An Analysis Of Large Multi-Unit Auctions With Bundling[1]

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Abstract
Should multi-unit auctions be used to sell multiple identical and indivisible units? In this paper we show that with sufficiently many bidders, the multi-unit format is more efficient than the bundle format in a broad range of situations. However, the revenue-performance depends on the fraction of demand that is met asymptotically. We also examine the absolute performance of the multi-unit auctions by providing bounds on the rate at which the multi-unit auction converges to competitive price.

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1. Introduction

During much of the last three decades auction theory has received a great deal of attention from economists. This is not only because an increasing number of economic transactions are becoming dependent on some form of auction or the other, but also because auctions provide insights into the process of price formation. Much of the past research concentrated on the study of single-object auctions.[2] In reality, however, a large number of situations involve either multiple goods that are sold separately or in lots, or (bundles of) goods that can be split into multiple parts. Some recent work has correctly pointed out that conclusions drawn about single-object auctions do not generally extend to their multi-object counterparts. This makes it necessary to develop a separate theory of multi-object auctions. But should we have a multi-object auction at the first place?

Conceivably, there are several reasons (some even regulatory) due to which multi-object auctions may need to be held. In this paper, we consider two standard bases on which auction markets are evaluated, *viz.*, efficiency and revenue, and identify those situations where there are performance based reasons to conduct a multi-object auction. A natural way to evaluate a multi-object auction is to compare its performance with that of the alternative – single-object auction for the bundle(s). Therefore, we focus on this comparison.

The problem of evaluating the performance of a multi-object auction relative to its bundled counterpart arises in two different contexts. First, an object (or a bundle of objects) may be split into multiple dissimilar objects, (*e.g.*, an apple and an orange). Second, the bundle may be split into multiple identical objects/units, (*e.g.*, two apples).[3] Each scenario gives rise to a different theoretical framework that must

[2] Throughout this paper we call auctions "single-object" or "multi-object" depending on whether a bidder has a value for only one of the objects or more than one object. Thus, an auction may offer multiple units of an object on sale, but if each bidder has value for only a single unit we refer to such an auction as a single-object auction just to avoid confusion with our multi-unit auction where each bidder wants more than a unit. If the units are bundles of smaller units and a bidder wants only one bundle, we refer to the auction as a single-object auction for bundles or a bundle auction.

[3] There are situations where a bundle of objects can be split into multiple units of dissimilar objects (*e.g.*, two apples and three oranges). The relevant conclusions for such scenarios follow from the results

be studied separately. Palfrey (1983) considered the first framework where the seller sells multiple dissimilar objects for which the bidders have independent private values. Clearly, under a *second-price rule* without reserves (or other rules that allocate the object to the bidder with the highest value) selling the objects separately is allocatively efficient while bundling is not. Palfrey (1983) and Chakraborty (1999) showed that whether bundling the objects before the auction generates a higher or lower expected revenue than selling the objects separately, depends, respectively, on whether the number of bidders is "small" or "large." Thus, objects should be sold separately for efficiency reasons regardless of the size of the auction, and for revenue reasons when there are sufficiently many bidders.

We consider the second scenario where the objects are essentially identical units. The results for dissimilar objects are not useful for drawing any conclusion in the multi-unit framework. In fact, the multi-unit framework gives rise to a much more complex problem because the equilibrium bidding strategies for multi-unit auctions cannot be described in a useful manner in most situations. For instance, Engelbrecht-Wiggans and Kahn (1998) showed that multi-unit uniform-price auctions give rise to *demand reduction* in which bidders tend to shade greater and greater amounts on their equilibrium bids for the successive units (relative to their values). The amount by which bidders shade their bids cannot be expressed in a closed form except in some special cases, thus making a somewhat general analysis difficult. Moreover, differential bidding on successive units in multi-unit auctions gives rise to inefficiency. Thus, unlike the case of Palfrey (1983), even efficiency is a non-trivial issue in this case.

There is a second reason for evaluating the performance of multi-unit auctions which goes beyond auction theory. Economists have long been interested in the efficiency properties of decentralizable market outcomes. Edgeworth (1981) showed that under complete information the competitive outcome is always Pareto optimal. Similarly, the long-run equilibrium of the full information neoclassical perfectly competitive market is also efficient. Unfortunately, real life markets do not satisfy the strong informational

obtained from the basic cases.

and other requirements of such markets that force agents to become price-takers, thus efficiency is more an exception than a rule in practical contexts. Still, models of price formation have been used to show that as the market becomes large the incentives for being strategic tend to go away thereby taking the market towards the efficient competitive market outcome. Gul and Postlewaite (1992), and Rustichini, Satterthwaite, and Williams (1994), for instance, have used double-auction models to show that as the market size increases without bounds the agents tend to become price-takes and the market-clearing price converges to the competitive price in the limit. In a contribution that is more relevant to this paper, Swinkels (2001) showed that multi-unit auctions for independent private values, when conducted without reserves, are efficient in the limit when the number of bidders goes to infinity. In that limiting case, again, bidders virtually exhibit a price-taking behavior and (in the absence of exogenous shocks) the auction price converges to the corresponding competitive price. This limit price has been described by Chakraborty and Engelbrecht-Wiggans (2004).

While convergence to competitive price in the limit is good news from efficiency point of view, short of the limiting case, the efficiency results and equilibrium price-taking behavior fail completely. Specifically, for *all* finite number of bidders, the uniform-price multi-unit auction is inefficient (in that all potential gains are not realized) because bidders act strategically, and not as price takers. Thus, while Swinkels' study is motivated by auctions with large number of bidders (that are always finite in real life), his results do not give any idea about how close large *but finite* markets actually come to the competitive outcome.

Swinkels (2001) recognized this need, and suggested identification of the rate of convergence of multi-unit auctions to efficiency through simple numerical examples as a way to fill this gap. In the absence of a closed form description of equilibrium strategies in multi-unit (uniform-price) auctions this serves as a second-best alternative for measuring the efficiency-performance of large multi-unit auctions relative to the competitive market. In this paper we take a different approach. Rather than calculating the actual rates numerically we identify lower and/or upper bounds, depending on what the situation allows, on the rates at which the prices in these auctions converge to the competitive price. This allows us to take a general approach towards measuring the maximum and, whenever possible, the minimum extent to which finite markets can come close to being competitive.

This is certainly not the first attempt to evaluate the multi-unit auction relative to the single-object auction for the bundle. Wilson (1979) considered some tractable examples of pure common value auctions without reserves for a perfectly divisible object.[4] He demonstrated that when bidders are allowed to submit continuous bid-price schedules for the different shares of the object under the uniform-price rule the problem of demand reduction may give rise to low revenues in the share auction relative to a single-object auction for the whole object. Moreover, he showed that demand reduction can increase in severity (in the sense that each bidder demands a smaller fraction of the item for a positive price) as the number of bidders increases. This prevents the seller from receiving any advantage from increased competition.

In a large number of situations, however, objects *cannot* be divided infinitely for physical or other practical reasons. In those situations there is a limit beyond which demands cannot reduce in equilibrium. For instance, in a uniform-price auction (with the highest-losing bid setting the price) for two identical but indivisible units a bidder can bid zero on the second unit. At the same time, bidding his true value on the first unit is a weakly dominant strategy under the uniform-price rule (see Engelbrecht-Wiggans and Kahn, 1998). Thus, in equilibrium, the demand cannot reduce below a unit while remaining positive. In short, the problem of evaluating the performance of a multi-unit auction for indivisible units relative to the bundle auction remains unsolved.

In this paper, we show that whenever there are sufficiently many bidders in the auction a multi-unit auction generally performs better than the bundle auction if a small fraction of demand is met asymptotically. A multi-unit auction offers a greater scope for efficiency and, thus, higher revenue by allowing more allocative flexibility. However, in small auctions demand reduction leaves both possibilities unfulfilled. When there

[4] Obviously, efficiency was not an issue in the pure common value model.

are a large number of bidders the likelihood of any bidder receiving more than one unit, and the price being set by a bidder's second or lower bid can be expected to become very small. In a broad range of situations the problem of demand reduction tends to become sufficiently irrelevant when there are many bidders, making it perform better than the bundle auction. In the following sections we give insights into when the above intuition *does* and *does not* hold.

2. The Auction Model

Consider a sequence of auctions $\{A_n\}_{n\geq 1}$. In auction A_n the seller wants to sell $2M_n$ identical (indivisible) units of an object in a single auction possibly under a reserve.[5] There are n risk-neutral bidders with *independent private values* for the objects. To avoid some trivial cases we assume $1 \leq M_n < n$. Bidder i has diminishing marginal values V_{1i} for the first unit and V_{2i} for the second, and value $V_{1i} + V_{2i}$ for the bundle.[6] The values (V_{1i}, V_{2i})-s are random vectors each having a joint distribution $F(v_1, v_2)$ with a density $f(v_1, v_2)$ on a support $S = [(v_1, v_2) : 0 \leq v_2 \leq v_1 \leq 1]$ that are independent across bidders.[7,8] The marginal distributions are denoted by F_1 and F_2, and the density functions denoted by f_1 and f_2. We denote the distribution of $V_1 + V_2$ by $F_{V_1+V_2}(\cdot)$, and the corresponding density function by $f_{V_1+V_2}(\cdot)$ on the support $[0, 2]$. By *upper end of the support* for a distribution on $[a, b]$ we mean b, the *lower end of the support* is a in that case.

Each bidder privately observes his values and then participates in the auction under a given rule. The auction is held under a *sealed-bid uniform-price rule* with the price set equal to the maximum of the highest-losing bid and reserve R_n. The remaining details

[5] Restriction to an even number of units is purely for expositional ease. This has no bearing on the actual results.

[6] Most results and conclusions in the paper extend straightforwardly, in their appropriate forms, to the more general case where each bidder has demand for m ($\leq M_n$) bundles, *i.e.*, $2m$ units.

[7] It is not difficult to see in our set-up that when bidders have increasing marginal values, instead, there is an equilibrium where bidders submit the same bid on both units in the multi-unit auction, thus effectively reducing it to a bundle auction.

[8] The existence of the density function everywhere is simply for the ease of exposition. A careful look at the proofs will show that all results continue to hold under weaker conditions and for a more general support.

of the rule depend on whether the auction is carried out under a *multi-unit* format or a *bundle* format.

In the *multi-unit auction* each bidder submits two sealed bids b_1 and b_2, the seller awards the units to the $2M_n$ highest bids in the auction provided that these bids are no less than the reserve R_n^M. Thus, a bidder receives one or two units depending on whether one or both his bids are no less than R_n^M, and are among the $2M_n$ highest bids in the auction. The price paid for every unit won is equal to the maximum of the third highest bid and R_n^M. We assume, without loss of generality, that $b_1 \geq b_2$.

In a *bundle auction* the seller sells M_n bundles each consisting of 2 units, and a bidder submits a single sealed bid for the bundle. The object, *i.e.*, the bundle, is awarded to the M_n highest bidders, provided their bids are no less than the reserve R_n^B, for a price equal to the maximum of the second-highest bid in the auction and R_n^B. This single-object rule, for obvious reasons, is known as the *second-price auction* (with reserve). Note that the uniform-price rule is a *Vickrey auction* (see Vickrey, 1961) for the bundle, but not for the multi-unit auction.[9]

We assume that the reserves are non-negative and exogenously given. The number of bidders, the auction format including the reserve price, and the value distributions are all common knowledge before the auction begins.

Equilibrium Strategies

In each auction we consider (symmetric Bayes-Nash) equilibria in weakly undominated strategies. A *strategy in the bundle auction* is a function $b : S \to [0,1]$ with the interpretation that all bids smaller than R_n^B are ignored by the seller. The bundle auction has an equilibrium in weakly dominant strategies of truthful bidding (*i.e.*, bidding the true value of the bundle), thus in equilibrium $b(v_1, v_2) = v_1 + v_2$.

A *strategy in the multi-unit auction* is a pair of functions $(b_1, b_2) : S \to S$. Thus $b_1(v_1, v_2)$ is the bid for the first unit and $b_2(v_1, v_2)$ is the bid for the second unit with $b_1(v_1, v_2) \geq b_2(v_1, v_2)$. Unlike the bundle auction, sincere bidding is a weakly domi-

[9]Note that the price in the bundle auction refers to the price for the entire bundle, and not the per unit price which is in fact the case in the multi-unit auction.

nant strategy in the multi-unit auction only for the first unit, so that in equilibrium $b_1(v_1, v_2) = v_1$. The equilibrium bidding strategy for the second unit cannot be described as a closed form mathematical expression except in special cases. If $M_n = 1$ a modification of the argument in Engelbrecht-Wiggans and Kahn (1998) can be applied to multi-unit auctions with reserves to show that when the marginal distribution F_1 satisfies the hazard rate condition

$$\frac{f_1(x)}{1 - F_1(x)} \leq \frac{1}{1 - x}, \forall x \in [R_n^M, 1) \tag{1}$$

there is an equilibrium that involves bidding $\min[v_2, R_n^M]$ on the second unit.[10] In that case the equilibrium bidding strategy is given by $b_1(v_1, v_2) = v_1$, $b_2(v_1, v_2) = \min\{v_2, R_n^M\}$. Among distributions that satisfy (1) is the uniform distribution on S. When the distribution does not satisfy the hazard rate condition (1) the equilibrium has the property that $b_2(v_1, v_2) < v_2$ for $v_2 > R_n^M$.[11]

In order that a strategy such as the one described above with "complete demand reduction" to be an equilibrium strategy for the general M_n a generalized version of condition (1),

$$\frac{f_1(x)}{1 - F_1(x)} \leq \frac{1}{(1 - x)(M_n - 1)} \quad \forall x \in (R_n^M, 1) \tag{2}$$

must be satisfied. Clearly, when M_n increases without bounds over $\{A_n\}_{n\geq 1}$ it is impossible to satisfy this condition for large enough auctions. In fact, the efficiency result of Swinkels (2001), adapted to our set-up, implies that if $\lim_{n\to\infty} \frac{M_n}{n} = \alpha \in (0, 1)$ then there exists v_2^* such that for all $v_2 \geq v_2^*$, $\lim_{n\to\infty} b_2(v_1, v_2) = v_2$, although $b_2(v_1, v_2) < v_2$ $\forall n$ and $\forall v_2 > R_n^M$.

Observe that since the complete equilibrium bidding strategy in the bundle auction is available, it is possible to write the expected revenue explicitly. Denoting the r-th highest of the random variables $X_1, .., X_n$ by $X_{r:n}$, the expected revenue in the equilibrium of the bundle auction is given by $E[\max\{(V_1 + V_2)_{2:n}, R_n^B\} I_{\{(V_1+V_2)_{1:n} \geq R_n^B\}}]$

[10] We maintain the convention that any bid below the reserve is ignored by the seller in the multi-unit auction, too.

[11] For a detailed study on the equilibrium of the multi-unit uniform-price auction see Engelbrecht-Wiggans and Kahn (1998).

where I_A denotes the indicator variable on the set A. The expected surplus generated in the auction also has a similar expression. A general formula for the multi-unit auction is not available. However, the expected revenue when (2) is satisfied can be written similarly.

(A1) Regularity condition

In some cases we need to assume that there is a $k \geq 1$ such that (i) $F_1(\cdot)$ and $F_{V_1+V_2}(\cdot)$ are k-times continuously differentiable in some left neighborhoods of 1 and 2, respectively, (ii) $f_1^{(l)}(1) = 0$ for $l = 0, 1, .., k-2$,[12] and $f_1^{(k-1)}(1) \neq 0$, and (iii) $f_{V_1+V_2}^{(l)}(2) = 0$ for $l = 0, 1, ..., k-1$, where $f_1^{(0)} \equiv f_1$ and $f_{V_1+V_2}^{(0)} \equiv f_{V_1+V_2}$.

Remark 1. It is straightforward to check that the condition is automatically satisfied if $f_1(1) \neq 0$ (this case corresponds to $k = 1$) and f_1 and $f_{V_1+V_2}$ are continuous in some left neighborhoods of the upper ends of their supports. In that case, $f_{V_1+V_2}(2) = 0$ and $f_{V_1+V_2}^{(1)}(2) = 0$. In fact, the condition $f_{V_1+V_2}(2) = 0$ always holds.

Sometimes due to a need for tractability of analysis in multi-unit auctions it is assumed that the values for the successive units are the higher and lower order statistics for two independent draws from a distribution with density $h(\cdot)$ (see, for instance, Katzman, 1999). It is routine to check that the regularity condition is satisfied in that special framework regardless of the particular distribution $h(\cdot)$ as long as $h(\cdot)$ is continuous in a left neighborhood of 1.

Remark 2. If (i) $f_{V_1}(\cdot)$ and $f_{V_1+V_2}(\cdot)$ are k-times differentiable in a left neighborhood of 1 and 2, respectively, and $f_{V_1}^{(k)}(1) = 0$, $l = 0, ..., k-1$, and (ii) $f_{V_2|V_1}(y|x)$ is uniformly bounded in y for $x > 1 - \delta^*$ for some $\delta^* > 0$, then $f_{V_1+V_2}^{(l)}(2) = 0$, $l = 0, ..., k$ and the regularity condition *(A1)* is satisfied whenever there is a $k > 1$ satisfying $f_1^{(k-1)}(1) \neq 0$. This sufficient condition for the regularity condition *(A1)* to hold has the intuitive description that the values V_1 and V_2 do not become increasingly and highly correlated conditional on V_1 tending to its highest possible value of 1. (Geometrically, this increasing correlation means that as V_1 approaches 1 the probability mass does not become

[12] For $k = 1$ the $f_1^{(l)}(1) = 0$ part of the condition is ignored.

too concentrated, and increasingly so, near the diagonal where $V_1 = V_2$.) Thus a very large class of joint distributions can be expected to satisfy the sufficient condition.

In order to derive a few of our results we make alternative assumptions about the upper and lower ends of the support. These assumptions are summarized below:

(A2)

There is a $\hat{k} \geq 1$ such that (i) $F_1(\cdot) + F_1(\cdot)$ and $F_{V_1+V_2}(\cdot)$ are both at least $\hat{k}$-times continuously differentiable in some left neighborhoods of 1 and 2, respectively, (ii) $f_1^{(l)}(1) + f_2^{(l)}(1) = 0$ for $l = 0, 1, .., \hat{k} - 2$,[13] and $f_1^{(\hat{k}-1)}(1) \neq 0$, and (iii) $f_{V_1+V_2}^{(l)}(2) = 0$ for $l = 0, 1, ..., \hat{k} - 1$, where $f_1^{(0)} + f_2^{(0)} \equiv f_1 + f_2$ and $f_{V_1+V_2}^{(0)} \equiv f_{V_1+V_2}$.

(A3)

There is a $\check{k} \geq 1$ such that (i) $F_1(\cdot) + F_2(\cdot)$ and $F_{V_1+V_2}(\cdot)$ are both at least $\check{k}$-times continuously differentiable in some right neighborhoods of 0, (ii) $f_1^{(l)}(0) + f_2^{(l)}(0) = 0$ for $l = 0, 1, .., \check{k} - 2$,[14] and $f_1^{(\check{k}-1)}(0) \neq 0$, and (iii) $f_{V_1+V_2}^{(l)}(0) = 0$ for $l = 0, 1, ..., \check{k} - 1$, where $f_1^{(0)} + f_2^{(0)} \equiv f_1 + f_2$ and $f_{V_1+V_2}^{(0)} \equiv f_{V_1+V_2}$.

(A4) Assumption on Reserves

As we point out later, if the reserves are not sufficiently well-behaved results can be obtained in virtually any direction. Therefore, we assume that either $\limsup R_n^M < 1$ or for every subsequence $\{R_{n_l}^M\}$ of reserves $\lim R_{n_l}^M = 1$ the rate of convergence is $o\left(n^{-\frac{1}{k}}\right)$, *i.e.*, $n^{\frac{1}{k}}\left(1 - R_n^M\right) \longrightarrow 0$, where $k = \min\left\{l : f_1^{(l-1)}(1) \neq 0\right\}$ and that $(2 - R_{n_l}^B) = o\left(n^{-\frac{1}{m}}\right)$ for every subsequence with $\lim R_{n_l}^B = 2$, *i.e.*, $n^{\frac{1}{k}}\left(1 - R_n^M\right)$ whenever $m = \min\left\{l : f_1^{(l-1)}(1) \neq 0\right\}$ exists.

3. Efficiency and Revenue

While the equilibrium property of bidding strategies (*i.e.*, differential bidding on successive units) in a multi-unit auction gives rise to allocative inefficiency in all finite

[13] For $\hat{k} = 1$ the $f_1^{(l)}(1) + f_2^{(l)}(1) = 0$ part of the condition is ignored.

[14] For $\check{k} = 1$ the $f_1^{(l)}(0) + f_2^{(l)}(0) = 0$ part of the condition is ignored.

auctions, the equilibrium in a bundle auction is also allocatively inefficient even if there is a zero reserve.[15] Thus, whether the multi-unit auction is more or less efficient than the bundle auction becomes a highly non-trivial question.

There are several ways in which inefficiency is measured in economics. We take one of the simplest measures of efficiency. Accordingly, we consider the difference between the maximum (feasible) expected social surplus $E[\phi_n]$ (with all agents weighted equally) that could be generated among all possible allocations, and $E[\alpha_n]$, the actual social surplus that a particular auction format is expected to generate. Convergence to efficiency in this context means that the above expected difference $E[\phi_n] - E[\alpha_n]$ converges to zero.[16] While $E[\phi_n]$ is the same in both the bundle and multi-unit auctions, the actual surplus expected to be generated $E[\alpha_n]$ generally differs across the two formats.

The expected actual surplus $E[\alpha_n]$ generated under the multi-unit auction cannot be expressed as a closed form mathematical expression or any other useful manner for our purpose. This makes it necessary to look for alternative approaches to evaluate the performance of multi-unit auctions. In order to evaluate the performance of the multi-unit auction against the bundle auction we shall make use of relevant bounds for and limit arguments on $E[\alpha_n]$ depending on what the situation demands. This approach turns out to be sufficient to evaluate the relative performance of the multi-unit auction. However, in order to evaluate the absolute efficiency performance of the multi-unit auctions one would need to calculate the rate at which $E[\phi_n] - E[\alpha_n]$ goes to zero in the limit. Since useful bounds on the rates at which the equilibrium strategies in the multi-unit auctions cannot be obtained, it is not possible to find appropriate bounds on the rate at which $E[\phi_n] - E[\alpha_n]$ converges to zero, *i.e.*, multi-unit auction converges to efficiency, much less the actual rates. Similar problem arises while evaluating the revenue performance of the multi-unit auction in the absolute sense.

This makes it necessary to look for alternative approach to get a sense of the ab-

[15] Compare this with the traditional view that all "standard rules" (without reserves) for single-object auctions give rise to allocative efficiency. The approach we take here is that whenever an object is divisible (for instance, it is a bundle of objects) with diminishing marginal values, even the single-object auction (for the whole) is allocatively inefficient.

[16] See Swinkels (1999) for a discussion of alternative measures of efficiency in this context.

solute performance of large multi-unit auctions, and insights into what elements determine that performance. The competitive market provides a standard for that purpose which is relevant in this case since the outcome in the multi-unit auction ultimately converges to the competitive market outcome. Therefore, a good measure of the absolute performance of the multi-unit auction is obtained by considering the rate at which convergence to the competitive market outcome happens. Even this is not possible to calculate in the absence of a description of the equilibrium bidding strategy. Recall that for all finite auctions $b_1(v_1, v_2) = v_1$ and $b_2(v_1, v_2) < v_2$, and that the price in the multi-unit auction is set by the $(2M_n + 1)$-th highest bid. Hence, a natural upper bound for the rate at which the price converges is given by the rate at which the $(2M_n+1)$-th highest value in the auction converges. Note that the convergence of price is related to the convergence to efficiency only in that efficiency is a property of the market in the limit when the price in the auction converges to the competitive market price. Thus, while the rate of convergence of the price does not directly reflect the rate at which $E[\phi_n] - E[\alpha_n]$ converges to zero, it certainly gives a sense of the rate at which the multi-unit auction converges to the efficient competitive market in the limit. While the calculation of this bound remains mathematically non-trivial and not obtainable directly from known mathematical results, it is still a tractable quantity.

An upper bound for the rate at which the price converges is also of interest from the revenue performance perspective since the expected revenue in the multi-unit auction cannot be calculated. Therefore, in what we do in the next three sections we present precise conditions under which the efficiency and revenue performance of the multi-unit auctions is better or worse relative to the bundle auction, and describe an upper bound for the rate at which the prices in the multi-unit auctions converge to the competitive price under different circumstances.

4. Auctions With A Vanishing Fraction Of Demand Met Asymptotically

We start with the case where a vanishing fraction of the demand is met asymptotically. *i.e.*, $\lim_n \frac{M_n}{n} = 0$. This can happen both with M_n being bounded or with $\lim_n M_n = \infty$. Let us consider the case of bounded supply first. To keep things simple let us fix, without loss of generality, $M_n = 1$, so that there are exactly two units on sale in all auctions.

4.1 Finite Supply

When exactly two units are on sale, both the multi-unit and the bundle auctions converge to the same limit revenue of 2. Therefore, comparison of the rates of convergence under the two formats is the appropriate way to proceed for evaluating the relative performance of the multi-unit auction. Our objective in this case is to show that multi-unit auctions perform better relative to the bundle auctions for all large n. So we need a lower bound for the rate of convergence in the multi-unit auction and show that this rate is still faster than the rate at which the bundle auctions converge in efficiency and to its limit revenue. In this case, the bidding strategy with "complete demand reduction" allows the construction of such a bound. As discussed earlier whenever (1) is satisfied the rate of convergence is actually attained by the multi-unit auctions. In the following Proposition we provide these rates of converge.

Proposition 1. Suppose that assumptions *(A1)* and *(A4)* are satisfied and that the number of units supplied is bounded above. The multi-unit auction converges to efficiency and to the limit revenue of 2 at a minimum rate of $O(n^{-\frac{1}{k}})$ where $k = \min\left\{l : f_1^{(l-1)}(1) \neq 0\right\}$. The bundle auction converges to efficiency and to its limit revenue of 2 at a rate of $O(n^{-\frac{1}{m}})$ for some $m \geq k+1$ whenever $m = \min\left\{l : f_{V_1+V_2}^{(l-1)}(2) \neq 0\right\}$ exists.

Proof. See Appendix.

The minimum rates at which a multi-unit auction converges to efficiency and to its limit revenue are faster than the same for the bundle auction. Hence it should not be surprising that a multi-unit auction becomes more efficient and generates a higher expected revenue than the bundle auction before actually reaching the identical limit as the following result suggests.

Proposition 2. Suppose that assumptions *(A1)* and *(A4)* are satisfied and that the number of units supplied is bounded above. The multi-unit auction is more efficient and generates a strictly higher expected revenue than the bundle-auction for all large number of bidders.

Proof. See Appendix.

Remark 3. Consider the role of assumption *(A4)*. Clearly, if in any auction there is a subsequence of reserves that increases to the upper end of the support with n at a sufficiently high speed, the auction can be inefficient even in the limit.[17] In fact, the reserves can be set to attain a continuous range of predetermined levels of inefficiency and expected revenue in the limit. Thus, if appropriate conditions are not imposed, the reserves can be chosen to violate both of the above results. Such a choice of reserves cannot be made if there is no subsequence of reserves that increases to the upper end of the support at a "sufficiently high" speed. Assumption *(A4)* simply gives an upper bound on the rate that is sufficient for the above results to hold. The bound can be made tighter for convergence to efficiency and of revenue to the upper end of the support. However, that will make the comparison between bundle and multi-unit auctions depend on the relationship between these rates. Thus, *(A4)* simply ensures that the reserves take a passive role so that our results hold otherwise unconditionally.

Remark 4. It is not difficult to construct an example of finite auctions with order-statistic valuations, and show that in a small market the relative revenue ranking of multi-unit and bundle auction can vary between a situation where no reserve is set and one where the reserves are set at the optimal levels. In large auctions, too, for every n there is a set of reserves such that the multi-unit auction generates a lower expected revenue than the bundle auction, regardless of how large n is. However, there is a difference between the example and this fact in that the nontrivial reserves considered in the example with order-statistic valuations are the optimal reserves for the auctions. When bidder values are the higher and lower order statistic from a distribution

[17] This is most easily verified in the context of a single-object auction for private values that are independently distributed as uniform [0,1]. A reserve equal to $1 - \frac{1}{n^2}$ when there are n bidders makes the expected surplus from the auction go to 0. Clearly, such a reserve is not optimal in the single-object auction. However, we have not made any assumption that the reserve prices have to be optimal, yet.

that satisfies (1), Chakraborty and Engelbrecht-Wiggans (2004) show that the optimal reserves are strictly dependent on the number of bidders and bounded away from the upper end of the support. In short, the optimal reserve in that case satisfies *(A4)*. Hence for all large n the multi-unit auction generates a higher expected revenue and efficiency both when the reserves are set optimally[18] and equal to zero, unlike the case of two bidders from the example. There is, however, no result available to guarantee that the optimal reserves in the multi-unit auction will satisfy (*A4*) even outside the above special structure.

Remark 5. Let us examine another boundary of our results that arises out of assumption (*A1*) being violated. From Remark 2 one would guess that one way to violate the regularity condition is to consider values that become highly correlated towards the upper end of the support which would imply that the behavior of $f_1(\cdot)$ does not differ very much from the behavior of $f_{X+Y}(\cdot)$ in the limit. Since it is not easy to construct such a joint distribution in a tractable manner while allowing it to satisfy condition (1), we consider the extreme case where both values are equal with probability 1, and has a distribution $H(\cdot)$ on $\{(v,v) : v \in [0,1]\}$. In that case, the equilibrium in the multi-unit auction involves bidding zero on the second unit if condition (1) is satisfied.

Standard arguments show that regardless of the number of bidders the multi-unit auction in this case generates a lower revenue than a bundle-auction. The intuition is simple. If the marginal value for the second unit is likely to be as large as that for the first unit (and increasingly so for the top few bidders who matter as the number of bidders increases), demand reduction (from the high bidders) can be very costly to the seller; so costly that even a large competitive effect with many bidders cannot compensate the loss.

4.2 Example: Sequential Auctions

A single-shot sale as a multi-unit auction is not the only way that multiple units of an object could be sold. A large number of auctions in practice, including auctions on the

[18] In the special case of order statistic valuations.

internet and those for government contracts, sell identical units in a sequential manner rather than a multi-unit format as discussed above. Let us examine what might happen in such auctions, again, relative to the single-object auction for the bundle. In order to keep things manageable, let us suppose that in this alternative format two auctions are held one after the other. In the first auction, sealed bids are obtained, one from each bidder, under the second-price auction rule. Once the auction is carried out and the object awarded, the price in the auction is announced and then the next sealed-bid second-price auction is held. The reserve is set equal to zero.

In spite of the apparently simple structure, this is an analytically difficult situation to explore. However, Katzman (1999) was able to obtain closed form bidding strategies for such auctions under the assumption that the values to a bidder are the higher and lower order statistics from two independent draws from a distribution with density $h(\cdot)$. He derived the equilibrium bidding strategies

$$\begin{aligned} b_1(v_1) &= \frac{(2n-3)\int_0^{v_1} xH(x)^{2n-4}h(x)dx}{H(v_1)^{2n-3}} \\ b_2(v_2) &= v_2 \end{aligned}$$

for the sequential auctions. Given the monotonicity of the bidding strategies it is easy to see that the auction is efficient regardless of the number of bidders. In other words, the sequential auction is more efficient than the bundle auction regardless of the number of bidders. Thus comparison of expected revenue with the bundle auction is all that remains to be done in this case.

The expected revenue from the sequential auction format is given by $E[b_1((V_1)_{2:n})] + E[b_2(X_{3:2n})]$ where $X_1, .., X_{2n}$ are *i.i.d.* $H(\cdot)$. We have

$$\begin{aligned} & E\left[b_1((V_1)_{2:n})\right] \\ = \quad & 2(n-1)(2n-3)n\int_0^1 tH(t)^{2n-4}\left(\frac{2}{3} - H(t) + \frac{1}{3}H(t)^3\right)h(t)dt \end{aligned}$$

and

$$\begin{aligned} & E[b_2(X_{3:2n})] \\ = \quad & \int_0^1 t\frac{2n(2n-1)(2n-2)}{2}H(t)^{2n-3}(1-H(t))^2h(t)dt \end{aligned}$$

Thus the expected revenue from the sequential auction is given by

$$
\begin{aligned}
&2(n-1)(2n-3)n\int_0^1 tH(t)^{2n-4}\left(\frac{2}{3}-H(t)+\frac{1}{3}H(t)^3\right)h(t)dt \\
&\qquad\qquad\qquad + \int_0^1 t2n(2n-1)(n-1)H(t)^{2n-3}(1-H(t))^2h(t)dt \\
= \quad & g_1(n)EX_{3:2n-1}+g_2(n)EX_{3:2n}
\end{aligned}
$$

It is easy to check that $g_1(n)+g_2(n)=2$ which implies that the revenue from the sequential auction converges to 2, as well.

Recall that the expected revenue from the bundle auction is given by

$$
(n-1)n\int_0^2 sF_{X+Y}(s)^{n-2}(1-F_{X+Y}(s))f_{X+Y}(s)ds = E[(X+Y)_{2:n}].
$$

Using our Lemmas, there is a n^* such that for all $n>n^*$ the difference

$$
\frac{4n}{3(2n-1)}EX_{3:2n-1}+\frac{2(4n-3)}{3(2n-1)}EX_{3:2n}-E[(X+Y)_{2:n}]>0
$$

for all $n>n^*$. Thus, for all large number of bidders, the sequential format dominates the bundle auction both in terms of efficiency and revenue.

4.3. Infinite Supply

A vanishing fraction of the demand could be met asymptotically even if the supply of units increases without bounds over $\{A_n\}_{n\geq 1}$, *i.e.*, $\lim M_n/n=0$ while $\lim M_n=\infty$. The price converges to the top of the support of value distribution in this case, as well. In the next Proposition we show that under appropriate assumptions the performance of large multi-unit auctions relative to their bundled counterpart continue to remain unchanged in this case, as well.

Proposition 3. Suppose that assumptions (A1) and (A2) are satisfied, and that a vanishing fraction of the demand is met asymptotically. Then for all large n the multi-unit auction generates a higher efficiency and expected revenue than the bundle auction.

Proof. See Appendix.

Since strategies with complete demand reduction are not equilibrium strategies for large auctions, a calculation of the rate of convergence in the multi-unit auction based on such a strategy is a lower bound that need not be attained. Nonetheless, this gives some idea of the rate of convergence and is sufficient for the purpose of showing the superiority of the efficiency and revenue performance of the multi-unit auction relative to the bundle auction.

Corollary 1. Suppose assumptions (A1) and (A2) are satisfied, and that a vanishing fraction of the demand is met asymptotically. The multi-unit auction converges to efficiency and the limit revenue at a rate no less than $O\left(\left(\frac{n}{M_n}\right)^{-\frac{1}{k}}\right)$. The bundle auction converges to efficiency and the limit revenue at a rate $O\left(\left(\frac{n}{M_n}\right)^{-\frac{1}{m}}\right)$ for some $m \geq k+1$ whenever $m \equiv \min\left\{l : f_{V_1+V_2}^{(l-1)}(2) \neq 0\right\}$ exists and the counterpart for (A2) is satisfied for the bundle auction.

Proof. Follows from Lemma 5 in the Appendix. ■

While we have calculated the minimum rates at which the auctions converge to their limit outcomes, the technique is useless when a positive fraction of demand is met asymptotically. Therefore, as discussed before, to have an idea of how close large multi-unit auctions come to its competitive market counterparts, we calculate an upper bound for the rate at which the price in the auction converges to the competitive market price that is associated with the efficient outcome. We end this section by providing this upper bound for both the cases of finite and infinite supply (with vanishing fraction of the demand satisfied asymptotically).

Proposition 4. Suppose that assumptions ... are satisfied and that a vanishing fraction of the demand is met asymptotically. The price in the multi-unit auction converges to its limit price of 1 at a rate not exceeding $O\left(n^{-\frac{1}{\hat{k}}}\right)$ when the supply is bounded above, $O\left(\left(\frac{n}{M_n}\right)^{\frac{1}{\hat{k}}}\right)$ when $M_n \to \infty$, where $\hat{k} \equiv \min\left\{l : \frac{1}{2}f_1^{(l-1)}(2) + \frac{1}{2}f_2^{(l-1)}(2) \neq 0\right\}$.[19]

[19] Compare this with the rate at which the price converges to its limit in the bundle auction which

Proof. Follows from Lemma 10. ■

Remark 6. The multi-unit auction under the Vickrey auction rule generates a higher expected revenue than under the uniform-price rule with complete demand reduction. Therefore, the revenue performance of multi-unit auction is better than the bundle auction under the Vickrey rule, as well, for all large n when a vanishing fraction of the the demand is met asymptotically. Moreover, it is easy to see that the multi-unit Vickrey auction is more efficient than the bundle auction regardless of the size of the auction and supply.

5. Large Supply With $\alpha \in (0,1)$

For the rest of the paper assume that the reserves in the auctions are set equal to zero.[20] In that case when a fraction $\alpha \in (0,1)$ of the demand is met asymptotically the price in the bundle auction converges to $F^{-1}_{V_1+V_2}(1-\alpha)$ - the probability limit of the order statistic $(V_1+V_2)_{M_n+1:n}$. Now consider the "multi-unit" auction. It follows from Swinkels (2001) that the prices in the sequence of multi-unit auctions converge to a limit, and from Chakraborty and Engelbrecht-Wiggans (2004) this limit price is known to be $\left(\frac{1}{2}F_{V_1}+\frac{1}{2}F_{V_2}\right)^{-1}(1-\alpha)$. When $\alpha \in (0,1)$ the limits cannot be guaranteed to be equal. As a result a direct comparison is possible as the following results states.

Proposition 5. Suppose a fraction $\alpha \in (0,1)$ of the demand is met asymptotically.

(i) Suppose also that the set $E = \{x : F_1(x) + F_2(x) = 2F_{V_1+V_2}(2x)\}$ is finite (countable). Then the multi-unit auction is more efficient relative to the bundle auction for all large number of bidders and for all but a finite (countable) number of α's.

(ii) Suppose also that either the set E defined in (i) is finite, or that *(A2)* and *(A3)* are satisfied. The multi-unit auction generates a larger expected revenue for all large number of bidders if α is small enough (*i.e.,* sufficiently close to 0). However, if α is large enough (*i.e.*, sufficiently close to 1) then bundle auction generates a strictly larger revenue.

---

has been already calculated above.

[20] Clearly, appropriate condition like *(A4)* can be imposed to maintain the passive role of reserves in all the cases, but that will not give any additional insight.

Proof. See Appendix.

In short, while the relative efficiency of large multi-unit auctions holds unambiguously, whether the relative revenue-performance of large multi-unit auctions is better or worse than bundle auction depends on the fraction of demand that is met asymptotically. This switch in performance takes place monotonically when the following regularity condition holds.

(A5) Another Regularity Condition
There is a unique $x^* \in (0, 1)$ such that

$$\begin{aligned} \frac{1}{2}F_{V_1}(x) + \frac{1}{2}F_{V_2}(x) &> F_{\frac{V_1+V_2}{2}}(x) \;\text{ if } x \in (0, x^*) \\ &< F_{\frac{V_1+V_2}{2}}(x) \;\text{ if } x \in (x^*, 1). \end{aligned}$$

Corollary 2. Suppose a fraction $\alpha \in (0, 1)$ of the demand is met asymptotically. The multi-unit auction generates a higher or lower revenue than the bundle auction for all large number of bidders depending on whether the fraction of demand met asymptotically is smaller or larger than α^* if and only if condition *(A5)* holds.[21]

Proof. Follows straightforwardly from Proposition 5. ■

Remark 7. The regularity condition *(A5)* implies that the convolution of random variables V_1 and V_2 reduces the probability mass towards its "center." In other words, the mixture distribution $\frac{1}{2}F_{V_1} + \frac{1}{2}F_{V_2}$ is larger than $F_{\frac{V_1+V_2}{2}}$ in the sense of a dispersion-based ranking. In fact, the proof of Proposition 4 involved showing that in general $\frac{1}{2}F_{V_1} + \frac{1}{2}F_{V_2} \succ F_{\frac{V_1+V_2}{2}}$ in the sense of convex ordering. What *(A5)* demands is an ordering in a stronger sense that involves the single-crossing property. Nonetheless, *(A5)* is satisfied by several distributions. Suppose that V_1 and V_2 are the higher and lower order statistics of two independent draws X and Y from a distribution $H(\cdot)$. Then *(A5)* is satisfied whenever X and Y satisfy the regularity condition of Chakraborty (1999)

[21] In this case when $\alpha = \alpha^*$ the revenue performace of multi-unit auction relative to the bundle auction is ambiguous even in large auctions.

(which is itself known to be satisfied by several classes of distributions). Incidentally, *(A5)* also means that $F_{\frac{V_1+V_2}{2}}$ has a greater kurtosis than $\frac{1}{2}F_{V_1}+\frac{1}{2}F_{V_2}$ (see van Zwet,1967)

Apparently, Corollary 3 is quite intuitive, when a large fraction of demand is met asymptotically, the auction is not "sufficiently competitive." As a result bundling of units becomes necessary to stimulate the price. This pattern is to some extent similar to that in the auctions for multiple dissimilar objects (Palfrey, 1983, and Chakraborty, 1999). In that case, separate auctions generate a higher revenue than bundle auction if and only if the number of bidders is larger than a certain threshold number of bidders. However, *(A5)* need not always hold. In other words, the switch in the revenue-performance of multi-unit auction relative to the bundle auction need not always be monotonic as the fraction of the demand met asymptotically increases (and, thus, the auction becomes less "competitive"). This is clearly illustrated by the following example:

Example

Let $X \sim U[0,1]$ and $Y \sim U[0.4, 0.6]$, with the distribution functions being denoted by G_X and G_Y. Suppose that V_1 and V_2 are the higher and lower order statistics for independent draws from G_X and G_Y. Then we have $\frac{1}{2}F_{V_1}(x)+\frac{1}{2}F_{V_2}(x) = \frac{1}{2}G_X(x)+\frac{1}{2}G_Y(x)$ and $F_{\frac{V_1+V_2}{2}}(x) = G_{\frac{X+Y}{2}}(x)$ where $G_{\frac{X+Y}{2}}(\cdot)$ is the distribution of $\frac{X+Y}{2}$. It follows that there are x_1, x_2, x_3 with $0 < x_1 < x_2 < x_3 < 1$ such that

$$\begin{aligned}
\frac{1}{2}F_{V_1}(x)+\frac{1}{2}F_{V_2}(x) - F_{\frac{V_1+V_2}{2}}(x) &> 0 \text{ for } x \in (0, x_1) \text{ and } x \in (x_2, x_3) \\
&< 0 \text{ for } x \in (x_1, x_2) \text{ and } x \in (x_3, 1).
\end{aligned}$$

Thus, *(A3)* is violated by F. This has the implication that if $\alpha \in (0, \alpha_1)$ or $\alpha \in (\alpha_2, \alpha_3)$ then multi-unit auction generates a higher expected revenue than bundle auction for all large number of bidders. However, if $\alpha \in (\alpha_1, \alpha_2)$ or $\alpha \in (\alpha_3, 1)$ then the bundle auction generates a strictly higher expected revenue than the multi-unit auction for all large number of bidders. In short, as the fraction of demand met asymptotically increases, it is no more the case that the revenue-performance of large multi-unit auctions changes in the monotonic manner that simple intuition on the competitive effect would make it appear.

It is also interesting to note that a direct application of the Theorem 1 of Watson and Gordon (1986) implies that the convolution of G_X and G_Y have the spread-reducing effect in the sense of the regularity condition of Chakraborty (1999). In short, monotonicity of the competitive effect as the fraction of demand met asymptotically decreases continues to hold with these same distributions in the context of the dissimilar object (as discussed above), further highlighting the difference between multi-unit set-up and the auctions with dissimilar objects.

Remark 8. Following Swinkels (2001) it follows that the uniform-price (with price set equal to the highest losing, or lowest winning bid), discriminatory-price, and Vickrey rules for multi-unit auctions are revenue equivalent in the limit (i.e., the limiting prices are equal). This implies that Proposition 5 and Corollary 2, and all the related discussions above hold when the auctions are carried out under the Vickrey and first-price rules.

Next we consider the rate at which the auctions converge to the corresponding competitive equilibrium prices.

Proposition 6. The prices in the multi-unit auction converge to the competitive price in the limit at a rate not exceeding $O\left(n^{-\frac{1}{2}}\right)$ when $\sqrt{n}\left(\frac{M_n}{n}-\alpha\right) \longrightarrow c \in R$, and $O\left(\frac{n}{M_n-n\alpha}\right)$ when $\sqrt{n}\left(\frac{M_n}{n}-\alpha\right) \longrightarrow \pm\infty$. The convergence of prices in the bundle auctions take place exactly at these rates.

Proof. The first part follows from Lemma 7 while the second part follows from Lemma 6. ■

Remark 9. Chakraborty and Engelbrecht-Wiggans (2004) demonstrate that when a positive fraction of the demand is met, *i.e.*, $\alpha \in (0,1)$, the optimal reserve in a multi-unit auction can exceed the limit price in the absence of reserves. In other words, when reserves are set optimally, the optimal reserves may determine the expected revenues for large auctions, rather than the limit price. In that case, the comparison of revenues essentially involves a comparison of the optimal reserves.

6. Vanishing Fraction Of Demand Not Met Asymptotically

Finally, we consider the case where $\alpha = 1$, *i.e.*, a vanishing fraction of demand is left *unfilled* asymptotically. The results in this case are almost mirror images of those corresponding to the case of $\alpha = 0$. However, there are several differences in the actual results, making it necessary to give a detailed description. Of course, by now it is almost expected that in this case under appropriate assumptions that correspond to *(A1)* bundle auction should perform better in revenue than the multi-unit auction for all large number of bidders. This is proved by showing that the rate of convergence under the bundle auction is slower than that in the multi-unit auction even if the bidders bid truthfully under the multi-unit auction. The calculation of the rate of convergence in that case involves, as in the previous section, the joint distribution $F(\cdot,\cdot)$. In this case, we have the following Proposition:

Proposition 7. Suppose assumption *(A3)* is satisfied. The prices in the multi-unit auction converge to 0 at a rate no slower than $O\left(n^{-\frac{1}{k}}\right)$ when $n - M_n$ is bounded above, and $O\left(\left(\frac{n}{M_n}\right)^{-\frac{1}{k}}\right)$ when $n - M_n \to \infty$. The price in the bundle auction converges to 0 at a rate $O\left(n^{-\frac{1}{\check{m}}}\right)$ when $n - M_n$ is bounded above, and $O\left(\left(\frac{n}{M_n}\right)^{-\frac{1}{\check{m}}}\right)$ when $n - M_n \to \infty$, whenever $\check{m} \equiv \min\{l : f^{(l-1)}_{V_1+V_2}(0) \neq 0\}$ exists.

Proof. The first statement follows from Corollary 5 and Corollary 6 in the Appendix. The second statement follows from Corollary 3 and Corollary 4 in the Appendix. ■

Proposition 8. Suppose that assumption *(A3)* is satisfied and that a vanishing fraction of the demand is left asymptotically unfulfilled. The multi-unit auction generates a smaller revenue than the bundle auction for all large number of bidders. However, the multi-unit auction generates a higher efficiency than the bundle auction for all large number of bidders.

Proof. The first part follows upon combining *(A3)* and Proposition 7.

To prove the second part observe that on a per unit basis both multi-unit and bundle auction generate the same limit surplus. Since the price goes to zero, therefore, the surplus increases as $n \longrightarrow \infty$. Finally, using Theorem 6 we have that the multi-unit auction converges faster, so that the result follows. ■

Remark 10. Proposition 8 is proved essentially by showing that an upper bound on the expected revenue from the multi-unit auction is smaller than the revenue from the bundle auction for all large n. This upper bound revenue is also larger than the revenue generated by the multi-unit auction under the Vickrey rule. Hence, Proposition 8 holds when the auctions are conducted under the Vickrey rule, as well.

7. Concluding Discussion

We have evaluated the performance of the multi-unit auction relative to the alternative of bundle auction, as well as, in the absolute sense by competitive market standards. In a broad class of situations a multi-unit auction dominates the bundle auction both in terms of efficiency whenever there are sufficiently many bidders. However, whether the multi-unit auction performs better or worse than the bundle auction in terms of expected revenue depends on whether a small or large fraction of the demand is met asymptotically. The switch in the revenue performance of multi-unit auctions as this fraction increases happens monotonically only in a special class of situations. In general, however, this switch need not always take place monotonically. The results continue to hold under the Vickrey auction rules and in part under the discriminatory price rule, as well.

As an interesting corollary these results give sufficient conditions for social objectives to become perfectly aligned with that of a monopolist in the context of multi-unit auctions under asymmetric information without pushing the market size to the limit. The necessary conditions simply involve additional constraints on the nature of the relationship between the reserves in the multi-unit and bundle auctions, and the distribution of values.

On the technical side, we have developed some useful results to analyze the asymptotics of an interesting class of problems. Ordering and ranking of actions can be an important element of market models even outside our framework (*e.g.*, the bilateral bargaining models). The typical analysis of such models make heavy use of results on order statistics. When each agent's action happens to be multivariate with a joint distribution before being pooled and ranked, there are no statistical results that can be used to analyze such order statistics. In course of proving our results on multi-unit auctions we have developed results on the behavior of such order statistics that is likely to make an asymptotic analysis of these other models possible. For instance, consider the multi-unit version of the double-bid auction model of Rustichini, Satterthwaite and Williams (1994). Suppose that there are M_n^B buyers and M_n^S sellers and $\alpha = \lim_{n\to\infty} M_n^S/M_n^B$. Each buyer has diminishing (or increasing) marginal values for m^B units and each seller has m^S units to sell with diminishing (or increasing) opportunity costs for the successive units. Let F_r^B and F_r^S be the marginal distribution corresponding to the r-th unit for the buyer and the seller, respectively. Whenever such a double-bid market converges to efficiency in the limit, similar techniques can be applied to show that the demand (respectively, supply) in the limit competitive market on a per buyer (respectively, per seller) basis is given by $n-\sum_{r=1}^{m^B} F_r^B(p)$ (respectively, $\sum_{r=1}^{m^S} F_r^S(p)$). The limit competitive price as characterized by $n-\sum_{r=1}^{m^B} F_r^B(p) = \alpha \sum_{r=1}^{m^S} F_r^S(p)$ that equates the demand and the supply in the limit competitive market can then be used to analyze the large double-bid market in question. In fact, a number of numerical examples indicate that as α increases from 0 to ∞ the ranking of the price in such large markets under bundle and multi-unit formats behave in a similar manner as do the prices in Corollary 2 when α increases from 0 to 1.

Any attempt to identify the rate of convergence more closely than we have done here has to involve dealing with the order statistics for bids that are drawn from multivariate distributions. Therefore, it is very likely that the results that we have presented in Lemmas 7 through 10 and Corollaries 5 and 6 will need to be used in some form to calculate the exact rates of convergence of the auction price. However, the extent to which the difficulty of describing the equilibrium strategies can actually be overcome for that purpose is not clear.

It would also be interesting to apply the techniques to specifically analyze the double bid auction market where each buyer and/or seller wants to buy/sell multiple units of an object. Using the steps outlined in our Lemmas to describe the rate of convergence and applying them to identify the general form of the competitive price in the limit market (as in Chakraborty and Engelbrecht-Wiggans, 2004b) are part of our future research agenda.

8. Appendix

Here we define some notations that will be used below in the statement of the result and their proofs. We shall denote weak convergence of distributions/random variables by $\xrightarrow{d}$ and convergence in probability of random variables by $\xrightarrow{P}$. Although in some cases almost sure convergence may hold true, we use weaker results so that similar proof would go though in cases where only, for example, convergence in probability holds true. All limits are taken, unless specified otherwise, as n approaches infinity. By $X \stackrel{d}{=} Y$ we mean that the random variables X and Y have the same distribution.

For a set A, we shall denote by $I_A(\omega)$, the indicator function of A, *i.e.*,

$$\begin{aligned} I_A(\omega) &= 1 \text{ if } \omega \in A \\ &= 0 \text{ if } \omega \in A^c; \end{aligned}$$

Gamma(α, β) will be used, depending on the context, to denote both a gamma variate and a gamma distribution with mean $\alpha\beta$ and variance $\alpha\beta^2$. Similarly, $N(\mu, \sigma^2)$ will correspond to the normal distribution with mean μ and variance σ^2, and $U(0,1)$ to the uniform distribution on $(0,1)$. The standard normal distribution function will be denoted by $\Phi(\cdot)$. The norm $\|\cdot\|$ will denote the total variation norm on the space of probability measures.

In the following $U_{j:n}$ will stand for the j-th highest order statistic from a random sample of size n from the uniform distribution on $(0,1)$. Hence $U_{1:n}$ and $U_{n:n}$ will stand for the maximum and the minimum, respectively. All results that we use for order statistics from $U(0,1)$ can be found in Reiss (1989). We summarize these in the following two lemmas - the first deals with weak convergence results and the second with the asymptotic behavior of moments.

Lemma 1. (i) For any positive integer j,

$$n(1-U_{j:n}) \xrightarrow{d} \text{Gamma}(j,1).$$

(ii) For any sequence of positive integers $\{j_n\}_{n\geq 1}$ satisfying

$$\frac{j_n}{n} \longrightarrow 0 \text{ and } j_n \longrightarrow \infty,$$

we have

$$\left(\frac{n}{j_n}\right)(1-U_{j:n}) \xrightarrow{P} 1.$$

(iii) For any sequence of positive integers $\{j_n\}_{n\geq 1}$ satisfying

$$j_n \longrightarrow \infty \quad \text{and} \quad (n-j_n) \longrightarrow \infty,$$

we have

$$\left(\frac{n}{\frac{j_n}{n}\left(1-\frac{j_n}{n}\right)}\right)^{\frac{1}{2}} \left(\left(1-\frac{j_n}{n}\right)-U_{j_n:n}\right) \xrightarrow{d} N(0,1)$$

Proof. (i) The proof follows simply as a weaker version of Lemma 5.1.5 of Reiss (1989).

(ii) To prove this part we use the fact that

$$\left(\frac{n}{\sqrt{j_n}}\right)\left(U_{j_n:n}-1+\frac{j_n}{n}\right) \xrightarrow{d} N(0,1),$$

which can be deduced from theorem 5.1.7 of Reiss (1989). Hence, using Slutsky's theorem we have

$$\left(\frac{n}{j_n}\right)(1-U_{j_n:n}) = \frac{1}{\sqrt{j_n}}\left[\left(\frac{n}{\sqrt{j_n}}\right)\left(1-\frac{j_n}{n}-U_{j_n:n}\right)\right]+1 \xrightarrow{d} 1.$$

(This result follows much more easily from the second part of the following Lemma.)

(iii) This is a standard asymptotic normality property of central sequences. See, for instance, Reiss (1989). ■

Lemma 2. (i) For any positive integer j,

$$n^{\zeta}E[(1-U_{j:n})^{\zeta}] \longrightarrow \frac{\Gamma(j+\zeta)}{\Gamma(j)}, \forall\zeta \geq 0.$$

(ii) For any sequence of positive integers $\{j_n\}_{n\geq 1}$ satisfying

$$\frac{j_n}{n} \longrightarrow 0 \text{ and } j_n \longrightarrow \infty,$$

we have

$$\left(\frac{n}{j_n}\right)^{\zeta} E[(1-U_{j_n:n})^{\zeta}) \longrightarrow 1, \forall\zeta \geq 0.$$

(iii) For every positive integer k and $j \in \{1, 2, ..., n\}$,

$$E[|U_{j:n}-\mu|^k] \leq 2k!5^k\sigma^k n^{-\frac{k}{2}},$$

and

$$\mu = 1 - \frac{j_n}{n+1} \text{ and } \sigma^2 = \mu(1-\mu).$$

Proof. Parts (i) and (ii) follow from expressing the concerned moments in terms of the beta function and then using the Stirling's approximation for the gamma function.

Part (iii) is the Lemma 3.1.3 of Reiss (1989). ■

We will need to use the Stein-Chen method for Poisson approximation and hence require the use of the following results which can be found in Lindvall (1992).

Lemma 3. Let $\{Z_i\}_{i\leq n}$ be a finite sequence of Bernoulli variables with expectations $\{p_i\}_{i\leq n}$, respectively. Let $S \equiv \sum_{i=1}^{n} Z_i$ and $\lambda \equiv E[S]$. Moreover, let $\{R_i\}_{i\leq n}$ and $\{T_i\}_{i\leq n}$ be such that

$$R_i \stackrel{d}{=} S \text{ and } 1+T_i \stackrel{d}{=} P[S \in \cdot|Z_i = 1).$$

Then,

$$\|P[S \in \cdot) - \text{Poisson}()\| \leq 2(1 \wedge \lambda^{-1}) \sum_{i=1}^{n} p_i E[|R_i - T_i|].$$

■

In the following two lemmas, $G(\cdot)$ will denote a univariate distribution function with density $g(\cdot)$ and support $[0, a]$. Moreover, we assume that $G(\cdot)$ is k-times differentiable in a left neighborhood of a with $g^{(l)}(a) = 0$ for $l = 0, 1, .., k-2$ and $g^{(k-1)}(a) \neq 0$. For such a $G(\cdot)$, let

$$C(G, k) \equiv \left(\frac{k!(-1)^{k-1}}{g^{(k-1)}(a)} \right)^{\frac{1}{k}}.$$

$V_{j:n}$, for a non-negative j between 0 and n, will denote the j-th highest order statistic from a random sample of size n from $G(\cdot)$.

Lemma 4. For a sequence of numbers $\{r_n\}_{n \geq 1}$ in $(0, 1)$ satisfying

$$\lim_{n \to \infty} n^{\zeta}(1 - r_n) > 0, \;\; \text{for some } \zeta < 1,$$

we have

$$n^{\frac{1}{k}} E\left[(a - V_{j:n}) I_{\{V_{j:n} > G^{-1}(r_n)\}}\right] \longrightarrow C(G, k) \frac{\Gamma\left(j + \frac{1}{k}\right)}{\Gamma(j)}.$$

Proof. Using the transformation $G^{-1}(\cdot)$, we see that

$$(a - V_{j:n}) I_{\{V_{j:n} > G^{-1}(r_n)\}} \stackrel{d}{=} \left(G^{-1}(1) - G^{-1}(U_{j:n}) I_{\{U_{j:n} > r_n\}}\right),$$

which converts the problem to one in order statistics from $U(0, 1)$. Now due to the fact that

$$\begin{aligned}
& \left| n^{\frac{1}{k}} E\left[G^{-1}(1) - G^{-1}(U_{j:n})\right] - n^{\frac{1}{k}} E\left[(G^{-1}(1) - G^{-1}(U_{j:n})) I_{\{U_{j:n} > r_n\}}\right] \right| \\
= \;& n^{\frac{1}{k}} E\left[G^{-1}(U_{j:n}) I_{\{U_{j:n} \leq r_n\}}\right] \\
\leq \;& a n^{\frac{1}{k}} P\left[U_{j:n} \leq r_n\right]
\end{aligned}$$

and that

$$\begin{aligned}
n^{\frac{1}{k}}P[U_{j:n} &\leq r_n] = n^{\frac{1}{k}}P[n(1-U_{j:n}) \geq n(1-r_n)) \\
&\leq \frac{n^{\frac{1}{k}}E\left[(n(1-U_{j:n})\right]}{(n(1-r_n))^{\beta}} \quad \text{(Markov's Inequality, } \beta > \frac{1}{k(1-\zeta)}) \\
&\leq \left(\frac{n^{\left(\frac{1}{k}-\beta(1-\zeta)\right)}}{(n^{\zeta}(1-r_n))^{\beta}}\right) E\left[(n(1-U_{j:n}))^{\beta}\right] \\
&\longrightarrow 0 \quad \text{(Using Lemma 2)}
\end{aligned}$$

it suffices to show that

$$n^{\frac{1}{k}}E\left[G^{-1}(1) - G^{-1}(U_{j:n})\right] \longrightarrow C(G,k)\frac{\Gamma\left(j+\frac{1}{k}\right)}{\Gamma(j)}.$$

Towards showing the weak convergence of $n^{\frac{1}{k}}(G^{-1}(1) - G^{-1}(U_{j:n}))$ and finding its weak limit, note that writing

$$X_n \equiv \frac{G^{-1}(1) - G^{-1}(U_{j:n})}{(1-U_{j:n})^{\frac{1}{k}}} \quad \text{and} \quad Y_n \equiv (n(1-U_{j:n}))^{\frac{1}{k}}.$$

we have

$$n^{\frac{1}{k}}\left(G^{-1}(1) - G^{-1}(U_{j:n})\right) = X_n \cdot Y_n$$

where by Lemma 1(i), $Y_n \xrightarrow{d} Y$ with $Y^k \stackrel{d}{=}\text{Gamma}(j,1)$ and $X_n \xrightarrow{P} C(G,k)$. The latter holds as $U_{j:n} \xrightarrow{P} 1$ and $X_n = \psi(U_{j:n})$ where $\psi(\cdot)$, defined by

$$\psi(x) \equiv \left(\frac{G^{-1}(1) - G^{-1}(x)}{(1-x)^{\frac{1}{k}}}\right), \forall x \in [0,1]$$

is a continuous function with the limit $C(G,k)$ at 1. The limit is derived using the Young's form of the Taylor's theorem and our assumptions on the behavior of $G(\cdot)$ at a. Hence, using Slutsky's theorem, we have

$$n^{\frac{1}{k}}\left(G^{-1}(1) - G^{-1}(U_{j:n})\right) \xrightarrow{d} C(G,k) \cdot Y.$$

Now all that remains is to show the L^1 convergence which we prove by showing that the sequence is bounded in the L^2 sense. Now $\{n^{\frac{1}{k}}(G^{-1}(1) - G^{-1}(U_{j:n}))\}_{n\geq 1}$ will be

L^2-bounded if so is $\{Y_n\}_{n\geq 1}$ as $\psi(\cdot)$ is non-negative and bounded (due to continuity on a compact interval). But $\{Y_n\}_{n\geq 1}$ is L^2-bounded by Lemma 2(i). The proof then becomes complete upon observing that $E[Y] = \frac{\Gamma\left(j+\frac{1}{k}\right)}{\Gamma(j)}$. ■

Lemma 5. For any sequence of positive integers $\{j_n\}_{n\geq 1}$ satisfying

$$\frac{j_n}{n} \longrightarrow 0 \quad \text{and} \quad j_n \longrightarrow \infty,$$

and a sequence of numbers $\{r_n\}_{n\geq 1}$ in $(0,1)$ satisfying

$$\lim_{n\longrightarrow\infty} \left(\frac{n}{j_n}\right)^{\zeta} (1-r_n) > 0, \quad \text{for some } \zeta < 1,$$

we have

$$\left(\frac{n}{j_n}\right)^{\frac{1}{k}} E\left((a - V_{j_n:n}) I_{\{V_{j_n:n} > G^{-1}(r_n)\}}\right) \longrightarrow C(G,k).$$

Proof. The proof follows along similar lines as Lemma 3 - the changes being using part of (ii) Lemma 1 and Lemma 2 instead of the part (i) used above and the rate of convergence now in terms of $j_n^{-1}n$ instead of n. ■

In the following corollaries, instead of the earlier assumptions on the behavior of $G(\cdot)$ at the upper end of its support, we will assume that it is k-times differentiable in a right neighborhood of 0 with $g^{(l)}(0) = 0$ for $l = 0, 1, ..., k-2$ and $g^{(k-1)}(0) \neq 0$. For such a $G(\cdot)$, define

$$C^*(G,k) \equiv \left(\frac{k!(-1)^{k-1}}{g^{(k-1)}(0)}\right)^{\frac{1}{k}}.$$

Corollary 3. For a positive integer j, we have

$$n^{\frac{1}{k}} E[V_{n+1-j:n}] \longrightarrow C^*(G,k) \frac{\Gamma\left(j+\frac{1}{k}\right)}{\Gamma(j)}.$$

Proof. If $F(\cdot)$ is defined by,

$$F(x) = 1 - G(a-x), \quad \forall x \in [0,a]$$

then

$$G^{-1}(y) = a - F^{-1}(y) = F^{-1}(1) - F^{-1}(1-y),\ \forall y \in [0,1],$$

F satisfies all the conditions of Lemma 4, and $C^*(G,k) = C(F,k)$. Moreover, as $U_{j:n} \overset{d}{=} 1 - U_{n+1-j:n}$, we have

$$V_{n+1-j:n} \overset{d}{=} G^{-1}(U_{n+1-j:n}) = F^{-1}(1) - F^{-1}(1-U_{n+1-j:n}) \overset{d}{=} F^{-1}(1) - F^{-1}(U_{j:n}) \overset{d}{=} a - V^*_{j:n},$$

where $V^*_{j:n}$ is the j-th highest order statistic from a random sample of size n from $F(\cdot)$. Hence the result follows from Lemma 4. ■

Corollary 4. For any sequence of positive integers $\{j_n\}_{n\geq 1}$ satisfying

$$\frac{j_n}{n} \longrightarrow 0 \text{ and } j_n \longrightarrow \infty,$$

we have

$$\left(\frac{n}{n-j_n}\right)^{\frac{1}{k}} E[V^*_{n-j_n:n}] \longrightarrow C^*(G,k).$$

Proof. By an argument similar to that of Corollary 3, the result follows from Lemma 5. ■

Lemma 6. Let G be a distribution function with density g and $V_{j:n}$, for a nonnegative integer j between 0 and n be the j-th highest order statistic from a random sample of size n from G. Let $\tau \in (0,1)$ be such that $g(G^{-1}(\tau)) > 0$. Then the following hold:

(i) For $\{j_n\}_{n\geq 1}$ satisfying,

$$\sqrt{n}\left(\frac{j_n}{n} - (1-\tau)\right) \longrightarrow c \in R,$$

we have

$$\sqrt{n}E[G^{-1}(\tau) - V_{j_n:n}] \longrightarrow \frac{c}{g(G^{-1}(\tau))}.$$

(ii) For $\{j_n\}_{n\geq 1}$ satisfying,

$$\sqrt{n}\left(\frac{j_n}{n} - (1-\tau)\right) \longrightarrow \pm\infty \text{ and } \frac{j_n}{n} \longrightarrow 1-\alpha,$$

we have

$$\left(\frac{n}{j_n - n(1-\tau)}\right) E[G^{-1}(\tau) - V_{j_n:n}] \longrightarrow \frac{1}{g(G^{-1}(\tau))}.$$

Proof. (i) We start by observing that $\sqrt{n}(\tau - U_{j_n:n}) \xrightarrow{d} N(c, \tau(1-\tau))$ since

$$\sqrt{n}(\tau - U_{j_n:n}) = \sqrt{n}\left(\left(1 - \frac{j_n}{n+1}\right) - U_{j_n:n}\right) + \sqrt{n}\left(\frac{j_n}{n+1} - (1-\tau)\right),$$

and

$$\sqrt{n}\left(\left(1 - \frac{j_n}{n+1}\right) - U_{j_n:n}\right) \xrightarrow{d} N(0, \tau(1-\tau)) \text{ and } \sqrt{n}\left(\frac{j_n}{n+1} - (1-\tau)\right) \longrightarrow c,$$

where the weak convergence of the first term follows from Lemma 1(iii). Now by the device of transformation, use of Young's form of the Taylor's theorem at $G^{-1}(\tau)$ and the above result, similar to the proof of Lemma 4, we have

$$\sqrt{n}E[G^{-1}(\tau) - V_{j_n:n}] \xrightarrow{d} N\left(\frac{c}{g(G^{-1}(\tau))}, \frac{\tau(1-\tau)}{(g(G^{-1}(\tau)))^2}\right).$$

For the convergence of the first moment we observe that sequence $\{\sqrt{n}(\tau - U_{j_n:n})\}_{n\geq 1}$ is L^2-bounded, using Minkowski's inequality we have

$$\left|\left(E\left[n\left(\left(1 - \frac{j_n}{n}\right) - U_{j_n:n}\right)^2\right]\right)^{\frac{1}{2}} - \left(E\left[n\left(\tau - U_{j_n:n}\right)^2\right]\right)^{\frac{1}{2}}\right| \leq n\left(\frac{j_n}{n} - (1-\tau)\right)^2 \longrightarrow c^2 < \infty$$

and the sequence $\left\{\sqrt{n}\left(\left(1 - \frac{j_n}{n}\right) - U_{j_n:n}\right)\right\}_{n\geq 1}$ is L^2-bounded by Lemma 2 (iii).

(ii) By arguments similar to the first part above, it can be shown that

$$\left(\frac{n}{j_n - n(1-\tau)}\right)(\tau - U_{j_n:n}) \xrightarrow{d} 1,$$

and hence

$$\left(\frac{n}{j_n - n(1-\tau)}\right)(G^{-1}(\tau) - V_{j_n:n}) \xrightarrow{d} \frac{1}{g(G^{-1}(\tau))}.$$

Use of Minkowski's inequality coupled with an application of Lemma 2(iii) proves, as above, the convergence of the first moment. ■

Let F denote a distribution with support as $[0,a]^2$,[22] for some positive a and F_1 and F_2 be its marginals. For $\{X_i = (X_i^1, X_i^2)\}_{1\leq j\leq n}$ a random sample of size n from F, we define $W_{j:2n}$ be the j-th highest value among the $2n$ values $\{X_i^k\}_{1\leq i\leq n, k=1,2}$, for $j = 1, 2, ..., 2n$. The following two lemmas and a corollary study the asymptotics of $W_{j_n:2n}$ under different assumptions on the behavior of $\{j_n\}_{n\geq 1}$.

As the sequence $W_{j_n:2n}$ is invariant with respect to permutations of the coordinates of X_i, we could equivalently work with $\{Y_i = (Y_i^1, Y_i^2) = (X_i^{\eta_i^1}, X_i^{\eta_i^2})\}_{i\geq 1}$ is an i.i.d. sequence of random uniform permutations of $(1,2)$. Note that Y_i is symmetric in its coordinates, *i.e.*, the joint distribution function is permutation invariant, which in particular implies that the marginals are identically equal to $G = \frac{1}{2}F_1 + \frac{1}{2}F_2$. By g we shall denote the first derivative of G.

Let σ_τ^2, for $\tau \in (0,1)$, denote the variance of $\frac{1}{2}I_{\{Y_i^1 > G^{-1}(\tau)\}} + \frac{1}{2}I_{\{Y_i^2 > G^{-1}(\tau)\}}$. It can be shown that

$$\sigma_\tau^2 = \tau(1-\tau) - \frac{1}{2}E[I_{\{Y_i^1 \leq G^{-1}(\tau) \leq Y_j^2\}}],$$

which reduces to $\frac{1}{2}\tau(1-\tau)$ in the case of independent coordinates.

We will also find use for the order statistics from a random sample of each of the coordinates - hence we denote by $W_{j_n:n}^1$ and $W_{j_n:n}^2$ the j-th highest 1-st and 2-nd coordinate of $\{Y_i\}_{1\leq i\leq n}$, respectively. Note that $W_{j_n:n}^1$ and $W_{j_n:n}^2$ are identically distributed though not necessarily independent.

Lemma 7. Let $\tau \in (0,1)$ be such that $g(G^{-1}(\tau)) > 0$. Then the following hold:

(i) For $\{j_n\}_{n\geq 1}$ satisfying,

$$\sqrt{n}\left(\frac{j_n}{n} - (1-\tau)\right) \longrightarrow c \in R,$$

we have

$$\sqrt{n}E[G^{-1}(\tau) - W_{2j_n:2n}) \longrightarrow \frac{c}{g(G^{-1}(\tau))}.$$

[22] In our case, we will be using support S. The results are presented in this case to recognize their usefulness beyond our auction framework.

(ii) For $\{j_n\}_{n\geq 1}$ satisfying,

$$\sqrt{n}\left(\frac{j_n}{n}-(1-\tau)\right) \longrightarrow \pm\infty \quad \text{and} \quad \frac{j_n}{n} \longrightarrow 1-\tau,$$

we have

$$\left(\frac{n}{j_n - n(1-\tau)}\right) E[G^{-1}(\tau) - W_{2j_n:2n}) \longrightarrow \frac{1}{g(G^{-1}(\tau))}.$$

Proof. (i) First, we show that

$$\sqrt{n}(G^{-1}(\tau) - W_{2j_n:2n}) \xrightarrow{d} N\left(\frac{c}{g(G^{-1}(\tau))}, \left[\frac{\sigma_\tau}{g(G^{-1}(\tau))}\right]^2\right).$$

Towards this end we employ another standard device of expressing the event

$$\{\sqrt{n}(G^{-1}(\tau) - W_{2j_n:2n}) \leq x\}$$

by

$$\left\{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}\sum_{k=1}^{2}\left[I_{\{Y_i^k \geq G^{-1}(\tau)-\frac{x}{\sqrt{n}}\}}\right] \geq \frac{j_n}{n}\right\}$$

and working with the latter event to establish the desired weak convergence. Rewriting the latter as

$$\underbrace{\frac{1}{2\sqrt{n}}\sum_{i=1}^{n}\sum_{k=1}^{2}\left[I_{\{Y_i^k \geq G^{-1}(\tau)\}} - (1-\tau)\right]}_{\xrightarrow{d} N(0,\sigma_\alpha^2)} + \underbrace{\frac{1}{2\sqrt{n}}\sum_{i=1}^{n}\sum_{k=1}^{2}\left[I_{\left\{Y_i^k \in \left[G^{-1}(\tau)-\frac{x}{\sqrt{n}}, G^{-1}(\tau)\right)\right\}}\right]}_{\xrightarrow{P} xg(G^{-1}(\tau))}$$

$$\geq \underbrace{\sqrt{n}\left(\frac{j_n}{n}-(1-\tau)\right)}_{\longrightarrow c},$$

where the first weak convergence follows using the central limit theorem on the zero mean, i.i.d. random variables $\frac{1}{2}\sum_{k=1}^{2}\left[I_{\{Y_i^k \geq G^{-1}(\tau)\}} - (1-\tau)\right]$ with variance σ_τ^2. The convergence of the second term follows by combining the convergence of its expectation, given by

$$G(G^{-1}(\tau)) - G\left(G^{-1}(\tau) - \frac{x}{\sqrt{n}}\right),$$

to $xg(G^{-1}(\tau))$ and that its variance, given by

$$\frac{1}{4}\left[G\left(G^{-1}(\tau)\right)G\left(G^{-1}(\tau)-\frac{x}{\sqrt{n}}\right)\right]\left[1-G\left(G^{-1}(\tau)\right)+G\left(G^{-1}(\tau)-\frac{x}{\sqrt{n}}\right)\right]=O\left(n^{-\frac{1}{2}}\right).$$

Having established the required convergences, by Slutsky's theorem and Pólya's theorem the probability of the inequality being satisfied converges to

$$1-\Phi\left(\frac{c-xg(G^{-1}(\tau))}{\sigma\tau}\right)=\Phi\left(\frac{x-\frac{c}{g(G^{-1}(\tau))}}{\frac{\sigma_\tau}{g(G^{-1}(\tau))}}\right),$$

which establishes the asymptotic behavior of $W_{2j_n:2n}$. Now, for the convergence of the first moment observe that

$$\left|W_{2j_n:2n}-G^{-1}(\tau)\right|\leq\max\left(\left|W^1_{j_n:n}-G^{-1}(\tau)\right|,\left|W^2_{j_n:n}-G^{-1}(\tau)\right|\right),$$

which in turn implies that

$$nE\left[\left(W_{2j_n:2n}-G^{-1}(\tau)\right)^2\right]\leq 2nE\left[\left(W^1_{j_n:n}-G^{-1}(\tau)\right)\right].$$

Moreover, since $nE\left[\left(W^1_{j_n:n}-G^{-1}(\tau)\right)^2\right]$ is bounded, as was $nE\left[\left(V_{j_n:n}-G^{-1}(\tau)\right)^2\right]$ in Lemma 6, the convergence of the first moment is established.

(ii) By arguments similar to part (ii) of Lemma 6 and part (i) above, the convergence of the first moment can be established. ■

In the following we study the behavior of $W_{j_n:2n}$ when $\frac{j_n}{n}\longrightarrow 0$ which will depend on the behavior of the previously defined G at the upper end of the support, *i.e.*, at a. We assume that $G(\cdot)$ is k-times differentiable in a left neighborhood of a with $g^{(l)}(a)=0$ for $l=0,1,...,k-2$ and $g^{(k-1)}(a)\neq 0$. We define $C(G,k)$ by,

$$C(G,k)=\left[\frac{(-1)^{k-1}k!}{g^{(k-1)}(a)}\right]^{\frac{1}{k}}$$

First, we study the case when $j_n=j$, for some positive integer j, for which we will need, for which we will need, for any $x\geq 0$, the asymptotic behavior of

$$S_n\equiv\sum_{i=1}^{n}\sum_{m=1}^{2}I_{\left\{Y_i^m\geq a-xn^{-\frac{1}{k}}\right\}}=\sum_{i=1}^{2n}Z_i^n,$$

where, for convenience, we define the sequence $\{Z_i^n\}_{1\le i\le 2n}$ to be

$$Z^n_{2(i-1)+m} = I_{\left\{Y_i^m \ge a - xn^{-\frac{1}{k}}\right\}}, \quad i = 1, 2, .., n \text{ and } m = 1, 2.$$

Observe that the summands are identically distributed Bernoulli variables with probability of taking the value 1, say p_n, satisfying

$$np_n = n\left(1 - G\left(a - xn^{-\frac{1}{k}}\right)\right) \longrightarrow \left[\frac{x}{C(G,k)}\right]^k.$$

The fact that $p_n = O(n^{-1})$ immediately points in the direction of a Poisson limiting distribution for S_n; but since the summands are independent, we need some condition based on a measure of dependence which will make the limiting distribution Poisson, the same that holds under independence. Note that the dependence in Z_i's is only between Z^n_{2i-1} and Z^n_{2i}.

Our choice for the measure of dependence is ϵ_n, defined as,

$$\epsilon_n = F\left(a - xn^{-\frac{1}{k}}, a - xn^{-\frac{1}{k}}\right) - \left[G\left(a - xn^{-\frac{1}{k}}\right)\right]^2, \forall n \ge 1.$$

It is a measure of dependence as it is the deviation of the probability mass function from that under independence as,

$$P\left[Z_1^n = i, Z_2^n - j)\right] = (p_n)^{i+j}(1-p_n)^{2-(i+j)} + (-1)^{i+j}\epsilon_n, \quad \forall i, j = 0, 1 \quad \text{and } \forall n \ge 1.$$

More specifically, the total variation distance between the distribution of (Z_1^n, Z_2^n) and that of two independent Bernoulli variables with parameter p_n is equal to $4\epsilon_n$. Also, note that $Cov(Z_1^n, Z_2^n) = \epsilon_n$ and the correlation coefficient between Z_1^n and Z_2^n, denoted by $\rho_{Z_1^n, Z_2^n}$, satisfies

$$\rho_{Z_1^n, Z_2^n} = \rho_{Z^n_{2i-1}, Z^n_{2i}} = \frac{\epsilon_n}{p_n(1-p_n)}, \ \forall n \ge 1.$$

Interestingly, even though ϵ_n can be negative, we have

$$0 \le \lim_{n\to\infty} \inf n\epsilon_n \le \lim_{n\to\infty} \sup n\epsilon_n \le \left[\frac{x}{C(G,k)}\right]^k.$$

The following lemma proves the Poisson convergence under the condition that $\lim_{n\to\infty} n\epsilon_n = 0$, which can be seen to be equivalent to both $\lim_{n\to\infty} \sup n\epsilon_n$ and, more importantly, $\lim_{n\to\infty} \rho_{Z_1^n, Z_2^n} = 0$. An interesting sufficient condition is $\epsilon_n \leq 0$, $\forall\, n \geq 0$, *i.e.*, negative dependence.

Even though there are a couple of other ways of proving the following Poisson convergence we choose the Stein-Chen method as it gives a better appreciation of the condition, $\lim_{n\to\infty} n\epsilon_n = 0$.

Lemma 8. If

$$\lim_{n\to\infty} n\epsilon_n = 0,$$

then

$$S_n \xrightarrow{d} \text{Poisson}\left(2\left[\frac{x}{C(G,k)}\right]^k\right).$$

Proof. First, using the Stein-Chen method for Poisson approximation, we will show that

$$\|P[S_n \in \cdot) - \text{Poisson}(2np_n)\| \leq 4\left(1 \wedge \frac{1}{2np_n}\right) np_n \left(p_n + \frac{|\epsilon_n|}{p_n}\right).$$

As $\left\{\left(Z_{2i-1}^n, Z_{2i}^n\right)\right\}_{i\leq n}$ are i.i.d. random vectors which are symmetric in their coordinates, it is sufficient to show that we can achieve a coupling of R_1 and T_1 of Lemma 3 in our case such that

$$E\left[|R_1 - T_1|\right] = \left(p_n + \frac{|\epsilon_n|}{p_n}\right).$$

One such coupling is the following:

$$R_1 \equiv \sum_{i=1}^{2n} Z_i^n \text{ and } T_1 \equiv Z_2^* + \sum_{i=3}^{2n} Z_i^n \quad \text{where } Z_2^* \stackrel{d}{=} P[Z_2^n \in \cdot | Z_1^n = 1]$$

where the coupling of (Z_1^n, Z_2^n, Z_2^*) is defined by the joint distribution of (Z_1, Z_2) and

$$P[Z_1^n = 1; Z_2^n = 0; Z_2^* = 0] = p_n(1-p_n) - \epsilon_n \quad \text{and} \quad P[Z_1^n = 1; Z_2^n = 1; Z_2^* = 1] = p_n^2 + \epsilon_n$$

$$P[Z_1^n = 0; Z_2^n = 0; Z_2^* = 1] = \frac{\epsilon_n}{p_n} \vee 0 \text{ and } P[Z_1^n = 0; Z_2^n = 1; Z_2^* = 0] = \left(-\frac{\epsilon_n}{p_n}\right) \vee 0$$

Now, since for discrete distributions we have equivalence of weak convergence and convergence in total variation, we have

$$\left\|\text{Poisson}(2np_n) - \text{Poisson}\left(2\left[\frac{x}{C(G,k)}\right]^k\right)\right\| \longrightarrow 0, \quad \text{as } np_n \longrightarrow \left[\frac{x}{C(G,k)}\right]^k.$$

Combining the above, we have the convergence to Poisson of S_n. ■

Lemma 9.

$$n^{\frac{1}{k}} E\left(a - W_{j:2n}\right) \longrightarrow 2^{-\frac{1}{k}} C(G,k) \frac{\Gamma\left(j+\frac{1}{k}\right)}{\Gamma(j)}.$$

Proof. First, we show that

$$n^{\frac{1}{k}}\left(a - W_{j:2n}\right) \xrightarrow{d} \left[\text{Gamma}\left(j, \frac{1}{2}C(G,k)^k\right)\right]^{\frac{1}{k}}.$$

Using the equivalence,

$$\left\{n^{\frac{1}{k}}\left(a - W_{j:2n}\right) \leq x\right\} = \left\{\sum_{i=1}^{2n} Z_i^n \geq j\right\}$$

and Lemma 8, we have

$$\begin{aligned} & P\left[n^{\frac{1}{k}}\left(a - W_{j:2n}\right) \leq x\right] \\ = \; & P\left[\sum_{i=1}^{2n} Z_i^n \geq j\right] \longrightarrow 1 - \exp\left\{-2\left[\frac{x}{C(G,k)}\right]^k\right\} \sum_{i=0}^{j-1} \frac{1}{i!}\left(2\left[\frac{x}{C(G,k)}\right]^k\right)^i, \end{aligned}$$

which is a restatement of the weak convergence above. The proof of the convergence of the first moment follows along the same lines as in Lemma 7. ■

Now we deal with the case when, $\frac{j_n}{n} \longrightarrow 0$ and $j_n \longrightarrow \infty$. Here we will need, for any $x \geq 0$, the asymptotic behavior of

$$S_n^* \equiv \sum_{i=1}^{n} \sum_{m=1}^{2} I_{\left\{Y_i^m \geq a - \left(\frac{j_n}{n}\right)^{\frac{1}{k}} x\right\}},$$

Observe that the summands are identically distributed Bernoulli variables with probability of taking the value 1, say p_n^*, satisfying

$$\left(\frac{n}{j_n}\right) p_n^* = \left(\frac{n}{j_n}\right)\left(1 - G\left(a - \left(\frac{j_n}{n}\right)^{\frac{1}{k}} x\right)\right) \longrightarrow \left[\frac{x}{C(G,k)}\right]^k$$

Also, the first two central moments of S_n^* are

$$E[S_n^*] = np_n^* \text{ and } Var(S_n^*) = 2n(p_n^*(1-p_n^*) + \epsilon_n^*),$$

where ϵ_n^* is defined as,

$$\epsilon_n^* = F\left(a - \left(\frac{j_n}{n}\right)^{\frac{1}{k}} x, a - \left(\frac{j_n}{n}\right)^{\frac{1}{k}} x\right) - \left[G\left(a - \left(\frac{j_n}{n}\right)^{\frac{1}{k}} x\right)\right]^2 \quad \forall n \geq 1.$$

Similar to ϵ_n, ϵ_n^* satisfies

$$0 \leq \lim_{n\to\infty} \inf \left(\frac{n}{j_n}\right) \epsilon_n^* \leq \left[\frac{x}{C(G,k)}\right]^k.$$

The above, in particular imply that

$$E\left[\frac{S_n^*}{j_n}\right] = \left(\frac{2n}{j_n}\right) p_n^* \longrightarrow 2\left[\frac{x}{C(G,k)}\right]^k$$

and

$$\lim_{n\to\infty} \sup Var\left(\frac{S_n^*}{j_n}\right) = \lim_{n\to\infty} \sup \frac{2}{j_n}\left[\left(\frac{n}{j_n}\right) p_n^*(1-p_n^*) + \left(\frac{n}{j_n}\right)\epsilon_n^*\right] = 0,$$

which together with Tchebycheff's inequality imply

$$\frac{S_n^*}{j_n} \xrightarrow{P} 2\left[\frac{x}{C(G,k)}\right]^k.$$

The above lead to the following lemma.

Lemma 10. For any sequence of positive integers $\{j_n\}_{n\geq 1}$ satisfying

$$\frac{j_n}{n} \longrightarrow 0 \text{ and } j_n \longrightarrow \infty,$$

we have

$$\left(\frac{n}{j_n}\right)^{\frac{1}{k}} E\left[a - W_{j_n:2n}\right] \longrightarrow 2^{-\frac{1}{k}} C(G,k).$$

Proof. First, we show that

$$\left(\frac{n}{j_n}\right)^{\frac{1}{k}} (a - W_{j_n:2n}) \xrightarrow{P} 2^{-\frac{1}{k}} C(G,k).$$

Using the equivalence,

$$\left\{ \left(\frac{n}{j_n} \right)^{\frac{1}{k}} (a - W_{j_n:2n}) \leq x \right\} = \{ S_n^* \geq j_n \},$$

and the discussion above, we have

$$\begin{aligned} \lim_{n \to \infty} P \left[n^{\frac{1}{k}} (a - W_{j_n:2n}) \leq x \right] &= 0 \text{ if } x < 2^{-\frac{1}{k}} C(G,k) \\ &= 1 \text{ if } x > 2^{-\frac{1}{k}} C(G,k) \end{aligned}$$

which is a restatement of the convergence in probability above. The proof of the convergence of the first moment follows along the same lines as in Lemma 7. ■

In the following corollaries, instead of the earlier assumptions on the behavior of $G(\cdot)$ at the upper end of its support, we will assume that it is k-times continuously differentiable in a right neighborhood of 0 with $g^{(l)}(0) = 0$ for $l = 0, 1, ..., k-2$ and $g^{(k-1)}(0) \neq 0$. For such a $G(\cdot)$, Let us define

$$C^*(G,k) \equiv \left(\frac{k!(-1)^{k-1}}{g^{(k-1)}(0)} \right).$$

Corollary 5. For a positive integer j, we have

$$n^{\frac{1}{k}} E\left[W_{2n+1-j_n:2n} \right] \longrightarrow 2^{-\frac{1}{k}} C^*(G,k) \frac{\Gamma\left(j + \frac{1}{k}\right)}{\Gamma(j)}.$$

Proof. The proof is similar to Corollary 3, using Lemma 9 instead of Lemma 4. ■

Corollary 6. For any sequence of positive integers $\{j_n\}_{n \geq 1}$ satisfying

$$\frac{j_n}{n} \longrightarrow 0 \text{ and } j_n \longrightarrow \infty,$$

we have

$$\left(\frac{n}{2n - j_n} \right)^{\frac{1}{k}} E\left[V_{2n+1-j_n:n} \right] \longrightarrow 2^{-\frac{1}{k}} C^*(G,k).$$

Proof. By an argument similar to that of Corollary 3, using Lemma 10 instead of Lemma 9. ■

Proof of Proposition 1. In this case, the maximum social surplus[23] can be no larger than $2(V_1)_{1:n}$ whereas the realized social surplus in a multi-unit auction can be no less than $(V_1)_{1:n} I_{\{(V_1)_{1:n} \geq R_n^M\}} + (V_1)_{2:n} I_{\{(V_1)_{2:n} \geq R_n^M\}}$. Therefore, upon taking a transformation and adding terms that cancel out, the maximum expected loss in social surplus is no less than

$$\begin{aligned}
& E[2(V_1)_{1:n}] - E[(V_1)_{1:n} I_{\{(V_1)_{1:n} \geq R_n^M\}} + (V_1)_{2:n} I_{\{(V_1)_{2:n} \geq R_n^M\}}] \\
= \; & E[1 - (V_1)_{1:n} I_{\{(V_1)_{1:n} \geq R_n^M\}}] + E[1 - (V_1)_{2:n} I_{\{(V_1)_{2:n} \geq R_n^M\}}] - 2E[1 - (V_1)_{1:n}] \\
= \; & E[F_1^{-1}(1) - F_1^{-1}(U_{1:n}) I_{\{U_{1:n} \geq F_1(R_n^M)\}}] + E[F_1^{-1}(1) - F_1^{-1}(U_{2:n}) I_{\{U_{2:n} \geq F_1(R_n^M)\}}] \\
& \qquad - 2E[F_1^{-1}(1) - F_1^{-1}(U_{1:n})]
\end{aligned}$$

Since R_n^M satisfies (A2), therefore, $F_1(R_n^M) \leq 1 - \frac{1}{n^\alpha}$ for some $\alpha < 1$. By Lemma 4 we have that each of the expectations in the last expression is $O(n^{-\frac{1}{k}})$ so that the expected loss in social surplus vanishes at least at the rate of $O(n^{-\frac{1}{k}})$, as well, by Slutsky's Theorem.

Similarly, the maximum possible expected loss in social surplus in a bundle auction can be no more than

$$\begin{aligned}
& E[2(V_1)_{1:n}] - E[(V_1 + V_2)_{1:n} I_{\{(V_1+V_2)_{1:n} \geq R_n^B\}}] \\
= \; & E[F_{V_1+V_2}^{-1}(2) - F_{V_1+V_2}^{-1}(U_{1:n}) I_{\{U_{1:n} \geq F_{V_1+V_2}(R_n^B)\}}] - 2E[F_1^{-1}(1) - F_1^{-1}(U_{1:n})].
\end{aligned}$$

R_n^B satisfies (A2) implies that $F_{V_1+V_2}(R_n^B) \leq 1 - \frac{1}{n^\alpha}$ for some $\alpha < 1$. Again, using Lemma 4 we have that $E[F_1^{-1}(1) - F_1^{-1}(U_{1:n})]$ is $O(n^{-\frac{1}{k}})$. Also, using the regularity condition and Lemma 3 we have that $E[F_{V_1+V_2}^{-1}(2) - F_{V_1+V_2}^{-1}(U_{1:n}) I_{\{U_{1:n} \geq F_{V_1+V_2}(R_n^B))\}}]$ is $O(n^{-\frac{1}{m}})$ where $m = \min\{l : f_{X+Y}^{(l)}(1) \neq 0\}$. Assumption (A2) implies that $m \geq k+1$. Therefore, the expected loss in social efficiency is at most $O(n^{-\frac{1}{m}})$ for some $m \geq k+1$.

The expected revenue from the multi-unit auction is no less than $2E[(V_1)_{3:n} I_{\{(V_1)_{3:n} \geq R_n^M\}}]$ whereas that from the bundle auction is is no less than $E[(V_1 + V_2)_{2:n} I_{\{(V_1+V_2)_{2:n} \geq R_n^B\}}]$.

[23]This is exactly the case when the support of the value distribution is given by $\{(v, v) : v \in [0, 1]\}$. In that case, whenever the distribution of the first value $F_1(\cdot)$, which in this case is also the distribution for the second value, satisfies the hazard condition (1) this efficiency-wise worst case scenario happens. If, however, one must stick to the support S then this is only a limiting scenario for distributions.

Applying Lemma 4 it follows that

$$\begin{aligned} & 2-2E[(V_1)_{3:n}I_{\{(V_1)_{3:n}\geq R_n^M\}}] \\ = \ & 2(1-E[(V_1)_{3:n}I_{\{(V_1)_{3:n}\geq R_n^M\}}]) \end{aligned}$$

is $O(n^{-\frac{1}{k}})$ so that the expected revenue under the multi-unit auction converges to 2 at a rate that is at least as fast as $O(n^{-\frac{1}{k}})$. On the other hand, $E[2-(V_1+V_2)_{2:n}I_{\{(V_1+V_2)_{2:n}\geq R_b\}}]$ is $O(n^{-\frac{1}{m}})$ which means that the expected revenue from the bundle auction converges to 2 at a minimum rate of $O(n^{-\frac{1}{m}})$ where $m>k$. ■

Proof of Proposition 2. The expected social surplus in the multi-unit auction is no less than $E[(V_1)_{1:n}I_{\{(V_1)_{1:n}\geq R_n^M\}}+(V_1)_{2:n}I_{\{(V_1)_{2:n}\geq R_n^M\}}]$. In a bundle auction an upper bound on the expected social surplus is given by $E[(V_1+V_2)_{1:n}]$. Then the expected difference in surplus is no less than the difference of these two expressions. Therefore,

$$\begin{aligned} & E[(V_1)_{1:n}I_{\{(V_1)_{1:n}\geq R_n^M\}}+(V_1)_{2:n}I_{\{(V_1)_{2:n}\geq R_n^M\}}]-E[(V_1+V_2)_{1:n}] \\ = \ & E[2-(V_1+V_2)_{1:n}]-E[1-(V_1)_{1:n}I_{\{(V_1)_{1:n}\geq R_n^M\}}]+E[1-(V_1)_{2:n}I_{\{(V_1)_{2:n}\geq R_n^M\}}] \end{aligned}$$

Since R_n^M satisfies (A2), $F_1(R_n^M)\leq 1-\frac{1}{n^\alpha}$ for some $\alpha<1$. By (a1) and Lemma 4 the second and the third expectation terms are $O\left(n^{-\frac{1}{k}}\right)$ whereas the first expectation term is no faster than $O\left(n^{-\frac{1}{k+1}}\right)$. Hence, the expression $n^{\frac{1}{k}}(E[(V_1)_{1:n1:n}I_{\{(V_1)_{1:n}\geq R_n^M\}}+(V_1)_{2:n}I_{\{(V_1)_{2:n}\geq R_n^M\}}]-E[(V_1+V_2)_{1:n}])\longrightarrow\infty$ by Slutsky's Theorem.

The expected revenue from the multi-unit auction minus the expected revenue from the bundle auction is larger than

$$\begin{aligned} & 2E[(V_1)_{3:n}I_{\{(V_1)_{3:n}\geq R_n^M\}}]-E[(V_1+V_2)_{1:n}] \\ = \ & E[2-(V_1+V_2)_{1:n}]-2E[1-(V_1)_{3:n}I_{\{(V_1)_{3:n}\geq R_n^M\}}] \end{aligned}$$

Using similar steps as above we have that

$$n^{\frac{1}{k}}\left(2E[(V_1)_{3:n}I_{\{(V_1)_{3:n}\geq R_n^M\}}]-E[(V_1+V_2)_{1:n}]\right)$$

goes to ∞ as $n\longrightarrow\infty$. The result then follows. ■

Proof of Proposition 3. The proof runs exactly along the lines of Proposition 1 and Proposition 2, except that Lemma 5 is applied instead of Lemma 4. ■

The following lemma demonstrates that the average of the marginal distributions of two random variables is greater in the sense of Lorenz ordering (denoted $\geq_{\text{Lorenz}}$) than the distribution of their average.

Lemma 11. Let $(X, Y) \sim F$, where $F(\cdot, \cdot)$ is a distribution function on $S \subseteq R^2$. Let $F_{\frac{X+Y}{2}}$ be the distribution function of the average of X and Y, F_X the marginal distribution of X, and F_Y the marginal distribution of Y. Defining F_M as the distribution function of the symmetric mixture of X and Y, *i.e.*,

$$F_M \equiv \frac{F_X + F_Y}{2},$$

we have

$$\int_t^1 F_M^{-1}(u)du \geq \int_t^1 F_{\frac{X+Y}{2}}^{-1}(u)du, \ \forall t \in [0, 1] \tag{3}$$

i.e., $F_M \geq_{\text{Lorenz}} F_{\frac{X+Y}{2}}$.

Proof. Let I be a Bernoulli variable with mean of $\frac{1}{2}$ and independent of (X, Y). Then

$$Z \equiv IX + (1 - I)Y \stackrel{d}{=} \frac{F_X + F_Y}{2} \quad \text{and} \quad E[Z|X + Y] = \frac{X + Y}{2}.$$

Hence by Theorem 3.4 of Arnold (1987) we have $Z \geq_{\text{Lorenz}} \frac{X+Y}{2}$, which, in particular, implies

$$\int_0^t F_M^{-1}(u)du \leq \int_0^t F_{\frac{X+Y}{2}}^{-1}(u)du, \ \forall t \in [0, 1]. \tag{4}$$

Combining the above with

$$\int_0^1 F_M^{-1}(u)du = E[Z] = E\left[\frac{X + Y}{2}\right] = \int_0^1 F_{\frac{X+Y}{2}}^{-1}(u)du$$

completes the proof. ■

Proof of Proposition 5. (i) Observe that in the limit bidders bid truthfully whenever the marginal value for a unit is larger than the price. Hence in the limit the social surplus generated per bidder is given by

$$\int_{(\frac{1}{2}F_{V_1}+\frac{1}{2}F_{V_2})^{-1}(1-\alpha)}^{1} x(f_1(x)+f_2(x))dx = 2\int_{1-\alpha}^{1} (\frac{1}{2}F_{V_1}+\frac{1}{2}F_{V_2})^{-1}(t)dt$$

in the multi-unit auction, and by

$$\int_{F^{-1}_{V_1+V_2}(1-\alpha)}^{2} xf_{V_1+V_2}(x)dx = \int_{1-\alpha}^{1} F^{-1}_{\frac{V_1+V_2}{2}}(t)dt$$

in the bundle auction. Upon applying Lemma 11 and the fact that E is finite (countable) we have

$$\int_{1-\alpha}^{1} \left(2\left(\frac{1}{2}F_{V_1}+\frac{1}{2}F_{V_2}\right)^{-1}(t) - F^{-1}_{V_1+V_2}(t)\right) dt > 0$$

for all but finitely (countably) many $\alpha \in (0,1)$. This proves that in the limit the per bidder surplus generated by the multi-unit auction is strictly larger than that from the bundle auction for all but finitely (countably) many $\alpha \in (0,1)$. Hence the same holds for all sufficiently large (but finite) number of bidders.

(ii) Combining (A2) with (3), and (A3) with (4) (so that $\frac{1}{2}F_{V_1}+\frac{1}{2}F_{V_2} \neq F_{\frac{V_1+V_2}{2}}$ in some right neighborhood of 0 and in some left neighborhood of 1) we have that there exist x_* and x^* ($x_* \leq x^*$) such that

$$\frac{1}{2}F_{V_1}(x)+\frac{1}{2}F_{V_2}(x) > F_{\frac{V_1+V_2}{2}}(x)\ \forall x \in (0,x_*)$$

$$\frac{1}{2}F_{V_1}(x)+\frac{1}{2}F_{V_2}(x) < F_{\frac{V_1+V_2}{2}}(x) \text{ for all } x \in (x^*,1).$$

These conditions along with the continuity and monotonicity of the distribution functions guarantee the existence of t_* and t^* such that

$$\begin{aligned}\left(\frac{1}{2}F_{V_1}+\frac{1}{2}F_{V_2}\right)^{-1}(t) &< F^{-1}_{\frac{V_1+V_2}{2}}(t),\ \forall t \in (0,t_*)\\ \left(\frac{1}{2}F_{V_1}+\frac{1}{2}F_{V_2}\right)^{-1}(t) &> F^{-1}_{\frac{V_1+V_2}{2}}(t),\ \forall t \in (t^*,1)\end{aligned}$$

and the result follows. ■

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