)}{S_2(0)}\right]^{-\alpha^{**}}. \end{aligned} \tag{9.17}$$

Here,

$$\begin{aligned} \kappa^{**} &= \frac{\lambda}{D^{**}(\beta^{**} - \alpha^{**})}, \\ D^{**} &= \tfrac{1}{2}\mathrm{Var}[X_2(1) - X_1(1)] = D^*, \end{aligned}$$

and $\alpha^{**} < 0$ and $\beta^{**} > 0$ are the zeros of the quadratic polynomial (9.9), which is

$$\ln[M_{\mathbf{X}(1)}((1-\xi, \xi)')] - (\lambda + \delta). \tag{9.18}$$

Because the polynomial (9.16) is the same as

$$\ln[M_{\mathbf{X}(1)}((\xi, 1-\xi)')] - (\lambda + \delta), \tag{9.19}$$

we see that

$$\alpha^* = 1 - \beta^{**} \tag{9.20}$$

and

$$\beta^* = 1 - \alpha^{**}. \tag{9.21}$$

Thus, $\kappa^* = \kappa^{**}$, and the right-hand sides of (9.14) and (9.17) are indeed the same.

Our other example in this section is a generalization of the dynamic fund protection model in Section 7. Here, the protection level L is generalized as a stochastic level given by $S_1(t)$. The time-t value of one unit of the investment fund is $S_2(t)$. Then the number-of-unit function is

$$n(t) = \max\left\{1, \max_{0 \le s \le t} \frac{S_1(s)}{S_2(s)}\right\}. \tag{9.22}$$

The investor's account value at time t is $n(t)S_2(t)$. The time-0 cost for providing the "protection" until time τ is determined by

$$\mathrm{E}[e^{-\delta\tau}[n(\tau)-1]S_2(\tau)]. \tag{9.23}$$

By writing $[n(\tau)-1]S_2(\tau)$ as

$$e^{X_2(\tau)}[S_1(0)\exp(\max_{0\leq s\leq\tau}[X_1(s)-X_2(s)])-S_2(0)]_+,$$

we can apply (9.7) with $\mathbf{h}=(0\ \ 1)'$ and $\mathbf{k}=(1\quad -1)'$. Then, the expectation (9.23) becomes

$$\mathrm{E}[e^{-\delta(\mathbf{h})\tau}[S_1(0)\exp[\max_{0\leq t\leq\tau}\mathbf{k}'\mathbf{X}(t)]-S_2(0)]_+;\mathbf{h}], \tag{9.24}$$

which is the time-0 value of a contingent fixed strike lookback call. Because $S_1(0)\leq S_2(0)$, we apply (5.4) with $K=S_2(0)$ and $S(0)=S_1(0)$ to obtain that the expectation (9.24) is

$$\frac{\lambda}{D^*}\frac{S_2(0)}{-\alpha^*\beta^*(\beta^*-1)}\left[\frac{S_1(0)}{S_2(0)}\right]^{\beta^*}, \tag{9.25}$$

where the quantities D^*, α^* and β^* are the same as those defined earlier in this section.

To check (9.25), we now show that it implies (7.6). Consider $S_1(0)=L$, $S_2(0)=S(0)$, $\mu_1=\sigma_1=0$, $\mu_2=\mu$, and $\sigma_2=\sigma$. The quadratic polynomial (9.16) simplifies as

$$D\xi^2-(\mu+\sigma^2)\xi-(\lambda+\delta-\mu-\frac{1}{2}\sigma^2),$$

which has the same discriminant as the polynomial on the LHS of (2.5). Thus, $\beta^*=1-\alpha$ and $\alpha^*=1-\beta$, and (9.25) matches the middle expression in (7.6).

We end this section with a *factorization formula*. As in (9.2), we let $g_t(\mathbf{X})$ denote a functional of an n-dimensional Brownian motion up to time t. The Brownian motion is stopped at time τ, an independent exponential random variable with mean $1/\lambda$. Then, for $\delta > -\lambda$,

$$\mathrm{E}[e^{-\delta\tau} g_\tau(\mathbf{X})] = \mathrm{E}[e^{-\delta\tau}] \times \mathrm{E}[g_{\tau_\delta}(\mathbf{X})], \tag{9.26}$$

where τ_δ is an exponential random variable with mean $1/(\lambda+\delta)$ and independent of the Brownian motion $\mathbf{X}(t)$. The proof is even simpler than that of (9.3). The LHS of (9.26) is

$$\begin{aligned}
&\int_0^\infty e^{-\delta t} g_t(\mathbf{X}) \lambda e^{-\lambda t} \mathrm{d}t \\
&= \frac{\lambda}{\lambda+\delta} \int_0^\infty g_t(\mathbf{X})(\lambda+\delta) e^{-(\lambda+\delta)t} \mathrm{d}t,
\end{aligned}$$

which is the RHS of (9.26). The factorization formula (9.26) is in fact true for the more general case where τ is a gamma random variable. Then, τ_δ is a gamma random variable with the same shape parameter, but the scale parameter is changed from λ to $\lambda+\delta$.

An immediate application of (9.26) is the derivation of the discounted joint density function $f^\delta_{X(\tau),M(\tau)}$ as a continuation of Remark 3.3. Here, $n = 1$. Equation (9.26) shows that

$$f^\delta_{X(\tau),M(\tau)} = \mathrm{E}[e^{-\delta\tau}] \times f_{X(\tau_\delta),M(\tau_\delta)}.$$

Also, formula 2.1.15.6 on page 271 of Borodin and Salminen (2002) can give us a formula for the joint density function $f_{X(\tau),M(\tau),m(\tau)}$, from which and (9.26), we obtain $f^\delta_{X(\tau),M(\tau),m(\tau)}$.

10 T-year contingent options

The options discussed in previous sections have no expiry date. Here, we want to value life-contingent options that will expire at a fixed time T, $T > 0$. Hence, we consider the defective probability density function

$$f_{T_x}(t)I_{(t<T)}, \qquad t > 0.$$

A first idea is to approximate this function by a linear combination of exponential probability density functions and to use the results of the previous sections. However, chances are that a large number of exponential probability density functions would be needed to obtain a reasonably good approximation (in particular to approximate 0 for $t > T$).

In this section we propose a practical method. We explain it with the put option as an example. Thus, the time-τ payoff is

$$[K - S(\tau)]_+ I_{(\tau \le T)}, \tag{10.1}$$

or

$$[K - S(\tau)]_+ - [K - S(\tau)]_+ I_{(\tau > T)}. \tag{10.2}$$

The time-0 cost of the T-*year deferred contingent put option* is

$$\begin{aligned} &\mathrm{E}[e^{-\delta\tau}[K - S(\tau)]_+ I_{(\tau>T)}] \\ &= \Pr(\tau > T)\mathrm{E}[e^{-\delta\tau}[K - S(\tau)]_+ | \tau > T] \\ &= e^{-(\lambda+\delta)T}\mathrm{E}[e^{-\delta(\tau-T)}[K - S(T)e^{X(\tau)-X(T)}]_+ | \tau > T]. \end{aligned} \tag{10.3}$$

By the memoryless property of the exponential random variable τ, it follows from (4.25) and (4.30) that the last conditional expectation in (10.3), given $S(T)$, is

$$\begin{aligned} &\frac{\kappa K}{-\alpha(1-\alpha)}\left[\frac{K}{S(T)}\right]^{-\alpha} I_{(S(T)>K)} \\ &+\left\{\frac{\kappa K}{\beta(\beta-1)}\left[\frac{S(T)}{K}\right]^{\beta} + \frac{\lambda}{\lambda+\delta}K - \mathrm{E}[e^{-\delta\tau+X(\tau)}]S(T)\right\} I_{(S(T)<K)}. \end{aligned} \tag{10.4}$$

To evaluate (10.3), or equivalently, to determine the expectation of (10.4), we can apply the factorization formula in the method of Esscher transforms (Gerber and Shiu

1994, p. 177; 1996, p. 188) and use Remark 2.1 which points out that $e^{-(\delta+\lambda)t}S(t)^{\xi}$ is a martingale for $\xi = \alpha$ and for $\xi = \beta$. Then,

$$\begin{aligned} e^{-(\delta+\lambda)T}\mathrm{E}[S(T)^{\alpha}I_{(S(T)>K)}] &= e^{-(\delta+\lambda)T}\mathrm{E}[S(T)^{\alpha}] \times \mathrm{E}[I_{(S(T)>K)};\alpha] \\ &= S(0)^{\alpha}\mathrm{Pr}[S(T) > K;\alpha]. \end{aligned} \tag{10.5}$$

Similarly,

$$e^{-(\delta+\lambda)T}\mathrm{E}[S(T)^{\beta}I_{(S(T)<K)}] = S(0)^{\beta}\mathrm{Pr}[S(T) < K;\beta]. \tag{10.6}$$

We also have

$$\mathrm{E}[I_{(S(T)<K)}] = \mathrm{Pr}[S(T) < K;0] \tag{10.7}$$

and

$$\mathrm{E}[S(T)I_{(S(T)<K)}] = S(0)e^{\vartheta T}\mathrm{Pr}[S(T) < K;1], \tag{10.8}$$

where ϑ is defined by (4.2). For each real number h, define

$$z_h = \frac{k - (\mu + h\sigma^2)T}{\sigma\sqrt{T}}, \tag{10.9}$$

where $k = \ln[K/S(0)]$ as in (4.14). Then,

$$\mathrm{Pr}[S(T) < K;h] = \Phi(z_h)$$

and

$$\mathrm{Pr}[S(T) > K;h] = \Phi(-z_h),$$

where $\Phi(z)$ is the standard normal c.d.f. Combining these results, we find (10.3) to be

$$\begin{aligned} &\frac{\kappa K}{-\alpha(1-\alpha)}\left[\frac{K}{S(0)}\right]^{-\alpha}\Phi(-z_{\alpha}) + \frac{\kappa K}{\beta(\beta-1)}\left[\frac{S(0)}{K}\right]^{\beta}\Phi(z_{\beta}) \\ &+e^{-(\lambda+\delta)T}\frac{\lambda}{\lambda+\delta}K\Phi(z_0) - e^{-(\lambda+\delta-\vartheta)T}\mathrm{E}[e^{-\delta\tau}S(\tau)]\Phi(z_1). \end{aligned} \tag{10.10}$$

We remark that z_0 and z_1 correspond to $-d_2$ and $-d_1$, respectively, in the finance literature. Also, with the definition

$$\varrho = \sqrt{\mu^2 + 4D(\lambda + \delta)}, \tag{10.11}$$

we have

$$z_\alpha = \frac{k + \varrho T}{\sigma\sqrt{T}} \tag{10.12}$$

and

$$z_\beta = \frac{k - \varrho T}{\sigma\sqrt{T}}. \tag{10.13}$$

Note that

$$\varrho = D(\beta - \alpha) = \lambda/\kappa; \tag{10.14}$$

see (2.13).

We are now ready to value the T-year K-strike contingent put option. According to (10.2), the expected discounted value of (10.1) is (4.25) or (4.30) minus (10.10), depending on whether the option is out-of-the-money or in-the-money. For the out-of-the-money case, $S(0) > K$, the valuation formula is

$$\begin{aligned}&\frac{\kappa K}{-\alpha(1-\alpha)}\left[\frac{K}{S(0)}\right]^{-\alpha}\Phi(z_\alpha) - \frac{\kappa K}{\beta(\beta-1)}\left[\frac{S(0)}{K}\right]^{\beta}\Phi(z_\beta)\\ &-e^{-(\lambda+\delta)T}\frac{\lambda}{\lambda+\delta}K\Phi(z_0) + e^{-(\lambda+\delta-\vartheta)T}\mathrm{E}[e^{-\delta\tau}S(\tau)]\Phi(z_1).\end{aligned} \tag{10.15}$$

For the in-the-money case, $S(0) < K$, the valuation formula is

$$\begin{aligned}&-\frac{\kappa K}{-\alpha(1-\alpha)}\left[\frac{K}{S(0)}\right]^{-\alpha}\Phi(-z_\alpha) + \frac{\kappa K}{\beta(\beta-1)}\left[\frac{S(0)}{K}\right]^{\beta}\Phi(-z_\beta)\\ &+\frac{\lambda}{\lambda+\delta}K[1 - e^{-(\lambda+\delta)T}\Phi(z_0)] - \mathrm{E}[e^{-\delta\tau}S(\tau)][1 - e^{-(\lambda+\delta-\vartheta)T}\Phi(z_1)].\end{aligned} \tag{10.16}$$

For $S(0) = K$ (at-the-money), (10.15) and (10.16) must give the same value. For a verification of this, observe that when $S(0) = K$, (4.19) and (4.31) give the same value; see also Remark 4.5.

Now consider the special case $\mu = \delta - D$, which is condition (4.12). Then, the last term in (10.15) and the last term in (10.16) simplify to $e^{-\lambda T} S(0)\Phi(z_1)$ and $S(0)[1 - e^{-\lambda T}\Phi(z_1)]$, respectively. Thus, expression (10.15) becomes

$$\frac{\kappa K}{-\alpha(1-\alpha)} \left[\frac{K}{S(0)}\right]^{-\alpha} \Phi(z_\alpha) - \frac{\kappa K}{\beta(\beta-1)} \left[\frac{S(0)}{K}\right]^{\beta} \Phi(z_\beta)$$
$$-e^{-(\lambda+\delta)T} \frac{\lambda}{\lambda+\delta} K\Phi(z_0) + e^{-\lambda T} S(0)\Phi(z_1). \tag{10.17}$$

which is a formula we need in the next section.

We now return to the "rollup" GMDB of Remark 4.6. But now we assume a fixed expiry date $T > 0$ and also that the probability that the policy has not lapsed by time t is given by the exponential function $e^{-\nu t}$, $0 < t < T$. Hence, the cost of this guarantee is

$$\mathrm{E}[e^{-\delta\tau}(Ke^{p\tau} - S(\tau))_+ e^{-\nu\tau} I_{(\tau<T)}]. \tag{10.18}$$

This is the same as

$$\mathrm{E}[e^{-(\delta-p+\nu)\tau}(K - e^{-p\tau}S(\tau))_+ I_{(\tau<T)}], \tag{10.19}$$

that is, the cost of the T-year K-strike put option, with the substitutions

$$\delta \leftarrow \delta - p + \nu, \quad \mu \leftarrow \mu - p. \tag{10.20}$$

From this and (10.15), it follows that the cost of the out-of-the-money guarantee is

$$\frac{\kappa K}{-\alpha(1-\alpha)} \left[\frac{K}{S(0)}\right]^{-\alpha} \Phi(z_\alpha) - \frac{\kappa K}{\beta(\beta-1)} \left[\frac{S(0)}{K}\right]^{\beta} \Phi(z_\beta)$$
$$-e^{-(\lambda+\delta-p+\nu)T} \frac{\lambda}{\lambda+\delta-p+\nu} K\Phi(z_0)$$
$$+e^{-(\lambda+\delta+\nu-\vartheta)T} \frac{\lambda}{\lambda+\delta+\nu-\vartheta} S(0)\Phi(z_1), \tag{10.21}$$

where $\alpha < 0$ and $\beta > 0$ are now the solutions of the equation

$$D\xi^2 + (\mu - p)\xi - (\lambda + \delta - p + \nu) = 0 \tag{10.22}$$

and z_h is defined by a modified (10.9) with μ replaced by $\mu - p$.

Remark 10.1: Ulm (2008) has attacked the problem of valuing the guarantee with a partial differential equation approach. To reconcile his result with our (10.21), observe that Ulm's analysis includes a maturity guarantee whose time-0 value is

$$e^{-(\delta+\lambda+\nu)T}\mathrm{E}[(Ke^{pT} - S(T))_+]. \tag{10.23}$$

Remark 10.2: Formula (4.29) can be generalized to the T-*year put-call-parity*:

$$\begin{aligned}
&\mathrm{E}[e^{-\delta\tau}[K - S(\tau)]_+ I_{(\tau\leq T)}] - \mathrm{E}[e^{-\delta\tau}[S(\tau) - K]_+ I_{(\tau\leq T)}] \\
&= \frac{\lambda}{\lambda+\delta}K[1 - e^{-(\lambda+\delta)T}] - \mathrm{E}[e^{-\delta\tau}S(\tau)][1 - e^{-(\lambda+\delta-\vartheta)T}].
\end{aligned} \tag{10.24}$$

From this and (10.15) or (10.16) we get the valuation formulas for the T-year K-strike contingent *call* option. Thus, the time-0 value of the T-year K-strike contingent call option can be obtained from (10.16) for the out-of-the-money $S(0) < K$ case,

$$\begin{aligned}
&-\frac{\kappa K}{-\alpha(1-\alpha)}\left[\frac{K}{S(0)}\right]^{-\alpha}\Phi(-z_\alpha) + \frac{\kappa K}{\beta(\beta-1)}\left[\frac{S(0)}{K}\right]^{\beta}\Phi(-z_\beta) \\
&+e^{-(\lambda+\delta)T}\frac{\lambda}{\lambda+\delta}K\Phi(-z_0) - e^{-(\lambda+\delta-\vartheta)T}\mathrm{E}[e^{-\delta\tau}S(\tau)]\Phi(-z_1),
\end{aligned} \tag{10.25}$$

and from (10.15) for the in-the-money $S(0) > K$ case,

$$\begin{aligned}
&\frac{\kappa K}{-\alpha(1-\alpha)}\left[\frac{K}{S(0)}\right]^{-\alpha}\Phi(z_\alpha) - \frac{\kappa K}{\beta(\beta-1)}\left[\frac{S(0)}{K}\right]^{\beta}\Phi(z_\beta) \\
&-\frac{\lambda}{\lambda+\delta}K[1 - e^{-(\lambda+\delta)T}\Phi(-z_0)] \\
&+\mathrm{E}[e^{-\delta\tau}S(\tau)][1 - e^{-(\lambda+\delta-\vartheta)T}\Phi(-z_1)].
\end{aligned} \tag{10.26}$$

Illustration: We consider T-year 90-strike life-contingent put options on a stock with initial price $S(0) = 100$. We assume $\delta = 8\%$ and $\mu = \delta - D$ as in (4.12). The option values are calculated by means of (10.17). First we assume that the distribution of T_x is exponential with mean 125/6. The results are shown in Table 2.

Then we assume that the probability density function of T_x is a combination of two exponential densities,

$$f_{T_x}(t) = 3 \times 0.08e^{-0.08t} - 2 \times 0.12e^{-0.12t}, \qquad t > 0. \tag{10.27}$$

Note that the mean of this T_x is also 125/6. The results are displayed in Table 3.

Table 2: Contingent put values $-$ T_x exponential

T	1	2	3	5	10	20	30	60	∞
$\sigma = 0.25$	0.080	0.241	0.421	0.764	1.378	1.860	1.973	2.005	2.006
$\sigma = 0.3$	0.122	0.359	0.626	1.150	2.148	3.026	3.269	3.353	3.354
$\sigma = 0.35$	0.167	0.485	0.845	1.564	2.983	4.324	4.729	4.887	4.890
$\sigma = 0.4$	0.215	0.616	1.072	1.993	3.854	5.688	6.274	6.515	6.521

Table 3: Contingent put values $-$ T_x combination of two exponentials

T	1	2	3	5	10	20	30	60	∞
$\sigma = 0.25$	0.010	0.055	0.134	0.356	0.962	1.608	1.770	1.808	1.809
$\sigma = 0.3$	0.015	0.081	0.199	0.538	1.525	2.708	3.053	3.153	3.154
$\sigma = 0.35$	0.021	0.109	0.268	0.732	2.141	3.948	4.526	4.711	4.713
$\sigma = 0.4$	0.026	0.138	0.339	0.934	2.784	5.259	6.093	6.375	6.378

11 De Moivre's law

In this section we consider the case where T_x has an uniform distribution between 0 and $\omega - x$, where ω is the maximal possible age. We consider an expiry date $T \leq \omega - x$. The value of the T-year guarantee is an integral with respect to the truncated probability density function of T_x. The latter can be obtained as the following limit:

$$\frac{1}{\omega - x} \lim_{\lambda \to 0} \frac{1}{\lambda} f_\tau(t) I_{(t<T)}, \tag{11.1}$$

where $f_\tau(t)$ is the exponential probability density function with parameter $\lambda > 0$. This mathematically trivial observation has an important consequence: results of the preceding section can be used to obtain results for the case of a uniform distribution. It suffices to divide the cost by $\omega - x$ and λ and take the limit for $\lambda \to 0$. Consider (10.15) and suppose that $\vartheta \neq \delta$ ($\mu \neq \delta - D$). By (10.14), we can replace κ in the first two terms of (10.15) by $\lambda/\varrho(\lambda)$, where the function $\varrho(\lambda)$ is defined by the RHS of (10.11). We also apply (4.10) to the last term of (10.15). Then, the limit is straightforward and the cost of the out-of-the-money T-year contingent put option is

$$\begin{aligned}\frac{1}{\omega - x}\Bigg\{ & \frac{K}{-\alpha(1-\alpha)\varrho}\left[\frac{K}{S(0)}\right]^{-\alpha}\Phi(z_\alpha) - \frac{K}{\beta(\beta-1)\varrho}\left[\frac{S(0)}{K}\right]^{\beta}\Phi(z_\beta) \\ & -\frac{1}{\delta}e^{-\delta T}K\Phi(z_0) + e^{-(\delta-\vartheta)T}\frac{1}{\delta-\vartheta}S(0)\Phi(z_1)\Bigg\}. \end{aligned} \tag{11.2}$$

where $\alpha < 0$ and $\beta > 0$ are the solutions of the quadratic equation (2.5) with $\lambda = 0$.

The special case $\vartheta = \delta$ ($\mu = \delta - D$) requires a more careful treatment. Here,

$$\alpha(\lambda) = \frac{-(\delta - D) - \varrho(\lambda)}{2D}, \tag{11.3}$$

$$\beta(\lambda) = \frac{-(\delta - D) + \varrho(\lambda)}{2D}. \tag{11.4}$$

Because

$$\varrho(0) = \sqrt{(\delta - D)^2 + 4D\delta} = \delta + D, \tag{11.5}$$

we have

$$\alpha(0) = -\delta/D \tag{11.6}$$

and

$$\beta(0) = 1. \tag{11.7}$$

Thus, after a division by λ, both the second and fourth terms in (10.17) tend to infinity (for $\lambda \to 0$). Hence, their limit cannot be taken separately, but only for their sum. For this purpose, we apply (4.32) to the second term in (10.17). Then, (10.17) becomes an expression with six terms. There are four terms, each of which has a λ in the numerator (recall that $\kappa = \lambda/\varrho(\lambda)$); their limits are obvious. The sum of the remaining two terms is

$$-K\left[\frac{S(0)}{K}\right]^{\beta(\lambda)}\Phi(z_{\beta(\lambda)}) + e^{-\lambda T}S(0)\Phi(z_1). \tag{11.8}$$

We apply the rule of Bernoulli-Hôpital to determine the limit ($\lambda \to 0$) of expression (11.8) divided by λ. Then, the limit is the derivative of (11.8) with respect to λ, evaluated at $\lambda = 0$,

$$\begin{aligned} &-K\left[\frac{S(0)}{K}\right]^{\beta(0)}\ln[S(0)/K]\beta'(0)\Phi(z_{\beta(0)}) \\ &-K\left[\frac{S(0)}{K}\right]^{\beta(0)}\varphi(z_{\beta(0)})[-\beta'(0)\sigma\sqrt{T}] - TS(0)\Phi(z_1), \end{aligned} \tag{11.9}$$

where $\varphi(z)$ is the standard normal density function. Differentiating (11.4) with respect to λ and applying condition (11.5) yields

$$\beta'(0) \;=\; \frac{\varrho'(0)}{2D} = \frac{1}{\delta + D}. \tag{11.10}$$

It follows from (11.7), (4.14) and (11.10) that (11.9) simplifies as

$$S(0)\left[\frac{k}{\delta + D}\Phi(z_1) + \frac{\sigma\sqrt{T}}{\delta + D}\varphi(z_1) - T\Phi(z_1)\right]. \tag{11.11}$$

Combining these results, we find that the cost of the T-year out-of-the-money contingent put option is

$$\begin{aligned} \frac{1}{\omega - x}\Bigg\{&\frac{K}{\delta}\frac{D^2}{(\delta + D)^2}\left[\frac{K}{S(0)}\right]^{\delta/D}\Phi(z_{-\delta/D}) - \frac{K}{\delta}e^{-\delta T}\Phi(z_0) + \frac{\sigma\sqrt{T}}{\delta + D}S(0)\varphi(z_1) \\ &-\left[T - \frac{k}{\delta + D} - \frac{\delta + 2D}{(\delta + D)^2}\right]S(0)\Phi(z_1)\Bigg\}. \end{aligned} \tag{11.12}$$

Finally, we consider the out-of-the-money "rollup" GMDB. If T_x has an exponential distribution, its cost is given by (10.21). Now suppose that T_x is uniformly distributed between 0 and $\omega - x$. To obtain the cost of the guarantee, we divide (10.21) by $\omega - x$ and λ and take the limit for $\lambda \to 0$. This procedure yields

$$\frac{1}{\omega - x}\left\{\frac{K}{-\alpha(1-\alpha)\varrho}\left[\frac{K}{S(0)}\right]^{-\alpha}\Phi(z_\alpha) - \frac{K}{\beta(\beta-1)\varrho}\left[\frac{S(0)}{K}\right]^{\beta}\Phi(z_\beta)\right.$$
$$- \frac{1}{\delta - p + \nu}e^{-(\delta-p+\nu)T}K\Phi(z_0)$$
$$\left. + e^{-(\delta+\nu-\vartheta)T}\frac{1}{\delta+\nu-\vartheta}S(0)\Phi(z_1)\right\}, \tag{11.13}$$

where $\alpha < 0$ and $\beta > 0$ are the solutions of the quadratic equation (10.22) with $\lambda = 0$, z_h is defined by a modified (10.9) with μ replaced by $\mu - p$ and

$$\varrho = \sqrt{(\mu - p)^2 + 4D(\delta - p + \nu)}.$$

Remark 11.1: The corresponding results in Ulm (2008) are for the special case $\vartheta = \delta$ and $\omega - x = T$. The methodologies in Ulm and in this paper are quite different.

12 Equity-linked death benefit reserves

As in Section 1, the exercise time of a life-contingent option is T_x, the time-until-death random variable of a policyholder with initial age x. It is convenient to use standard actuarial notation of life contingencies. Correspondingly,

$$\begin{aligned} {}_tp_x &= \Pr(T_x > t), \\ {}_tp_x\mu_{x+t} &= \frac{\mathrm{d}}{\mathrm{d}t}\Pr(T_x < t), \end{aligned} \tag{12.1}$$

where μ_{x+t} is the force of mortality at time t.

The death benefit is equity-linked. The death benefit is $b(t, s)$, if death occurs at time t and the stock price is s at that time. We assume that $S(t) = S(0)e^{X(t)}$, with $X(t)$ as in Section 2.

We assume that reserves are defined as expected discounted values of future benefits. Let $V(t, s)$ denote the time-t value of such a reserve if the policyholder survives to time t and if the stock price at that time is s. Then,

$$V(t,s) = \mathrm{E}[e^{-\delta(T_x - t)} b(T_x, S(T_x)) | T_x > t, S(t) = s], \tag{12.2}$$

where δ is the valuation force of interest.

We shall derive a partial differential equation for the function $V(t, s)$. Let $h > 0$. By conditioning on what happens within h time units after time t, we see that $V(t, s)$ is

$$\begin{aligned} &\int_0^h e^{-\delta u}\, {}_u p_{x+t} \mu_{x+t+u} \mathrm{E}[b(t+u, S(t+u)) | S(t) = s] \mathrm{d}u \\ &+ e^{-\delta h}\, {}_h p_{x+t} \mathrm{E}[V(t+h, S(t+h)) | S(t) = s]. \end{aligned} \tag{12.3}$$

Because the sum of these two terms does not depend on h, the sum of their derivatives vanishes. Thus

$$\begin{aligned} &e^{-\delta h}\, {}_h p_{x+t} \mu_{x+t+h} \mathrm{E}[b(t+h, S(t+h)) | S(t) = s] \\ &- e^{-\delta h} (\delta + \mu_{x+t+h})\, {}_h p_{x+t} \mathrm{E}[V(t+h, S(t+h)) | S(t) = s] \\ &+ e^{-\delta h}\, {}_h p_{x+t} \frac{\mathrm{d}}{\mathrm{d}h} \mathrm{E}[V(t+h, S(t+h)) | S(t) = s] = 0. \end{aligned} \tag{12.4}$$

Setting $h = 0$, we use the *infinitesimal generator* of the process $S(t)$ and obtain the equation

$$\mu_{x+t} b(t,s) - (\delta + \mu_{x+t}) V(t,s) + V_t(t,s) + \vartheta s V_s(t,s) + D s^2 V_{ss}(t,s) = 0, \tag{12.5}$$

where $\vartheta = \mu + D$ as in (4.2). This PDE generalizes *Thiele's* ODE, which is for the case where $b(t, s)$ does not depend on the stock price s, implying $V_s = V_{ss} = 0$.

The change of the reserve between t and $t + \mathrm{d}t$ is

$$\begin{aligned} \mathrm{d}V(t, S(t)) &= V(t + \mathrm{d}t, S(t + \mathrm{d}t)) - V(t, S(t)) \\ &= V_t(t, S(t))\mathrm{d}t + V_s(t, S(t))\mathrm{d}S(t) + D S(t)^2 V_{ss}(t, S(t))\mathrm{d}t. \end{aligned} \tag{12.6}$$

Given $S(t) = s$, its expectation is

$$\mathrm{E}[\mathrm{d}V|S(t) = s] = V_t(t,s)\mathrm{d}t + \vartheta s V_s(t,s)\mathrm{d}t + Ds^2 V_{ss}(t,s)\mathrm{d}t. \tag{12.7}$$

With this, we can rewrite (12.5) in the following appealing form:

$$V(t,s)\delta\mathrm{d}t = [b(t,s) - V(t,s)]\mu_{x+t}\mathrm{d}t + \mathrm{E}[\mathrm{d}V|S(t) = s]. \tag{12.8}$$

Thus the interest on the reserve is the sum of the instantaneous cost of insurance based on the net amount at risk $b(t,s) - V(t,s)$ and the expected change of the reserve. Note that the net amount at risk can be negative. This is in particular the case whenever $b(t,s) = 0$.

The company is committed to maintain the reserve value calculated by (12.2) at each point in time. For the following analysis, we assume that the mortality risk is covered by a (possibly fictitious) life insurance of the amount $b - V$ against a continuous premium equal to the instantaneous cost of insurance based on the net amount at risk. We shall now assume that there are two ways to invest the reserves, either risk-free at the force $r > 0$, or else in the stock with price process $S(t)$. We assume that the funds can be shifted continuously and without any transaction costs. An *investment strategy* is given by $\varphi(t, S(t))$, the fraction of $V(t, S(t))$ that is invested in stock at time t. Let $\Delta = \Delta(t)$ denote the *cumulative reserve deficit* at time t. If (x) survives to age $x + t + \mathrm{d}t$, the differential $\mathrm{d}\Delta(t)$ denotes the *reserve deficit* that occurs between t and $t + \mathrm{d}t$. It is the sum of the instantaneous cost of insurance and the change of the reserve, reduced by the return on the investment. Thus

$$\begin{aligned}\mathrm{d}\Delta(t) &= [b(t,S(t)) - V(t,S(t))]\mu_{x+t}\mathrm{d}t \\ &\quad + \mathrm{d}V - \varphi V\frac{\mathrm{d}S}{S} - (1-\varphi)Vr\mathrm{d}t,\end{aligned} \tag{12.9}$$

with $\mathrm{d}V$ given by (12.6). If the reserve deficit is positive, the company injects this amount to reach the reserve value at time $t + \mathrm{d}t$. If it is negative, the absolute value of this amount can be released from the reserve at time $t + \mathrm{d}t$ as a profit. From (12.9) and (12.6) we see that

$$\begin{aligned}\mathrm{d}\Delta(t) &= [b(t,S(t)) - V(t,S(t))]\mu_{x+t}\mathrm{d}t \\ &\quad -[\varphi\frac{V}{S} - V_s]\mathrm{d}S - [(1-\varphi)Vr - V_t - DS^2V_{ss}]\mathrm{d}t. \qquad (12.10)\end{aligned}$$

Then (12.5) leads to

$$\mathrm{d}\Delta(t) = -[\varphi\frac{V}{S} - V_s]\mathrm{d}S - [SV_s\vartheta + (1-\varphi)Vr - V\delta]\mathrm{d}t. \qquad (12.11)$$

It is judicious to introduce

$$\varepsilon(t,s) = \frac{sV_s(t,s)}{V(t,s)}, \qquad (12.12)$$

the *elasticity* of the reserve with respect to the stock price. Then (12.11) can be written in the following more suggestive form:

$$\begin{aligned}\mathrm{d}\Delta(t) &= (\varepsilon - \varphi)V[\frac{\mathrm{d}S}{S} - \vartheta\mathrm{d}t] \\ &\quad +V[\delta - \varphi\vartheta - (1-\varphi)r]\mathrm{d}t. \qquad (12.13)\end{aligned}$$

We note that

$$\mathrm{E}[\mathrm{d}S(t)|S(t)] = S(t)\vartheta\mathrm{d}t. \qquad (12.14)$$

Hence

$$\mathrm{E}[\mathrm{d}\Delta(t)|S(t)] = V[\delta - \varphi\vartheta - (1-\varphi)r]\mathrm{d}t. \qquad (12.15)$$

This result could perhaps have been anticipated.

Formula (12.13) shows that by choosing $\varphi(t,S(t)) = \varepsilon(t,S(t))$, the company can eliminate the dependence of $\mathrm{d}\Delta(t)$ on $\mathrm{d}S(t)$. For this particular investment strategy, (12.13) reduces to

$$\mathrm{d}\Delta(t) = V[\delta - \varepsilon\vartheta - (1-\varepsilon)r]\mathrm{d}t. \qquad (12.16)$$

In other words, the stochastic differential (12.13) becomes an ordinary differential.

For the following discussion we assume that the company uses the investment strategy with $\varphi(t, S(t)) = \varepsilon(t, S(t))$. First we note that the sign of $\varepsilon(t, s)$ is the same as the sign of $V_s(t, s)$. If the time-t elasticity is between 0 and 1, the reserve is invested in risk-free asset and in stock. If it is greater than one, money is borrowed at the risk-free rate so that more than the reserve is invested in stock. If it is negative, the company has a short position on the stock so that more than the reserve is invested in risk-free asset. If, for any given t, $b(t, s)$ is a non-decreasing function of s, $V_s(t, s)$ and with that $\varepsilon(t, s)$ are positive. This is in particular the case for a call option, where $b(t, s) = (s - K)_+$. As an illustration, consider the out-of-the-money situation with exponential exercise time. It follows from (4.19) that the elasticity $\varepsilon(t, s)$ is the positive constant β whenever $s < K$. Likewise, if $b(t, s)$ is a non-increasing function of s, $V_s(t, s)$ and with that $\varepsilon(t, s)$ are negative. This is in particular the case for put options, where $b(t, s) = (K - s)_+$. In the out-of-the-money situation with exponential exercise time, it follows from (4.25) that the elasticity $\varepsilon(t, s)$ is the negative constant α whenever $s > K$.

We assume that $r < \vartheta$ (if $r > \vartheta$, no risk-averse investor would buy the stock). The company prefers at any time a negative reserve deficit, resulting in a stream of funds released from the reserve. This is the condition that (12.16) is negative, or

$$\delta < r + \varepsilon(t, S(t))(\vartheta - r). \tag{12.17}$$

We note that it can turn out that the initial reserve $V(0, S(0))$ is greater than the value of the option. The resulting loss at the start of the contract will be compensated by a stream of funds released from the reserve during the life of the policy.

Appendix

In this Appendix, we list formulas of various kinds of barrier options, which we obtained with the help of Mathematica. In order to write the formulas in a compact way, we introduce the following notation.

$$\begin{aligned}
A_1(n) &= \frac{\lambda}{D}\frac{S(0)^n}{(n-\alpha)(\beta-n)}, \\
A_2(n) &= \frac{\lambda}{D}\frac{L^n}{(n-\alpha)(\beta-n)}\left[\frac{S(0)}{L}\right]^{\beta}, \\
A_3(n) &= \frac{\lambda}{D}\frac{L^n}{(n-\alpha)(\beta-n)}\left[\frac{L}{S(0)}\right]^{-\alpha}, \\
A_4 &= \frac{\lambda}{D}\frac{K^n}{(n-\alpha)(\beta-\alpha)}\left[\frac{K}{S(0)}\right]^{-\alpha} = \frac{\kappa K^n}{n-\alpha}\left[\frac{K}{S(0)}\right]^{-\alpha}, \\
A_5 &= \frac{\lambda}{D}\frac{K^{n-\alpha}L^{\alpha}}{(n-\alpha)(\beta-\alpha)}\left[\frac{S(0)}{L}\right]^{\beta} = \frac{\kappa K^{n-\alpha}L^{\alpha}}{n-\alpha}\left[\frac{S(0)}{L}\right]^{\beta}, \\
A_6 &= \frac{\lambda}{D}\frac{K^n}{(\beta-n)(\beta-\alpha)}\left[\frac{S(0)}{K}\right]^{\beta} = \frac{\kappa K^n}{\beta-n}\left[\frac{S(0)}{K}\right]^{\beta}, \\
A_7 &= \frac{\lambda}{D}\frac{K^{-(\beta-n)}L^{\beta}}{(\beta-n)(\beta-\alpha)}\left[\frac{L}{S(0)}\right]^{-\alpha} = \frac{\kappa K^{-(\beta-n)}L^{\beta}}{\beta-n}\left[\frac{L}{S(0)}\right]^{-\alpha}, \\
A_8 &= \frac{\lambda}{D}\frac{K}{-\alpha(1-\alpha)(\beta-\alpha)}\left[\frac{K}{S(0)}\right]^{-\alpha} = \frac{\kappa K}{-\alpha(1-\alpha)}\left[\frac{K}{S(0)}\right]^{-\alpha},
\end{aligned}$$

$$\begin{aligned}
A_9 &= \frac{\lambda}{D}\frac{K^{1-\alpha}L^{\alpha}}{-\alpha(1-\alpha)(\beta-\alpha)}\left[\frac{S(0)}{L}\right]^{\beta} \\
&= \frac{\kappa K^{1-\alpha}L^{\alpha}}{-\alpha(1-\alpha)}\left[\frac{S(0)}{L}\right]^{\beta}, \\
A_{10} &= \frac{\lambda}{D}\frac{K}{\beta(\beta-1)(\beta-\alpha)}\left[\frac{S(0)}{K}\right]^{\beta} = \frac{\kappa K}{\beta(\beta-1)}\left[\frac{S(0)}{K}\right]^{\beta}, \\
A_{11} &= \frac{\lambda}{D}\frac{K^{-(\beta-1)}L^{\beta}}{\beta(\beta-1)(\beta-\alpha)}\left[\frac{L}{S(0)}\right]^{-\alpha} \\
&= \frac{\kappa K^{-(\beta-1)}L^{\beta}}{\beta(\beta-1)}\left[\frac{L}{S(0)}\right]^{-\alpha}. \qquad \text{(A.1)}
\end{aligned}$$

Note that $A_1(n) = \mathrm{E}[e^{-\delta\tau}S(\tau)^n]$, A_4 is the RHS of (4.23), A_6 is the RHS of (4.15), A_8 is the RHS of (4.25), and A_{10} is the RHS of (4.19).

Up-and-out all-or-nothing call option

We evaluate (6.4) for $b(s)$ defined by (4.13). The option value (6.4) is

$$\begin{cases} 0, & \text{if } L < K, \\ \frac{\lambda}{D}\int_0^\ell [\int_k^y S(0)^n e^{nx} e^{-\alpha x} \mathrm{d}x] e^{-(\beta-\alpha)y} \mathrm{d}y, & \text{if } L \geq K \text{ and } S(0) > K, \\ \frac{\lambda}{D}\int_k^\ell [\int_k^y S(0)^n e^{nx} e^{-\alpha x} \mathrm{d}x] e^{-(\beta-\alpha)y} \mathrm{d}y, & \text{if } L \geq K \text{ and } S(0) \leq K \end{cases}$$

$$= \begin{cases} 0, & \text{if } L < K, \\ A_1(n) - A_2(n) - A_4 + A_5, & \text{if } L \geq K \text{ and } S(0) > K, \\ A_6 - A_2(n) + A_5, & \text{if } L \geq K \text{ and } S(0) \leq K. \end{cases} \tag{A.2}$$

Up-and-out call option

By applying (A.2) with $n = 0$ and $n = 1$, we have that the value of the up-and-out call option is

$$\begin{cases} 0, & \text{if } L < K, \\ A_1(1) - A_2(1) - A_1(0)K + A_2(0)K + A_8 - A_9, & \text{if } L \geq K \text{ and } S(0) > K, \\ A_2(0)K + A_{10} - A_2(1) - A_9, & \text{if } L \geq K \text{ and } S(0) \leq K. \end{cases} \tag{A.3}$$

Up-and-out all-or-nothing put option

With $b(s)$ defined by (4.22), the option value (6.4) is

$$\begin{cases} \frac{\lambda}{D}\int_0^\ell [\int_{-\infty}^y S(0)^n e^{nx} e^{-\alpha x} \mathrm{d}x] e^{-(\beta-\alpha)y} \mathrm{d}y, & \text{if } L < K, \\ \frac{\lambda}{D}\int_0^\ell [\int_{-\infty}^k S(0)^n e^{nx} e^{-\alpha x} \mathrm{d}x] e^{-(\beta-\alpha)y} \mathrm{d}y, & \text{if } L \geq K \text{ and } S(0) > K, \\ \frac{\lambda}{D}\{\int_0^k [\int_{-\infty}^y S(0)^n e^{nx} e^{-\alpha x} \mathrm{d}x] e^{-(\beta-\alpha)y} \mathrm{d}y & \\ \quad + \int_k^\ell [\int_{-\infty}^k S(0)^n e^{nx} e^{-\alpha x} \mathrm{d}x] e^{-(\beta-\alpha)y} \mathrm{d}y\}, & \text{if } L \geq K \text{ and } S(0) \leq K \end{cases}$$

$$= \begin{cases} A_1(n) - A_2(n), & \text{if } L < K, \\ A_4 - A_5, & \text{if } L \geq K \text{ and } S(0) > K, \\ A_1(n) - A_5 - A_6, & \text{if } L \geq K \text{ and } S(0) \leq K. \end{cases} \tag{A.4}$$

Note that we can check the answers by put-call parity. If we add the option values of (A.2) and (A.4), we obtain the value of the up-and-out option with payoff $S(\tau)^n$,

$$\frac{\lambda}{D}\int_0^{\ell}\left[\int_{-\infty}^{y} S(0)^n e^{nx} e^{-\alpha x}\mathrm{d}x\right] e^{-(\beta-\alpha)y}\mathrm{d}y = A_1(n) - A_2(n).$$

Up-and-out put option

By applying (A.4) with $n = 0$ and $n = 1$, we have that the value of the up-and-out put option is

$$\begin{cases} A_1(0)K - A_2(0)K - A_1(1) + A_2(1), & \text{if } L < K, \\ A_8 - A_9, & \text{if } L \geq K \text{ and } S(0) > K, \\ A_1(0)K - A_1(1) + A_{10} - A_9, & \text{if } L \geq K \text{ and } S(0) \leq K. \end{cases} \tag{A.5}$$

Up-and-in all-or-nothing call option

We evaluate (6.5) for $b(s)$ defined by (4.13). The option value (6.5) is

$$\begin{cases} \frac{\lambda}{D}\int_k^{\infty}[\int_k^y S(0)^n e^{nx} e^{-\alpha x}\mathrm{d}x] e^{-(\beta-\alpha)y}\mathrm{d}y & \text{if } L < K, \\ \frac{\lambda}{D}\int_{\ell}^{\infty}[\int_k^y S(0)^n e^{nx} e^{-\alpha x}\mathrm{d}x] e^{-(\beta-\alpha)y}\mathrm{d}y & \text{if } L \geq K \end{cases}$$

$$= \begin{cases} A_6, & \text{if } L < K, \\ A_2(n) - A_5, & \text{if } L \geq K. \end{cases} \tag{A.6}$$

Up-and-in call option

By applying (A.6) with $n = 0$ and $n = 1$, we have that the value of the up-and-in call option is

$$\begin{cases} A_{10}, & \text{if } L < K, \\ A_2(1) + A_9 - A_2(0)K, & \text{if } L \geq K. \end{cases} \tag{A.7}$$

Up-and-in all-or-nothing put option

With $b(s)$ defined by (4.22), the option value (6.5) is

$$\begin{cases} \frac{\lambda}{D}\{\int_\ell^k [\int_{-\infty}^y S(0)^n e^{nx} e^{-\alpha x} \mathrm{d}x] e^{-(\beta-\alpha)y} \mathrm{d}y \\ \quad + \int_k^\infty [\int_{-\infty}^k S(0)^n e^{nx} e^{-\alpha x} \mathrm{d}x] e^{-(\beta-\alpha)y} \mathrm{d}y\}, & \text{if } L < K, \\ \frac{\lambda}{D}\int_\ell^\infty [\int_{-\infty}^k S(0)^n e^{nx} e^{-\alpha x} \mathrm{d}x] e^{-(\beta-\alpha)y} \mathrm{d}y, & \text{if } L \geq K \end{cases}$$

$$= \begin{cases} A_2(n) - A_6, & \text{if } L < K, \\ A_5, & \text{if } L \geq K. \end{cases} \tag{A.8}$$

Note that we can check the answers by put-call parity. If we add the option values of (A.6) and (A.8), we obtain the value of the up-and-in option with payoff $S(\tau)^n$,

$$\frac{\lambda}{D}\int_\ell^\infty \left[\int_{-\infty}^y S(0)^n e^{nx} e^{-\alpha x} \mathrm{d}x\right] e^{-(\beta-\alpha)y} \mathrm{d}y = A_2(n).$$

Up-and-in put option

By applying (A.8) with $n = 0$ and $n = 1$, we have that the value of the up-and-in put option is

$$\begin{cases} A_2(0)K + A_{10} - A_2(1), & \text{if } L < K, \\ A_9, & \text{if } L \geq K. \end{cases} \tag{A.9}$$

Down-and-out all-or-nothing call option

We evaluate (6.8) for $b(s)$ defined by (4.13). The option value (6.8) is

$$\begin{cases} \frac{\lambda}{D}\int_\ell^0 [\int_k^\infty S(0)^n e^{nx} e^{-\beta x} \mathrm{d}x] e^{(\beta-\alpha)y} \mathrm{d}y, & \text{if } K \geq S(0), \\ \frac{\lambda}{D}\{\int_\ell^k [\int_k^\infty S(0)^n e^{nx} e^{-\beta x} \mathrm{d}x] e^{(\beta-\alpha)y} \mathrm{d}y \\ \quad + \int_k^0 [\int_y^\infty S(0)^n e^{nx} e^{-\beta x} \mathrm{d}x] e^{(\beta-\alpha)y} \mathrm{d}y\}, & \text{if } L < K < S(0), \\ \frac{\lambda}{D}\int_\ell^0 [\int_y^\infty S(0)^n e^{nx} e^{-\beta x} \mathrm{d}x] e^{(\beta-\alpha)y} \mathrm{d}y, & \text{if } K \leq L \end{cases}$$

$$= \begin{cases} A_6 - A_7, & \text{if } K \geq S(0), \\ A_1(n) - A_4 - A_7, & \text{if } L < K < S(0), \\ A_1(n) - A_3(n), & \text{if } K \leq L. \end{cases} \tag{A.10}$$

Down-and-out call option

By applying (A.10) with $n = 0$ and $n = 1$, we have that the value of the down-and-out call option is

$$\begin{cases} A_{10} - A_{11}, & \text{if } K \geq S(0), \\ A_1(1) - A_1(0)K + A_8 - A_{11}, & \text{if } L < K < S(0), \\ A_1(1) - A_3(1) - A_1(0)K + A_3(0)K, & \text{if } K \leq L. \end{cases} \tag{A.11}$$

Down-and-out all-or-nothing put option

With $b(s)$ defined by (4.22), the option value (4.15) is

$$\begin{cases} \frac{\lambda}{D}\int_\ell^0 [\int_y^k S(0)^n e^{nx} e^{-\beta x} \mathrm{d}x] e^{(\beta-\alpha)y} \mathrm{d}y, & \text{if } K \geq S(0), \\ \frac{\lambda}{D}\int_\ell^k [\int_y^k S(0)^n e^{nx} e^{-\beta x} \mathrm{d}x] e^{(\beta-\alpha)y} \mathrm{d}y, & \text{if } L < K < S(0), \\ 0, & \text{if } K \leq L \end{cases}$$

$$= \begin{cases} A_1(n) - A_3(n) - A_6 + A_7, & \text{if } K \geq S(0), \\ A_4 - A_3(n) + A_7, & \text{if } L < K < S(0), \\ 0, & \text{if } K \leq L. \end{cases} \tag{A.12}$$

Note that we can check the answers by put-call parity. If we add the option values of (A.10) and (A.12), we obtain the value of the down-and-out option with payoff $S(\tau)^n$,

$$\frac{\lambda}{D}\int_\ell^0 \left[\int_y^\infty S(0)^n e^{nx} e^{-\beta x} \mathrm{d}x\right] e^{(\beta-\alpha)y} \mathrm{d}y = A_1(n) - A_3(n).$$

Down-and-out put option

By applying (A.12) with $n = 0$ and $n = 1$, we have that the value of the down-and-out put option is

$$\begin{cases} A_1(0)K - A_3(0)K + A_{10} - A_{11} - A_1(1) + A_3(1), & \text{if } K \geq S(0), \\ A_8 - A_3(0)K + A_3(1) - A_{11}, & \text{if } L < K < S(0), \\ 0, & \text{if } K \leq L. \end{cases} \tag{A.13}$$

Down-and-in all-or-nothing call option

We evaluate (6.9) for $b(s)$ defined by (4.13). The option value (6.9) is

$$\begin{cases} \frac{\lambda}{D}\int_{-\infty}^{\ell}[\int_{k}^{\infty} S(0)^n e^{nx}e^{-\beta x}\mathrm{d}x]e^{(\beta-\alpha)y}\mathrm{d}y, & \text{if } K \geq L, \\ \frac{\lambda}{D}\{\int_{-\infty}^{k}[\int_{k}^{\infty} S(0)^n e^{nx}e^{-\beta x}\mathrm{d}x]e^{(\beta-\alpha)y}\mathrm{d}y & \\ \quad +\int_{k}^{\ell}[\int_{y}^{\infty} S(0)^n e^{nx}e^{-\beta x}\mathrm{d}x]e^{(\beta-\alpha)y}\mathrm{d}y\}, & \text{if } K < L \end{cases}$$

$$= \begin{cases} A_7, & \text{if } K \geq L, \\ A_3(n) - A_4, & \text{if } K < L. \end{cases} \tag{A.14}$$

Down-and-in call option

By applying (A.14) with $n = 0$ and $n = 1$, we have that the value of the down-and-in call option is

$$\begin{cases} A_{11}, & \text{if } K \geq L, \\ A_3(1) + A_8 - A_3(0)K, & \text{if } K < L. \end{cases} \tag{A.15}$$

Down-and-in all-or-nothing put option

With $b(s)$ defined by (4.22), the option value (6.9) is

$$\begin{cases} \frac{\lambda}{D}\int_{-\infty}^{\ell}[\int_{y}^{k} S(0)^n e^{nx}e^{-\beta x}\mathrm{d}x]e^{(\beta-\alpha)y}\mathrm{d}y & \text{if } K \geq L, \\ \frac{\lambda}{D}\int_{-\infty}^{k}[\int_{y}^{k} S(0)^n e^{nx}e^{-\beta x}\mathrm{d}x]e^{(\beta-\alpha)y}\mathrm{d}y & \text{if } K < L \end{cases}$$

$$= \begin{cases} A_3(n) - A_7, & \text{if } K \geq L, \\ A_4, & \text{if } K < L. \end{cases} \tag{A.16}$$

Note that we can check the answers by put-call parity. If we add the option values of (A.14) and (A.16), we obtain the value of the down-and-in option with payoff $S(\tau)^n$,

$$\frac{\lambda}{D}\int_{-\infty}^{\ell}\left[\int_{y}^{\infty} S(0)^n e^{nx}e^{-\beta x}\mathrm{d}x\right]e^{(\beta-\alpha)y}\mathrm{d}y = A_3(n).$$

Down-and-in put option

By applying (A.16) with $n = 0$ and $n = 1$, we have that the value of the down-and-in put option is

$$\begin{cases} A_3(0)K + A_{11} - A_3(1), & \text{if } K \geq L, \\ A_8, & \text{if } K < L. \end{cases} \tag{A.17}$$

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