A NOTE ON NON-NEGATIVE ARMA PROCESSES

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(NON-NEGATIVE ARMA PROCESSES)

Abstract. Recently, there are much works on developing models suitable for analyzing the volatility of a discrete-time process. Within the framework of Auto-Regressive Moving-Average (ARMA) processes, we derive a necessary and sufficient condition for the kernel to be non-negative. This condition is in terms of the generating function of the ARMA kernel which has a simple form. We discuss some useful consequences of this result and delineate the parametric region of stationarity and non-negative kernel for some lower-order ARMA models.

Key words and phrases: Absolutely monotonicity, ARMA, generating function, stochastic volatility.

1. Introduction

Over the past two decades, there are many econometric models done in the literature for modeling the volatility of an asset return, See Tsay (2002). Some representative models include the ARCH (autoregressive conditional heteroscedastic) model (Engle, 1982), the GARCH (generalized ARCH) model (Bollerslev, 1986), the EGARCH (exponential GARCH) model (Nelson, 1991), the CHARMA (conditional heteroscedastic autoregressive moving-average) model (Tsay, 1987), and the RCA (random coefficient autoregressive) model (Nicholls and Quinn, 1982).

These models adopt the observation driven approach in modeling the conditional variance of the log returns of an asset as some function of past returns. In contrast, the stochastic volatility model uses a parameter driven approach that models the conditional variance in terms of its lags, see Melino and Turnbull (1990), Harvey *et al.* (1994), and Jacquier *et al.* (1994). See Cox (1981) for further discussion of the parameter driven versus observation driven approach.

Let $\{r_t\}$ be a time series of the log returns of an asset, the stochastic volatility model states that

$$r_t = \mu_t + \sigma_t \epsilon_t, \tag{1.1}$$

$$(1 - \alpha_1 B - \cdots - \alpha_m B^m) \log(\sigma_t^2) = \alpha_0 + \nu_t, \tag{1.2}$$

where B is the backward shift operator defined by $B^j X_t = X_{t-j}$, ϵ_ts are iid $N(0,1)$, ν_ts are iid $N(0, \sigma_\nu^2)$, $\{\epsilon_t\}$ and $\{\nu_t\}$ are independent, αs are parameters, and all zeros of the polynomial $1 - \sum_{i=1}^m \alpha_i B^i$ are greater than 1 in modulus. That $\log(\sigma_t^2)$ rather than σ_t^2 is modeled in equation (1.2) is mainly for the reason of circumventing the positivity constraint on the conditional variances. An alternative approach consists of modeling the conditional variances directly by some constrained ARMA model whose time series realizations are non-negative almost surely. Even if we use non-negative innovations νs, this alternative approach raises the issue of studying when an ARMA model always admits non-negative realizations.

Besides volatility in financial data, non-negative time series occur in many other fields, e.g. in longitudinal studies, actuarial data and clinical trials. In the continuous-time framework, Barndorff-Nielsen and Shephard (2001) and Brockwell and Marquardt (2003) considered a class of continuous-time stochastic volatility models for financial assets where the volatility processes are defined as solutions to the Ornstein-Uhlenbeck (OU) processes driven by non-decreasing Lévy processes, whose realizations are always non-negative if the kernel is non-negative. Tsai and Chan (2004) derived a necessary and sufficient condition for the kernel to be non-negative, as well as some related results.

A discrete-time causal ARMA model admits a moving-average representation in terms of the convolution of its kernel and the innovation sequence. Its realizations are always non-negative if the ARMA kernel is non-negative and the innovations are non-negative random variables. Here, we study the problem of characterizing the non-negativity of the kernel of a causal ARMA model. We derive a necessary and sufficient condition for the kernel of a discrete-time ARMA model to be non-negative. We then derive some readily verifiable necessary (and sufficient) conditions for some lower-order ARMA models to admit a non-negative kernel. These results are useful for the statistical analysis of discrete-time ARMA processes with non-negative time series data.

The main results are stated in Section 2. We characterize the parametric region of causality, invertibility and non-negative kernel for some lower-order ARMA models in Section 3. These characterizations are pertinent for general volatility modeling with a possibly non-monotone autocorrelation function for the volatility process. We conclude in Section 4. All proofs are collected in an appendix.

2. Main results

Let $\{X_t\}$ be an ARMA(p, q) process, i.e.,

$$\phi(B)X_t = \theta(B)Z_t, \qquad t = 0, \pm1, \pm2, ...,$$

where $\{Z_t\}$ is a white noise with mean 0 and variance σ^2, $\phi(z) = 1-\phi_1 z-\cdots-\phi_p z^p$, $\theta(z) = 1+\theta_1 z+\cdots+\theta_q z^q$. We assume that $\phi(\cdot)$ and $\theta(\cdot)$ have no common zeros and all roots of $\phi(z) = 0$ and $\theta(z) = 0$ are outside the unit circle, so $\{X_t\}$ is a causal and invertible process. By Section 3.1 of Brockwell and Davis(1991), causality implies that there exists a sequence of constants $\{\psi_j\}$ such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \qquad t = 0, \pm1, \pm2, ... \tag{2.1}$$

The sequence of ψ-weights can be extended to a doubly-infinite sequence by defining $\psi_j = 0$ for negative j. Then (2.1) can be written as $X_t = (\psi * Z)(t)$ where $*$ denotes the convolution between the kernel sequence ψ and the innovation process Z, both of which are doubly-infinite sequences whose jth elements are respectively equal to ψ_j and Z_j.

The auto-covariance function of the causal ARMA(p, q) process is given by

$$\gamma(k) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|},$$

see, e.g., Theorem 3.2.1 of Brockwell and Davis (1991). If the ψ_js are non-negative, and the driving noise process $\{Z_t\}$ is non-negative, the process X will be non-negative as is necessary if it is to represent volatility. In this paper, we are interested in studying conditions under which all ψ_js are non-negative.

We shall characterize the non-negativity of the ψ_js for any ARMA(p,q) process in terms of its generating function. For this purpose, we first recall the definition of the generating function (See Chapter XI of Feller, 1968). Let $\{p_i\}_{i=0}^{\infty}$ be a sequence of real numbers. If

$$u(x) = p_0 + p_1 x + p_2 x^2 + \cdots$$

converges in some interval $-x_0 < x < x_0$, then $u(x)$ is called the generating function of the sequence $\{p_j\}$. For the $\{\psi_j\}$ defined in (2.1), its generating function equals (Theorem 3.1.1 of Brockwell and Davis, 1991),

$$\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\theta(z)}{\phi(z)}, \qquad |z| \le 1.$$

The significance of the generating function of the ψ_js lies in the well-known result that the non-negativity of the ψ_js is equivalent to the absolutely monotonicity of its generating function. First, we recall the definition of absolutely monotonicity; see Chapter VII of Feller (1971) and Chapter IV of Widder (1946) for further discussion. A continuous function $f(x)$ is absolutely monotone in the interval $0 \le x < 1$ if it has non-negative derivatives of all orders there:

$$f^{(n)}(x) \ge 0, \qquad 0 < x < 1, \qquad n = 0, 1, 2, \ldots.$$

Let $\lambda_j, j = 1, \cdots, p$, be the roots of $\phi(z) = 0$, where $1 < |\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_p|$. Let R be the set of real numbers, $i = \sqrt{-1}$ and $\bar{\lambda}$ be the complex conjugate of λ. Now we can state the main results.

THEOREM 2.1. *(a) The kernel of an ARMA(p,q) process is non-negative, i.e. the ψ_js are non-negative, if and only if its generating function, $\psi(z) = \theta(z)/\phi(z), 0 \le z < 1$, is absolutely monotone.*

(b) For an AR(p) process, if all λ_js are real and > 1, then the ψ_js are non-negative.

(c) For an ARMA(p,q) process, if $\{\psi_j\}$ are non-negative, then λ_1 is real and $\lambda_1 > 1$.

(d) For an AR(p) process, if $\{\psi_j\}$ are non-negative, then $\sum_{j=1}^{p} \lambda_j^{-1} \ge 0$, λ_1 is real, and $\lambda_1 > 1$.

(e) For an AR(1) process, $\{\psi_j\}$ are non-negative if and only if λ_1 is real and $\lambda_1 > 1$.

(f) For an AR(2) process, $\{\psi_j\}$ are non-negative if and only if $\lambda_1^{-1} + \lambda_2^{-1} \ge 0$, λ_1 is real, and $\lambda_1 > 1$.

(g) For an AR(3) process, there are two cases:

case (g.1) $\lambda_j \in R, j = 1, 2, 3$: $\{\psi_j\}$ *are non-negative if and only if* $\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} \geq 0$ *and* $\lambda_1 > 1$.

case (g.2) $\lambda_2 = \bar{\lambda}_3 = |\lambda_2|e^{i\theta} = a + bi$, *where* $a, b \in R$, $b > 0$, *and* $0 < \theta < \pi$:

(g.2.1) if $\theta = 2\pi/r$ *for some integer* $r \geq 3$, *then* $\{\psi_j\}$ *are non-negative if and only if* $1 < \lambda_1 \leq |\lambda_2|$.

(g.2.2) if $\theta \notin \{2\pi/r | \, r = 3, 4, ...\}$, *then* $\{\psi_j\}$ *are non-negative if and only if* $|\lambda_2|/\lambda_1 \geq x_0 > 1$, *where* x_0 *is the root of* $f_{n,\theta}(x) = 0$, *where*

$$f_{n,\theta}(x) = x^{n+2} - x\,\frac{\sin((n+2)\theta)}{\sin\theta} + \frac{\sin((n+1)\theta)}{\sin\theta}, \tag{2.2}$$

and n *is the smallest positive integer such that* $\sin((n+1)\theta) < 0$ *and* $\sin((n+2)\theta) > 0$.

(g.2.3) if $a \geq \lambda_1 > 1$, *then* $\{\psi_j\}$ *are non-negative.*

(h) For an AR(4) process, if $\lambda_j \in R, j = 1, 2, 3$, *and* 4, *then* $\{\psi_j\}$ *are non-negative if and only if* $\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} + \lambda_4^{-1} \geq 0$ *and* $\lambda_1 > 1$.
(j) For an AR(p) process, if for each pair of complex roots of $\phi(z) = 0$, *there exists a unique real root such that case (g.2) holds, then* $\{\psi_j\}$ *are non-negative.*
(k) For an MA(q) process, $\{\psi_j\}$ *are non-negative if and only if* $\theta_j \geq 0, 1 \leq j \leq q$.
(l) For an ARMA(1,q) process, $\{\psi_j\}$ *are non-negative if and only if* $\phi_1 \geq 0$ *and* $\psi_j = \phi_1\psi_{j-1} + \theta_j \geq 0, 1 \leq j \leq q$, *where* $\psi_0 = 1$.

The proof of Theorem 2.1 is given in the Appendix. We note that the obtained results for the case of real eigenvalues suggest the following conjecture that if the λ_js are real, a necessary and sufficient condition for the kernel to be non-negative is that the sum of the reciprocal of the λ_js is non-negative.

3. Parametric region of causality, invertibility, and non-negativity of the ψ_js

In this section, we discuss the parametric region of causality, invertibility, and non-negativity of the ψ_js.

3.1 *AR(1) processes*

By Theorem 2.1 (e), a necessary and sufficient condition for the non-negativity of $\{\psi_j\}$ is that $\lambda_1 > 1$. But $\phi_1 = 1/\lambda_1$, so the condition is equivalent to $0 < \phi_1 < 1$. If $\phi_1 = 0$, then the AR(1) process degenerates to a white noise. Therefore, the condition becomes $0 \leq \phi_1 < 1$.

3.2 *AR(2) processes*

The parametric region for an AR(2) process to be causal can be defined in terms of the following three inequalities (see, e.g., equation (3.9) of Chan, 2001):

$$\phi_1 + \phi_2 < 1, \qquad \phi_2 - \phi_1 < 1, \qquad \phi_2 > -1. \tag{3.1}$$

We claim that the non-negative parametric region is:

$$\phi_1 + \phi_2 < 1, \qquad \phi_1^2 + 4\phi_2 \geq 0, \qquad \phi_1 \geq 0. \tag{3.2}$$

We now prove condition (3.2). By Theorem 2.1 (f), a necessary and sufficient condition for the non-negativity of $\{\psi_j\}$ is that $\lambda_1 > 1$ and $\lambda_1^{-1} + \lambda_2^{-1} \geq 0$. But $1 < \lambda_1 \leq |\lambda_2|$ implies $\lambda_1^{-1} \geq |\lambda_2|^{-1} \geq -\lambda_2^{-1}$, which means the condition $\lambda_1 > 1$ is redundant. The condition $\lambda_1^{-1} + \lambda_2^{-1} \geq 0$ is same as $\phi_1 \geq 0$. The condition that λ_1 and λ_2 are real is equivalent to $\phi_1^2 + 4\phi_2 \geq 0$. Therefore, condition (3.2) is the non-negative parametric region. Note also that condition (3.2) is similar to Figure 3.1 of Chan (2001).

3.3 *Other cases*

The parametric region of non-negative kernel for an AR(3) process is completely characterized by part (g) of Theorem 2.1. Unfortunately, the characterization is implicit, and best determined in practice by numerical methods. The parametric region of non-negative kernel for a finite-order MA process is quite simple as it simply requires all the MA coefficients to be non-negative, see part (k) of Theorem 2.1. That of an ARMA(1,q) model also follows easily from part (l) of Theorem 2.1.

4. Summary

While the parametric region of non-negative kernel is simple for a finite-order MA process, the situation for a finite-order AR process is much more complex. We have derived some readily verifiable necessary and sufficient conditions for an AR(p) process to admit a non-negative kernel where $p \leq 3$. For higher order AR processes, we give some sufficient condition for a non-negative kernel. An interesting future research problem is to derive some readily verifiable necessary and sufficient condition for the kernel of a finite-order ARMA process to be non-negative.

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Appendix 1

Proof of Theorem 2.1

(a) This follows from Theorem 2 of Feller (1971, p. 223).
(b) We first prove the case for $p = 1$. In this simplest case, $\psi(z) = (1 - z/\lambda_1)^{-1}$, where $0 \leq z < 1$. Therefore, for $n \geq 0$, $0 < z < 1$, and $\lambda_1 > 1$,

$$\psi^{(n)}(z) = \frac{n!}{\lambda_1^n}\left(1 - \frac{z}{\lambda_1}\right)^{-(n+1)} \geq 0. \tag{A1}$$

This proves that $\psi(z)$ is absolutely monotone in the interval $0 \leq z < 1$. Therefore, the ψ_js are non-negative because of (a). The proof for a general order p follows readily from (i) the factorization $\psi(s) = \prod_i (1-s/\lambda_i)^{-1}$, (ii) any function $(1-s/\lambda)^{-1}$ is absolutely monotone in the interval $0 \leq s < 1$ for $\lambda > 1$, and (iii) the product of two absolutely monotone functions is still absolutely monotone (Theorem 2a of Widder, 1946, p. 145).
(c) We prove the necessary condition, first for the simple case that $p > q$ and all roots of $\psi(z) = 0$ are distinct. By equation (4.8) of Feller (1968, p. 276),

$$\psi_n = \sum_{i=1}^{p} \frac{r_i}{\lambda_i^{n+1}}, \tag{A2}$$

where $r_i = -\theta(\lambda_i)/\phi^{(1)}(\lambda_i)$. If λ_1 is real, then by the theorem on page 277 of Feller (1968), ψ_n is given asymptotically, as $n \to \infty$, by $\psi_n \sim r_1 \lambda_1^{-(n+1)}$, the sign $\sim$ indicating that the ratio of the two sides tends to 1. Therefore, λ_1 must > 1 in order for the ψ_js to be non-negative. If λ_1 is a complex number, then the first two terms in the sum are dominating and for sufficiently large n, the sign of the sum is the same as that of the sum of the first two terms. Denote the sum of the first two terms by h_n which equals $2\mathrm{Re}(r_1 \bar{\lambda}_1^{n+1})/|\lambda_1^{n+1}|^2$, where $\mathrm{Re}(\cdot)$ denotes the real part of the complex number in the parenthesis and recall that $\bar{z}$ denotes the complex conjugate of the complex number z. Then by an argument similar to that of the proof of Theorem 2(d) of Tsai and Chan (2004), we can show that there exists infinitely many n for which the numerator of h_n is negative (see supplementary page i for a proof) and hence ψ_ns is negative. Therefore, the non-negativity of the ψ_ns implies that λ_1 must be real and > 1. The proof for the case that $\psi(z) = 0$ has multiple roots and the case that $p \leq q$ can be proved similarly, (see also pages 277 and 285 of Feller, 1968) and hence omitted.

(d) Note that $\phi(s) = 1 - \sum_{i=1}^{p} \phi_i s^i = \prod_i (1 - s/\lambda_i)$. Comparing the coefficients of both sides, we get $\phi_1 = \sum_{j=1}^{p} \lambda_j^{-1}$. But by equation (3.3.5) of Brockwell and Davis (1991), $\psi_1 = \phi_1$. Therefore, $\sum_{j=1}^{p} \lambda_j^{-1}$ must be ≥ 0. The condition on λ_1 follows from part (c).

(e) This follows readily from parts (b) and (d).

(f) We claim that $\lambda_1^{-1} + \lambda_2^{-1} \geq 0$ and $\lambda_1 > 1$ are sufficient conditions for the non-negativity of $\{\phi_j\}$. We first prove the sufficiency. If $\lambda_2 > 1$, the sufficiency follows from (i) the factorization $\psi(s) = (1-s/\lambda_1)^{-1}(1-s/\lambda_2)^{-1}$, (ii) any function $(1 - s/\lambda)^{-1}$ is completely monotone for $0 \leq s < 1$ and $\lambda > 1$, and (iii) the aforementioned result of Widder (1946, p. 145). Therefore, we only consider the case that $\lambda_2 < -1$. Now note that the condition $\lambda_1^{-1} + \lambda_2^{-1} \geq 0$ is the same as $1 < \lambda_1 \leq -\lambda_2$. By equation (A2) and the fact that $\phi_2 = (-\lambda_1\lambda_2)^{-1}$, we have

$$\begin{aligned}\psi_n &= \frac{1}{\phi_2(\lambda_1 - \lambda_2)} \left(\frac{1}{\lambda_1^{n+1}} - \frac{1}{\lambda_2^{n+1}} \right) \\ &\geq \frac{1}{\phi_2(\lambda_1 - \lambda_2)} \left(\frac{1}{\lambda_1^{n+1}} - \frac{1}{(-\lambda_2)^{n+1}} \right) \\ &\geq 0.\end{aligned}$$

This proves the sufficiency. The necessity of the conditions follow from part (d).

(g) First consider case (g.1). The necessity follows from part (d). Now, we prove the sufficiency. If λ_2 or λ_3 is > 1, then again the sufficiency follows from (i) the factorization $\psi(s) = \prod_i (1 - s/\lambda_i)^{-1}$, (ii) the results for $p = 1$ and 2 and the fact that if $\lambda_2 < -1$, the condition $\lambda_1^{-1} + \lambda_2^{-1} \geq 0$ is the same as $1 < \lambda_1 \leq -\lambda_2$, and (iii) the aforementioned result of Widder (1946, p. 145). Therefore, we only consider the case that $1 < \lambda_1 \leq -\lambda_2 \leq -\lambda_3$. First consider the simple case that $\lambda_2 \neq \lambda_3$. Now, $\phi_3 = (\lambda_1\lambda_2\lambda_3)^{-1} > 0$, and by equation (A2),

$$\psi_n = \frac{r_1}{\lambda_1^{n+1}} + \frac{r_2}{\lambda_2^{n+1}} + \frac{r_3}{\lambda_3^{n+1}}, \tag{A3}$$

where $r_1 = \{\phi_3(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\}^{-1} > 0$, $r_2 = \{\phi_3(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)\}^{-1} < 0$, $r_3 = \{\phi_3(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)\}^{-1} > 0$, and $|r_2| \geq |r_3|$. If $n = 2k$, then

$$\psi_n = \frac{|r_1|}{\lambda_1^{2k+1}} + \frac{|r_2|}{|\lambda_2|^{2k+1}} - \frac{|r_3|}{|\lambda_3|^{2k+1}} \geq 0.$$

If $n = 2k+1$, then by using $r_1 + r_2 + r_3 = 0$, we have

$$
\begin{aligned}
\psi_{2k+1} &= r_1 \lambda_1^{-(2k+2)} + r_2 \lambda_2^{-(2k+2)} + r_3 \lambda_3^{-(2k+2)} \\
&= r_2(\lambda_2^{-(2k+2)} - \lambda_1^{-(2k+2)}) + r_3(\lambda_3^{-(2k+2)} - \lambda_1^{-(2k+2)}) \\
&= \frac{\lambda_1^{2k+2} - \lambda_2^{2k+2}}{\phi_3(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_1\lambda_2)^{2k+2}} + \frac{\lambda_1^{2k+2} - \lambda_3^{2k+2}}{\phi_3(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_1\lambda_3)^{2k+2}}.
\end{aligned}
$$

Multiplying both sides of the above equation by $\phi_3(\lambda_1\lambda_2\lambda_3)^{2k+2}\prod_{1\le i<j\le 3}(\lambda_i - \lambda_j)$, we get

$$
\begin{aligned}
&\psi_{2k+1}\phi_3(\lambda_1\lambda_2\lambda_3)^{2k+2}(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) \\
&= -\lambda_3^{2k+2}(\lambda_1 - \lambda_3)(\lambda_1^{2k+2} - \lambda_2^{2k+2}) + \lambda_2^{2k+2}(\lambda_1 - \lambda_2)(\lambda_1^{2k+2} - \lambda_3^{2k+2}) \\
&= (\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)\left\{-(\lambda_1 + \lambda_2)\lambda_3^{2k+2}\sum_{i=0}^{k}\lambda_2^{2i}\lambda_1^{2k-2i} + (\lambda_1 + \lambda_3)\lambda_2^{2k+2}\sum_{i=0}^{k}\lambda_3^{2i}\lambda_1^{2k-2i}\right\} \\
&= (\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)\left\{-\sum_{i=0}^{k}\lambda_1^{2k-2i+1}\lambda_2^{2i}\lambda_3^{2i}(\lambda_3^{2k-2i+2} - \lambda_2^{2k-2i+2})\right. \\
&\qquad\qquad\qquad\qquad\left. -\sum_{i=0}^{k}\lambda_1^{2k-2i}\lambda_2^{2i+1}\lambda_3^{2i+1}(\lambda_3^{2k-2i+1} - \lambda_2^{2k-2i+1})\right\}.
\end{aligned}
$$

Note that: (i) $\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} \ge 0$ is equivalent to $|\lambda_2\lambda_3| \ge \lambda_1(|\lambda_2| + |\lambda_3|)$; and (ii) $\lambda_2 - \lambda_3 > 0$. Dropping $(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)$ from both sides of the above equation, we get

$$
\begin{aligned}
&\psi_{2k+1}\phi_3(\lambda_1\lambda_2\lambda_3)^{2k+2}(\lambda_2 - \lambda_3) \\
&= -\sum_{i=0}^{k}\lambda_1^{2k-2i+1}|\lambda_2\lambda_3|^{2i}(|\lambda_3|^{2k-2i+2} - |\lambda_2|^{2k-2i+2}) \\
&\quad + \sum_{i=0}^{k}\lambda_1^{2k-2i}|\lambda_2\lambda_3|^{2i+1}(|\lambda_3|^{2k-2i+1} - |\lambda_2|^{2k-2i+1}) \\
&\ge -\sum_{i=0}^{k}\lambda_1^{2k-2i+1}|\lambda_2\lambda_3|^{2i}(|\lambda_3|^{2k-2i+2} - |\lambda_2|^{2k-2i+2}) \\
&\quad + \sum_{i=0}^{k}\lambda_1^{2k-2i+1}|\lambda_2\lambda_3|^{2i}(|\lambda_2| + |\lambda_3|)(|\lambda_3|^{2k-2i+1} - |\lambda_2|^{2k-2i+1}) \\
&= \sum_{i=0}^{k}\lambda_1^{2k-2i+1}|\lambda_2\lambda_3|^{2i}(|\lambda_2||\lambda_3|^{2k-2i+1} - |\lambda_3||\lambda_2|^{2k-2i+1}) \\
&\ge 0.
\end{aligned}
$$

This proves that if $1 < \lambda_1 \le -\lambda_2 < -\lambda_3$, then $\psi_n \ge 0$ for all non-negative integer n. The case when $1 < \lambda_1 \le -\lambda_2 = -\lambda_3$ can be proved similarly, and hence omitted. This completes the proof for case (g.1).

Now consider case (g.2). By equation (A3), we have

$$\begin{aligned}
&\phi_3|\lambda_1-\lambda_2|^2\lambda_1^{n+1}|\lambda_2|^{2n+2}\psi_n\\
&=|\lambda_2|^{2n+2}+\frac{\lambda_1^{n+2}(\lambda_3^{n+1}-\lambda_2^{n+1})}{\lambda_3-\lambda_2}-\frac{\lambda_1^{n+1}(\lambda_3^{n+2}-\lambda_2^{n+2})}{\lambda_3-\lambda_2}\\
&=|\lambda_2|^{2n+2}+\frac{\lambda_1^{n+2}|\lambda_2|^{n+1}\sin((n+1)\theta)}{|\lambda_2|\sin\theta}-\frac{\lambda_1^{n+1}|\lambda_2|^{n+2}\sin((n+2)\theta)}{|\lambda_2|\sin\theta}\\
&=\lambda_1^{n+2}\;|\lambda_2|^n\;\left\{\left|\frac{\lambda_2}{\lambda_1}\right|^{n+2}-\left|\frac{\lambda_2}{\lambda_1}\right|\frac{\sin((n+2)\theta)}{\sin\theta}+\frac{\sin((n+1)\theta)}{\sin\theta}\right\}. \qquad \text{(A4)}
\end{aligned}$$

Note that $\phi_3=(\lambda_1\lambda_2\lambda_3)^{-1}=\lambda_1^{-1}|\lambda_2|^{-2}>0$ and by part (c), it is necessary that $|\lambda_2|\ge\lambda_1>1$. Therefore, equations (2.2) and (A4) imply that $\psi_n\ge 0$ if and only if there exists some $x_n\ge 1$ such that $f_{n,\theta}(x)\ge 0$ for $x\ge x_n$ and $|\lambda_2|/\lambda_1\ge x_n>1$. For simplicity, we will write $f_n(x)$ for $f_{n,\theta}(x)$ below.

We claim that, for any integer $n\ge 0$,

$$f_n(x) \text{ is an increasing function for } x\ge 1. \qquad \text{(A5)}$$

We will prove claim (A5) later. Now, for each integer $n\ge 0$, there are two possible cases for $f_n(1)$.

Case (B1): if $f_n(1)\ge 0$, then claim (A5) implies that $f_n(x)\ge 0$ for $x\ge 1$ and therefore $\psi_n\ge 0$ if and only if $|\lambda_2|\ge\lambda_1>1$.

Case (B2): if $f_n(1)<0$, then claim (A5) implies that there exists exactly one root, say x_n, of $f_n(x)=0$ on $(1,\infty)$, and so $f_n(x)\ge 0$ if and only if $x\ge x_n$. In other words, $\psi_n\ge 0$ if and only if $|\lambda_2|/\lambda_1\ge x_n>1$.

Note that $\sin\theta>0$ for $0<\theta<\pi$, therefore, $f_n(1)\ge 0$ if and only if $\sin\theta-\sin((n+2)\theta)+\sin((n+1)\theta)\ge 0$. We also note that, for $\sin\alpha\ge 0$ and $\sin\beta\ge 0$,

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\sin\beta\cos\alpha\le\sin\alpha+\sin\beta. \qquad \text{(A6)}$$

The equality holds if and only if $\alpha\equiv 0 \pmod{2\pi}$ or $\beta\equiv 0 \pmod{2\pi}$.

Now, for each fixed θ and every integer n, there are three possibilities.

Case (θ1): if $\sin((n+1)\theta)\ge 0$, then by inequality (A6), we get $f_n(1)\ge 0$, and therefore, this is case (B1), i.e., $\{\psi_n\}$ are non-negative if and only if $|\lambda_2|\ge\lambda_1>1$.

Case (θ2): if $\sin((n+1)\theta)<0$ and $\sin((n+2)\theta)\le 0$, then again by inequality (A6), $\sin\theta-\sin((n+2)\theta)=\sin\theta+\sin(-(n+2)\theta)\ge\sin(-(n+1)\theta)=-\sin((n+1)\theta)$, which implies $f_n(1) \geq 0$, and again, this is case (B1), i.e., $\{\psi_n\}$ are non-negative if and only if $|\lambda_2| \geq \lambda_1 > 1$.

Case ($\theta 3$): if $\sin((n+1)\theta) < 0$ and $\sin((n+2)\theta) > 0$, then this is case (B2), i.e., $f_n(1) < 0$, and therefore, we need to first find the root, say x_n, of $f_n(x) = 0$ on $(1, \infty)$, and then, $\{\psi_n\}$ are non-negative if and only if $|\lambda_2|/\lambda_1 \geq x_n > 1$.

Now, (g.2.1) follows from the above discussion because in this case it is impossible that $\sin((n+1)\theta) < 0$ and $\sin((n+2)\theta) > 0$ for some integer n, and therefore, it can only be Case ($\theta 1$) or Case ($\theta 2$). These imply $\{\psi_n\}$ are non-negative if and only if $|\lambda_2| \geq \lambda_1 > 1$.

Now, we prove (g.2.2). Again, it suffices to consider θ and n which are of Case ($\theta 3$), i.e., the case when $\sin((n+1)\theta) < 0$ and $\sin((n+2)\theta) > 0$. Let n and x_0 be as defined in the statement of (g.2.2). We need the following claim to be valid for all non-negative integer r.

Claim (A): if $\sin((r+1)\theta) < 0$ and $\sin((r+2)\theta) > 0$, then $f_r(x) \geq 0$ for $x \geq x_0 > 1$.

It can be easily seen that (g.2.2) follows from the above claim. Now, we prove Claim (A) by mathematical induction. Claim (A) clearly holds for all non-negative $m \leq n+1$, by the definition of n and case ($\theta 1$). Assume then that Claim (A) holds for $0 \leq r < m$ where $m \geq n+2$. Therefore, $f_r(x) \geq 0$ for $x \geq x_0$ and $0 \leq r < m$. We now show that Claim (A) also holds for the integer m, and therefore, Claim (A) holds for all non-negative integer r, by mathematical induction. The induction assumption implies there are δ and ϵ such that $0 < \delta < \theta$, $0 < \epsilon < \theta$, $(n+1)\theta \equiv -\delta \pmod{2\pi}$, and $(m+1)\theta \equiv -\epsilon \pmod{2\pi}$. These imply $(m-n)\theta \equiv \delta - \epsilon \pmod{2\pi}$. Now, consider two cases.

Case (m1): if $-\theta < \delta - \epsilon < 0$, then $\sin((m-n)\theta) = \sin(\delta - \epsilon) < 0$ and $\sin((m-n+1)\theta) = \sin(\theta + \delta - \epsilon) > 0$, which imply

$$f_{m-n-1}(x) = x^{m-n+1} - x\frac{\sin((m-n+1)\theta)}{\sin\theta} + \frac{\sin((m-n)\theta)}{\sin\theta} \geq 0,$$

and thus $x^{m-n+1} \geq x\sin(\theta + \delta - \epsilon)/\sin\theta + \sin(\epsilon - \delta)/\sin\theta \geq 0$. Therefore,

$$\begin{aligned}
xf_m(x) &= x^{m+3} - x^2\frac{\sin((m+2)\theta)}{\sin\theta} + x\frac{\sin((m+1)\theta)}{\sin\theta} \\
&\geq \left(x\frac{\sin(\theta+\delta-\epsilon)}{\sin\theta} + \frac{\sin(\epsilon-\delta)}{\sin\theta}\right)\left(x\frac{\sin(\theta-\delta)}{\sin\theta} + \frac{\sin\delta}{\sin\theta}\right) \\
&\quad -x^2\frac{\sin(\theta-\epsilon)}{\sin\theta} - x\frac{\sin\epsilon}{\sin\theta}
\end{aligned}$$

$$
\begin{aligned}
&= x^2\left(\frac{\sin(\theta+\delta-\epsilon)\sin(\theta-\delta)}{\sin^2\theta} - \frac{\sin(\theta-\epsilon)}{\sin\theta}\right) + \frac{\sin(\epsilon-\delta)\sin\delta}{\sin^2\theta}\\
&\quad + x\left(\frac{\sin(\epsilon-\delta)\sin(\theta-\delta)+\sin\delta\sin(\theta+\delta-\epsilon)}{\sin^2\theta} - \frac{\sin\epsilon}{\sin\theta}\right)\\
&= \frac{\sin\delta\sin(\epsilon-\delta)}{\sin^2\theta}\{(x-\cos\theta)^2+\sin^2\theta\}\\
&\geq 0.
\end{aligned}
$$

Case (m2): if $0<\delta-\epsilon<\theta$, then $\sin((m-n)\theta)=\sin(\delta-\epsilon)>0$ and $\sin((m-n-1)\theta)=\sin(\delta-\epsilon-\theta)<0$, which imply

$$
f_{m-n-2}(x) = x^{m-n} - x\frac{\sin((m-n)\theta)}{\sin\theta} + \frac{\sin((m-n-1)\theta)}{\sin\theta} \geq 0,
$$

and thus $x^{m-n}\geq x\sin(\delta-\epsilon)/\sin\theta+\sin(\theta+\epsilon-\delta)/\sin\theta\geq 0$. Therefore,

$$
\begin{aligned}
f_m(x) &\geq \left(x\frac{\sin(\delta-\epsilon)}{\sin\theta}+\frac{\sin(\theta+\epsilon-\delta)}{\sin\theta}\right)\left(x\frac{\sin(\theta-\delta)}{\sin\theta}+\frac{\sin\delta}{\sin\theta}\right)\\
&\quad - x\frac{\sin(\theta-\epsilon)}{\sin\theta} - \frac{\sin\epsilon}{\sin\theta}\\
&= x^2\frac{\sin(\delta-\epsilon)\sin(\theta-\delta)}{\sin^2\theta}\\
&\quad + x\left(\frac{\sin(\theta-\delta)\sin(\theta+\epsilon-\delta)+\sin\delta\sin(\delta-\epsilon)}{\sin^2\theta} - \frac{\sin(\theta-\epsilon)}{\sin\theta}\right)\\
&\quad + \frac{\sin\delta\sin(\theta+\epsilon-\delta)}{\sin^2\theta} - \frac{\sin\epsilon}{\sin\theta}\\
&= \frac{\sin(\delta-\epsilon)\sin(\theta-\delta)}{\sin^2\theta}\{(x-\cos\theta)^2+\sin^2\theta\}\\
&\geq 0.
\end{aligned}
$$

This completes the proof for Claim (A) and therefore (g.2.2) follows if we can prove claim (A5). Now we prove claim (A5). It suffices to show $f_n'(x)\geq 0$ for $x\geq 1$, where f_n' denotes the first derivative of f_n. The fact that $\sin\theta>0$ implies that, for $x\geq 1$,

$$
\begin{aligned}
f_n'(x) &= (n+2)x^{n+1} - \frac{\sin((n+2)\theta)}{\sin\theta}\\
&\geq \frac{(n+2)\sin\theta-\sin((n+2)\theta)}{\sin\theta}\\
&\geq 0.
\end{aligned}
$$

The last inequality follows from the fact that, for $0\leq\alpha\leq\pi$ and integer $n\geq 1$, we have $\sin\alpha\geq 0$ and

$$
\sin((n+1)\alpha) = \sin(n\alpha)\cos\alpha + \sin\alpha\cos(n\alpha)
$$

$$
\begin{aligned}
&\le \{\sin\alpha\cos((n-1)\alpha)+\sin((n-1)\alpha)\cos\alpha\}\cos\alpha+\sin\alpha\\
&\le 2\sin\alpha+\sin((n-1)\alpha)\cos^2\alpha\\
&= 2\sin\alpha+\sin((n-2)\alpha)\cos^3\alpha+\sin\alpha\cos^2\alpha\cos((n-2)\alpha)\\
&\le 3\sin\alpha+\sin((n-2)\alpha)\cos^3\alpha\\
&\quad\vdots\\
&\le (n-1)\sin\alpha+\sin(2\alpha)\cos^{n-1}\alpha\\
&= (n-1)\sin\alpha+2\sin\alpha\cos^n\alpha\\
&\le (n+1)\sin\alpha.
\end{aligned}
$$

This proves claim (A5) and therefore (g.2.2).

We now prove (g.2.3). Note that $a \ge \lambda_1 > 1$ implies $|\lambda_2|\cos\theta \ge \lambda_1 > 1$, or equivalently

$$
\left|\frac{\lambda_2}{\lambda_1}\right| \ge \frac{1}{\cos\theta} > 1. \tag{A7}
$$

Equation (A4) and inequality (A7) imply that (g.2.3) is equivalent to the following:

$$
f_n(x) \ge 0, \qquad \text{for} \quad x \ge 1/\cos\theta > 1. \tag{A8}
$$

Condition (A8) is true if we can show: (i) $f_n(x)$ is an increasing function for $x \ge 1/\cos\theta > 1$; and (ii) $f_n(1/\cos\theta) \ge 0$. To prove claim (i), it suffices to show $f_n'(x) \ge 0$ for $x \ge 1/\cos\theta > 1$. For $x \ge 1/\cos\theta > 1$,

$$
\begin{aligned}
f_n'(x) &= (n+2)x^{n+1} - \frac{\sin((n+2)\theta)}{\sin\theta}\\
&\ge \frac{n+2}{\cos^{n+1}\theta} - \frac{\sin((n+2)\theta)}{\sin\theta}\\
&= \frac{(n+2)\sin\theta - \sin\theta\cos^{n+1}\theta(\cos((n+1)\theta)) - \cos\theta\cos^{n+1}\theta(\sin((n+1)\theta))}{\cos^{n+1}\theta\sin\theta},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
(\sin\theta)(\cos^{n+1}\theta)f_n'(x) &\ge (n+1)\sin\theta+\sin\theta\{1-\cos^{n+1}\theta\cos((n+1)\theta)\}\\
&\quad -\cos^{n+2}\theta\sin((n+1)\theta)\\
&\ge (n+1)\sin\theta-\cos^{n+2}\theta\sin((n+1)\theta)\\
&= (n+1)\sin\theta-\cos^{n+2}\theta\sin\theta\cos(n\theta)-\cos^{n+2}\theta\sin(n\theta)\cos\theta\\
&= n\sin\theta+\sin\theta\{1-\cos^{n+2}\theta\cos(n\theta)\}-\cos^{n+3}\theta\sin(n\theta)\\
&\ge n\sin\theta-\cos^{n+3}\theta\sin(n\theta)\\
&\quad\vdots
\end{aligned}
$$

$$
\begin{aligned}
&\geq \sin\theta - \cos^{2n+2}\theta \sin\theta \\
&\geq 0.
\end{aligned}
$$

This proves claim (i). Claim (ii) follows from the fact that

$$
\begin{aligned}
f_n\left(\frac{1}{\cos\theta}\right) &= \frac{1}{\cos^{n+2}\theta} - \frac{\sin((n+2)\theta)}{\cos\theta\sin\theta} + \frac{\sin((n+1)\theta)}{\sin\theta} \\
&= \frac{1}{\cos^{n+2}\theta} + \frac{\sin((n+1)\theta)\cos\theta - \sin\theta\cos((n+1)\theta) - \sin((n+1)\theta)\cos\theta}{\sin\theta\cos\theta} \\
&= \frac{1 - \cos^{n+1}\theta\cos((n+1)\theta)}{\cos^{n+2}\theta} \\
&\geq 0.
\end{aligned}
$$

This completes the proof for (g.2.3).

(h) The necessity follows from part (d). Now, we prove the sufficiency. Note that there are 8 cases: (h1) $1 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$; (h2) $1 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq -\lambda_4$; (h3) $1 < \lambda_1 \leq \lambda_2 \leq -\lambda_3 \leq \lambda_4$; (h4) $1 < \lambda_1 \leq \lambda_2 \leq -\lambda_3 \leq -\lambda_4$; (h5) $1 < \lambda_1 \leq -\lambda_2 \leq \lambda_3 \leq \lambda_4$; (h6) $1 < \lambda_1 \leq -\lambda_2 \leq \lambda_3 \leq -\lambda_4$; (h7) $1 < \lambda_1 \leq -\lambda_2 \leq -\lambda_3 \leq \lambda_4$; (h8) $1 < \lambda_1 \leq -\lambda_2 \leq -\lambda_3 \leq -\lambda_4$. For cases (h1)-(h6), the sufficiency follows from (i) the factorization $\psi(s) = \prod_i (1 - s/\lambda_i)^{-1}$, (ii) the results for $p = 1, 2$, and 3 and the fact that if $\lambda_2 < -1$, the condition $\lambda_1^{-1} + \lambda_2^{-1} \geq 0$ is the same as $1 < \lambda_1 \leq -\lambda_2$, and (iii) the aforementioned result of Widder (1946, p. 145). Therefore, we only need to prove cases (h7) and (h8). We first consider the simple case that λ_2, λ_3, and λ_4 are all distinct, i.e., $1 < \lambda_1 \leq -\lambda_2 < -\lambda_3 < -\lambda_4$. The cases with multiple roots can be proved similarly, and hence omitted. We need to consider odd n and even n separately. By equation (A2),

$$
\psi_n = \frac{r_1}{\lambda_1^{n+1}} + \frac{r_2}{\lambda_2^{n+1}} + \frac{r_3}{\lambda_3^{n+1}} + \frac{r_4}{\lambda_4^{n+1}}, \tag{A9}
$$

where $r_1 = \{\phi_4(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)\}^{-1}$, $r_2 = \{\phi_4(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)\}^{-1}$, $r_3 = \{\phi_4(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_4)\}^{-1}$, $r_4 = \{\phi_4(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)\}^{-1}$, and $\phi_4 = (-\lambda_1\lambda_2\lambda_3\lambda_4)^{-1}$. Note that $r_1 + r_2 + r_3 + r_4 = 0$. For $n = 2k+1$,

$$
\begin{aligned}
\psi_{2k+1} &= r_1\lambda_1^{-(2k+2)} + r_2\lambda_2^{-(2k+2)} + r_3\lambda_3^{-(2k+2)} + r_4\lambda_4^{-(2k+2)} \\
&= r_2(\lambda_2^{-(2k+2)} - \lambda_1^{-(2k+2)}) + r_3(\lambda_3^{-(2k+2)} - \lambda_1^{-(2k+2)}) + r_4(\lambda_4^{-(2k+2)} - \lambda_1^{-(2k+2)}) \\
&= \{\phi_4(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)\}^{-1}(\lambda_1\lambda_2)^{-(2k+2)}(\lambda_1^{2k+2} - \lambda_2^{2k+2}) \\
&\quad + \{\phi_4(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_4)\}^{-1}(\lambda_1\lambda_3)^{-(2k+2)}(\lambda_1^{2k+2} - \lambda_3^{2k+2}) \\
&\quad + \{\phi_4(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)\}^{-1}(\lambda_1\lambda_4)^{-(2k+2)}(\lambda_1^{2k+2} - \lambda_4^{2k+2}).
\end{aligned}
$$

Multiplying both sides of the above equation by $\phi_4(\lambda_1\lambda_2\lambda_3\lambda_4)^{2k+2}\prod_{1\le i<j\le 4}(\lambda_i-\lambda_j)$, we get

$$\begin{aligned}
&\psi_{2k+1}\phi_4(\lambda_1\lambda_2\lambda_3\lambda_4)^{2k+2}\prod_{1\le i<j\le 4}(\lambda_i-\lambda_j)\\
=&-(\lambda_3\lambda_4)^{2k+2}(\lambda_1^{2k+2}-\lambda_2^{2k+2})(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)(\lambda_3-\lambda_4)\\
&+(\lambda_2\lambda_4)^{2k+2}(\lambda_1^{2k+2}-\lambda_3^{2k+2})(\lambda_1-\lambda_2)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)\\
&-(\lambda_2\lambda_3)^{2k+2}(\lambda_1^{2k+2}-\lambda_4^{2k+2})(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3).
\end{aligned}$$

Dropping $\prod_{j=2}^4(\lambda_1-\lambda_j)$ from both sides of the above equation, we get

$$\begin{aligned}
&\psi_{2k+1}\phi_4(\lambda_1\lambda_2\lambda_3\lambda_4)^{2k+2}\prod_{2\le i<j\le 4}(\lambda_i-\lambda_j)\\
=&-(\lambda_3\lambda_4)^{2k+2}(\lambda_1+\lambda_2)(\lambda_3-\lambda_4)\sum_{i=0}^{k}\lambda_1^{2i}\lambda_2^{2k-2i}\\
&+(\lambda_2\lambda_4)^{2k+2}(\lambda_1+\lambda_3)(\lambda_2-\lambda_4)\sum_{i=0}^{k}\lambda_1^{2i}\lambda_3^{2k-2i}\\
&-(\lambda_2\lambda_3)^{2k+2}(\lambda_1+\lambda_4)(\lambda_2-\lambda_3)\sum_{i=0}^{k}\lambda_1^{2i}\lambda_4^{2k-2i}.
\end{aligned}\tag{A10}$$

Note that $(\lambda_1+\lambda_2)(\lambda_3-\lambda_4)+(\lambda_1+\lambda_4)(\lambda_2-\lambda_3)=(\lambda_1+\lambda_3)(\lambda_2-\lambda_4)$, therefore, equation (A10) becomes

$$\begin{aligned}
&\psi_{2k+1}\phi_4(\lambda_1\lambda_2\lambda_3\lambda_4)^{2k+2}\prod_{2\le i<j\le 4}(\lambda_i-\lambda_j)\\
=&-(\lambda_1+\lambda_2)(\lambda_3-\lambda_4)\lambda_4^{2k+2}\left(\lambda_3^{2k+2}\sum_{i=0}^{k}\lambda_1^{2i}\lambda_2^{2k-2i}-\lambda_2^{2k+2}\sum_{i=0}^{k}\lambda_1^{2i}\lambda_3^{2k-2i}\right)\\
&+(\lambda_1+\lambda_4)(\lambda_2-\lambda_3)\lambda_2^{2k+2}\left(\lambda_4^{2k+2}\sum_{i=0}^{k}\lambda_1^{2i}\lambda_3^{2k-2i}-\lambda_3^{2k+2}\sum_{i=0}^{k}\lambda_1^{2i}\lambda_4^{2k-2i}\right)\\
=&-(\lambda_1+\lambda_2)(\lambda_3-\lambda_4)\lambda_4^{2k+2}\sum_{i=0}^{k}\lambda_1^{2i}\lambda_2^{2k-2i}\lambda_3^{2k-2i}(\lambda_3^{2i+2}-\lambda_2^{2i+2})\\
&+(\lambda_1+\lambda_4)(\lambda_2-\lambda_3)\lambda_2^{2k+2}\sum_{i=0}^{k}\lambda_1^{2i}\lambda_3^{2k-2i}\lambda_4^{2k-2i}(\lambda_4^{2i+2}-\lambda_3^{2i+2}).
\end{aligned}$$

Dropping $(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)$ from both sides of the above equation, we get

$$\begin{aligned}
&\phi_4(\lambda_2-\lambda_4)(\lambda_1\lambda_2\lambda_3\lambda_4)^{2k+2}\psi_{2k+1}\\
=&(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)\lambda_4^{2k+2}\sum_{i=0}^{k}\lambda_1^{2i}\lambda_2^{2k-2i}\lambda_3^{2k-2i}\sum_{j=0}^{i}\lambda_2^{2j}\lambda_3^{2i-2j}\\
&-(\lambda_1+\lambda_4)(\lambda_3+\lambda_4)\lambda_2^{2k+2}\sum_{i=0}^{k}\lambda_1^{2i}\lambda_3^{2k-2i}\lambda_4^{2k-2i}\sum_{j=0}^{i}\lambda_3^{2j}\lambda_4^{2i-2j}
\end{aligned}$$

$$
\begin{aligned}
&= \lambda_1^{2k}\lambda_2^{2k}\lambda_3^{2k}\lambda_4^{2k+2}(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)\sum_{i=0}^{k}\sum_{j=0}^{i}\lambda_1^{-(2k-2i)}\lambda_2^{-(2i-2j)}\lambda_3^{-2j} \\
&\quad -\lambda_1^{2k}\lambda_2^{2k+2}\lambda_3^{2k}\lambda_4^{2k}(\lambda_1+\lambda_4)(\lambda_3+\lambda_4)\sum_{i=0}^{k}\sum_{j=0}^{i}\lambda_1^{-(2k-2i)}\lambda_3^{-(2i-2j)}\lambda_4^{-2j}. \qquad \text{(A11)}
\end{aligned}
$$

Note that for both cases (h7) and (h8), $\phi_4(\lambda_2-\lambda_4)>0$, and

$$
\begin{aligned}
&\lambda_4^2(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)-\lambda_2^2(\lambda_1+\lambda_4)(\lambda_3+\lambda_4) \\
&= \lambda_4^2\{\lambda_2^2+(\lambda_1+\lambda_3)\lambda_2+\lambda_1\lambda_3\}-\lambda_2^2\{\lambda_4^2+(\lambda_1+\lambda_3)\lambda_4+\lambda_1\lambda_3\} \\
&= (\lambda_1+\lambda_3)\lambda_2\lambda_4(\lambda_4-\lambda_2)+\lambda_1\lambda_3(\lambda_4^2-\lambda_2^2) \\
&= |\lambda_4-\lambda_2||\lambda_1\lambda_2\lambda_3\lambda_4|\left(\lambda_1^{-1}+\lambda_2^{-1}+\lambda_3^{-1}+\lambda_4^{-1}\right) \\
&\geq 0. \qquad \text{(A12)}
\end{aligned}
$$

Therefore, equations (A11) and (A12) imply that for cases (h7) and (h8), $\psi_{2k+1}\geq 0$. This completes the proof for cases (h7) and (h8) when $n=2k+1$.

Next, we consider case (h7) when $n=2k$. Note that $\phi_4=(-\lambda_1\lambda_2\lambda_3\lambda_4)^{-1}<0$, $r_1>0$, $r_2<0$, $r_3>0$, $r_4<0$, $|\lambda_1|\geq|\lambda_4|$, and $|\lambda_2|\geq|\lambda_3|$, where the r_js are defined below equation (A9). Therefore, equation (A9) implies that

$$
\psi_{2k}=\tfrac{|r_1|}{\lambda_1^{2k+1}}+\tfrac{|r_2|}{|\lambda_2|^{2k+1}}-\tfrac{|r_3|}{|\lambda_3|^{2k+1}}-\tfrac{|r_4|}{|\lambda_4|^{2k+1}}\geq 0.
$$

Finally, we consider case (h8) when $n=2k$. Let $\eta_n=(\lambda_1\lambda_2\lambda_3\lambda_4)^{n+1}\prod_{1\leq i<j\leq 4}(\lambda_i-\lambda_j)$, and note that $\phi_4>0$ and $\eta_n<0$, then by equation (A9),

$$
\begin{aligned}
-\eta_n\phi_4\psi_n &= -(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)(\lambda_2\lambda_3\lambda_4)^{n+1} \\
&\quad +(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)(\lambda_3-\lambda_4)(\lambda_1\lambda_3\lambda_4)^{n+1} \\
&\quad -(\lambda_1-\lambda_2)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_1\lambda_2\lambda_4)^{n+1} \\
&\quad +(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_1\lambda_2\lambda_3)^{n+1} \\
&= (|\lambda_3|-|\lambda_2|)(|\lambda_4|-|\lambda_2|)(|\lambda_4|-|\lambda_3|)|\lambda_2\lambda_3\lambda_4|^{2k+1} \\
&\quad +(|\lambda_3|+\lambda_1)(|\lambda_4|+\lambda_1)(|\lambda_4|-|\lambda_3|)|\lambda_1\lambda_3\lambda_4|^{2k+1} \\
&\quad -(|\lambda_2|+\lambda_1)(|\lambda_4|+\lambda_1)(|\lambda_4|-|\lambda_2|)|\lambda_1\lambda_2\lambda_4|^{2k+1} \\
&\quad +(|\lambda_2|+\lambda_1)(|\lambda_3|+\lambda_1)(|\lambda_3|-|\lambda_2|)|\lambda_1\lambda_2\lambda_3|^{2k+1} \\
&= (|\lambda_3|-|\lambda_2|)(|\lambda_4|-|\lambda_2|)(|\lambda_4|-|\lambda_3|)|\lambda_2\lambda_3\lambda_4|^{2k+1} \\
&\quad +\lambda_1^{2k+3}\{(|\lambda_4|-|\lambda_3|)|\lambda_3\lambda_4|^{2k+1}-(|\lambda_4|-|\lambda_2|)|\lambda_2\lambda_4|^{2k+1} \\
&\qquad\qquad +(|\lambda_3|-|\lambda_2|)|\lambda_2\lambda_3|^{2k+1}\} \\
&\quad +\lambda_1^{2k+2}\{(|\lambda_4|^2-|\lambda_3|^2)|\lambda_3\lambda_4|^{2k+1}-(|\lambda_4|^2-|\lambda_2|^2)|\lambda_2\lambda_4|^{2k+1}
\end{aligned}
$$

$$+(|\lambda_3|^2 - |\lambda_2|^2)|\lambda_2\lambda_3|^{2k+1}\}$$

$$+\lambda_1^{2k+1}\{(|\lambda_4| - |\lambda_3|)|\lambda_3\lambda_4|^{2k+2} - (|\lambda_4| - |\lambda_2|)|\lambda_2\lambda_4|^{2k+2}$$

$$+(|\lambda_3| - |\lambda_2|)|\lambda_2\lambda_3|^{2k+2}\}. \tag{A13}$$

Now, note that, for integer $k \geq 1$,

$$\begin{aligned}
&(|\lambda_4| - |\lambda_3|)|\lambda_3\lambda_4|^k - (|\lambda_4| - |\lambda_2|)|\lambda_2\lambda_4|^k + (|\lambda_3| - |\lambda_2|)|\lambda_2\lambda_3|^k \\
&= (|\lambda_4| - |\lambda_3|)(|\lambda_3\lambda_4|^k - |\lambda_2\lambda_4|^k) - (|\lambda_3| - |\lambda_2|)(|\lambda_2\lambda_4|^k - |\lambda_2\lambda_3|^k) \\
&= (|\lambda_4| - |\lambda_3|)(|\lambda_3| - |\lambda_2|)\left(|\lambda_4|^k \sum_{i=0}^{k-1} |\lambda_3|^i |\lambda_2|^{k-1-i} - |\lambda_2|^k \sum_{i=0}^{k-1} |\lambda_3|^i |\lambda_4|^{k-1-i}\right) \\
&\geq 0,
\end{aligned} \tag{A14}$$

and

$$\begin{aligned}
&(|\lambda_4|^2 - |\lambda_3|^2)|\lambda_3\lambda_4|^k - (|\lambda_4|^2 - |\lambda_2|^2)|\lambda_2\lambda_4|^k + (|\lambda_3|^2 - |\lambda_2|^2)|\lambda_2\lambda_3|^k \\
&= (|\lambda_4|^2 - |\lambda_3|^2)(|\lambda_3\lambda_4|^k - |\lambda_2\lambda_4|^k) - (|\lambda_3|^2 - |\lambda_2|^2)(|\lambda_2\lambda_4|^k - |\lambda_2\lambda_3|^k) \\
&= (|\lambda_4| - |\lambda_3|)(|\lambda_3| - |\lambda_2|)\left\{(|\lambda_4| + |\lambda_3|)|\lambda_4|^k \sum_{i=0}^{k-1} |\lambda_3|^i |\lambda_2|^{k-i-1}\right. \\
&\qquad \left. -(|\lambda_3| + |\lambda_2|)|\lambda_2|^k \sum_{i=0}^{k-1} |\lambda_3|^i |\lambda_4|^{k-i-1}\right\} \\
&\geq 0.
\end{aligned} \tag{A15}$$

Equation (A13) and inequalities (A14) and (A15) imply that $\psi_{2k} \geq 0$. This completes the proof for cases (h7) and (h8). Therefore, the proof of (h) is completed.

(j) This follows from the aforementioned result of Widder (1946, p. 145).

(k) This follows from the fact that $\psi(z) = \theta(z) = \sum_{j=0}^{q} \theta_j z^j$.

(l) This follows from equations (3.3.3) and (3.3.4) of Brockwell and Davis (1991), namely, $\psi_j = \phi_1\psi_{j-1} + \theta_j$, for $1 \leq j \leq q$, where $\psi_0 = 1$, and $\psi_j = \phi_1\psi_{j-1}$, for $j \geq q + 1$.

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