A Note on the Non-negativity of Continuous-time ARMA and GARCH Processes

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Abstract

A general approach for modeling the volatility process in continuous-time is based on the convolution of a kernel with a non-decreasing Lévy process, which is non-negative if the kernel is non-negative. Within the framework

of Continuous-time Auto-Regressive Moving-Average (CARMA) processes, we derive a sufficient condition for the kernel to be non-negative, based on which we propose a numerical method for checking the non-negativity of a kernel function. We discuss how to adapt this approach to solving a similar problem with the second approach to modeling volatility via the COntinuous-time Generalized Auto-Regressive Conditional Heteroscedastic (COGARCH) processes.
Some key words: DIRECT; global optimization; kernel; Lévy process; Volatility.

1 Introduction

Prompted by the need for analyzing financial time series, there has been recently much work on developing models suitable for analyzing the volatility of a continuous-time process, see Andersen & Lund (1997), Comte & Renault (1998), Barndorff-Nielsen & Shephard (2001), Brockwell (2004), Klüppelberg et al. (2004), Brockwell & Marquardt (2005) and Brockwell et al. (2006). There are, at least, two approaches to modeling a continuous-time volatility process. In the first approach, the volatility process is modeled as some continuous-time Auto-Regressive Moving-Average (CARMA) process driven by a Lévy process, e.g. a compound Poisson process; see Barndorff-Nielsen & Shephard (2001), Brockwell (2004) and Brockwell & Marquardt (2005). Conditional on the volatility process, the observed process (after suitable transformation) is modeled as some diffusion process. Thus, there is no direct feedback to the volatility process from the observed process. For financial applications of the non-negative Lévy-driven CARMA processes, see Roberts et al. (2004), or a working paper (available at `http://www.econ.duke.edu/~get/wpapers/sc.pdf.`) by Todorov & Tauchen (2005). The second approach attempts to directly model the volatility process in terms of the current and past values of the observed process, i.e. to lift the discrete-time Generalized Auto-Regressive Conditional Heteroscedastic (GARCH) model to the continuous-time setting. This leads to the development of the COntinuous-time Generalized Auto-Regressive Conditional Heteroscedastic (COGARCH) processes proposed by Klüppelberg et al. (2004) and Brockwell et al. (2006).

For volatility modeling, the continuous-time process must be non-negative. A stationary Lévy-driven CARMA process can be shown to be the convolution of a kernel function with a Lévy-driving process, which is non-negative if the kernel is non-negative and the Lévy-driving process is a non-decreasing process. Tsai & Chan (2005) showed that the kernel of a CARMA process is non-negative if and only if its Laplace transform is completely monotone. Based on this characterization, Tsai & Chan (2005) gave some more readily verifiable necessary and sufficient conditions for the kernel to be non-negative for some lower order CARMA processes. However, analogous results are lacking for higher order cases. Here, we obtain some new sufficient (necessary) conditions for the non-negativity of the kernel of a CARMA process, based on which we propose a numerical method for verifying the non-negativity of the kernel of a general CARMA process.

The rest of this paper is organized as follows. In § 2, we briefly review the Lévy-driven CARMA processes. The main result is stated in § 3. We illustrate our numerical approach for verifying the non-negativity of a CARMA process in § 4. The COGARCH processes are reviewed in § 5. We point out that the approach outlined in § 3 can be adapted to solve the non-negativity problem for a COGARCH process. The proof of the main result is deferred to the appendix.

2 CARMA Processes

We now recall the Lévy-driven CARMA(p, q) process introduced by Brockwell (2000, 2001, and 2004). The Lévy process is defined in terms of infinitely divisible distributions. Let $\phi(u)$ be the characteristic function of a distribution. We say that the distribution is infinitely divisible if, for every positive integer n, $\phi(u)$ is the nth power of some characteristic function. Let $\mathbb{R}$ be a set of real numbers. For every infinitely divisible distribution, we can define a stochastic process $\{X_t, t \in \mathbb{R}\}$, called a Lévy process, such that $X_0 \equiv 0$, and it has independent and stationary increments with $(\phi(u))^t$ as the characteristic function of $X_{t+s} - X_s$, for any $s \in \mathbb{R}$ and $t \geq 0$. For more information on Lévy processes, see Protter (1991), Bertoin (1996), Sato (1999), and Applebaum (2004). Heuristically, a Lévy-driven CARMA(p, q) process $\{Y_t\}$ is defined as the solution of a p-th order stochastic differential equation with suitable initial condition, and driven by a Lévy process and its derivatives up to and including order $0 \leq q < p$. Specifically, for $t \in \mathbb{R}$,

$$Y_t^{(p)} - \alpha_p Y_t^{(p-1)} - \cdots - \alpha_1 Y_t - \alpha_0 = \sigma\{L_t^{(1)} + \beta_1 L_t^{(2)} + \cdots + \beta_q L_t^{(q+1)}\}, \quad (1)$$

where $\{L_t, t \in \mathbb{R}\}$ is a Lévy process with $L_0 \equiv 0$ and $EL_1^2 = 1$; the superscript $^{(j)}$ denotes j-fold differentiation with respect to t, i.e. $dY_t^{(j-1)} =$

$Y_t^{(j)}dt, j = 1, ..., p-1$. Note that the derivatives may not be well-defined in the usual sense; here their use merely serves as a shorthand for a vector integral equation to be defined below. We assume that $\sigma > 0$, $\alpha_1 \neq 0$, and $\beta_q \neq 0$.

Equation (1) can be equivalently cast in terms of the *observation* and *state* equations (see Brockwell, 2001):

$$\begin{aligned} Y_t &= \beta' X_t, \qquad t \in \mathbb{R}, \\ dX_t &= (AX_t + \alpha_0 \delta)dt + \sigma \delta dL_t, \end{aligned} \tag{2}$$

where the superscript prime denotes taking the transpose,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_p \end{bmatrix}, \quad X_t = \begin{bmatrix} X_t^{(0)} \\ X_t^{(1)} \\ \vdots \\ X_t^{(p-2)} \\ X_t^{(p-1)} \end{bmatrix}, \quad \delta = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 \\ \beta_1 \\ \vdots \\ \beta_{p-2} \\ \beta_{p-1} \end{bmatrix},$$

$\beta_j = 0$ for $j > q$. The stationary mean of Y_t, if it exists, can be shown to equal $-\alpha_0/\alpha_1$, see Tsai & Chan (2005). The stationary mean must be non-negative for the process to model conditional variances; for simplicity, we henceforth assume that $\alpha_0 = 0$.

Provided all the eigenvalues of A have negative real parts, the process $\{X_t\}$ defined by

$$X_t = \sigma \int_{-\infty}^{t} \exp\{A(t-u)\} \delta dL_u$$

is the strictly stationary solution of (2) for $t \in (-\infty, \infty)$ with the corresponding CARMA process given by

$$Y_t = \sigma \int_{-\infty}^{\infty} g(t-u) dL_u, \qquad -\infty < t < \infty, \tag{3}$$

where $g(t) = \beta' \exp(At)\delta I_{[0,\infty)}(t)$. Henceforth in this paper, let $\lambda_1, \cdots, \lambda_p$ be the roots of $\alpha(z) = 0$. Without loss of generality, assume $\mathrm{Re}(\lambda_p) \leq \cdots \leq \mathrm{Re}(\lambda_2) \leq \mathrm{Re}(\lambda_1) < 0$, where $\mathrm{Re}(\lambda_i)$ denotes the real part of λ_i. In the case when $\lambda_1, ..., \lambda_p$ are distinct and $\mathrm{Re}(\lambda_j) < 0$, for $j = 1, ..., p$, Brockwell & Marquardt (2005) showed that, for $u \geq 0$,

$$g(u) = \sum_{r=1}^{p} \frac{\beta(\lambda_r)}{\alpha^{(1)}(\lambda_r)} \exp(\lambda_r u), \tag{4}$$

where $\alpha^{(1)}$ denotes its first derivative, $\alpha(z) = z^p - \alpha_p z^{p-1} - \cdots - \alpha_1$ and $\beta(z) = 1 + \beta_1 z + \beta_2 z^2 + \cdots + \beta_q z^q$. Equation (4) implies that the kernel function is Lipschitz continuous. Recall that the characteristic equation of A, i.e. $\det(A - zI) = 0$, equals $\alpha(z) = 0$. We assume that all roots of $\alpha(z) = 0$ and those of $\beta(z) = 0$ have negative real parts, and the two equations have no common roots. The condition on the roots of $\alpha(z) = 0$ is necessary for the stationarity of the process whereas that on $\beta(z) = 0$ is for the process to be of minimum phase and is akin to the invertibility condition for discrete-time processes. We claim that if the CARMA(p, q) process $\{Y_t\}$ is stationary, then $\alpha_j < 0$, for $j = 1, ..., p$. The cases of $p = 1$ and $p = 2$ can be checked by algebra. For higher order cases, we note that the characteristic polynomial $z_p - \alpha_p z^{p-1} - \cdots - \alpha_1$ can be factorized into products of real polynomials of degree not greater than two, all of which have positive coefficients based on the arguments presented for orders one and two. Similarly, it can be shown that for $\{Y_t\}$ to be of minimum phase, it is necessary that $\beta_j > 0$, for $j = 1, ..., q$.

3 Main Results

¿From (3), the process $\{Y_t\}$ is non-negative if (i) the kernel g is non-negative, and (ii) the driving Lévy process L is non-decreasing. Tsai & Chan (2005) characterised the non-negativity of the kernel for any CARMA(p, q) process in terms of the complete monotonicity of its Laplace transform. They made use of this characterization to show that a necessary condition for the kernel g of a stationary CARMA(p, q) process to be non-negative is that λ_1 is real, and $\lambda_1 < 0$. Furthermore, for CARMA processes of lower orders, they derived some readily verifiable necessary and sufficient conditions for the kernel g to be non-negative. However, similar readily verifiable conditions for the general case are lacking. What is more intriguing is that given a particular set of CARMA parameters, the non-negativity of the corresponding kernel requires checking the values of the function over the unbounded interval $[0, \infty)$, which may be a numerically infeasible task. Interestingly, it is shown in the main result below that under some mild conditions, the kernel is non-negative over $[0, \infty)$ if and only if it is non-negative over some finite interval $[0, u^*]$, with a tractable end-point u^*. Importantly, the non-negativity of a Lipschitz-continuous kernel over a bounded interval can be numerically determined using some global optimization scheme; one such useful scheme is the DIRECT method (Jones et al. 1993 and Kelley, 1999). We illustrate in § 4 the use of this approach to verify whether a kernel function is non-negative or not.

THEOREM 1. *Let $p > q \geq 0$ and $p \geq 2$. Assume the CARMA(p, q) process $\{Y_t\}$ is stationary and that the roots of the characteristic equation $\alpha(z) = 0$*

are distinct. Then Conditions (5), (6), (7), and (8) are sufficient for the kernel g to be non-negative, while Conditions (5), (6), and (7) are necessary for the kernel g to be non-negative:

$$\lambda_1 \textit{ is real, and } \lambda_1 < 0, \tag{5}$$

$$\beta(\lambda_1) > 0, \tag{6}$$

$$g(u) \geq 0, \textit{ for } 0 \leq u \leq u^*, \tag{7}$$

$$Re(\lambda_1) > Re(\lambda_2), \tag{8}$$

where $u^* = \{(p-1)r^* - r_1\}/[r_1\{Re(\lambda_1) - Re(\lambda_2)\}]$, $r^* = \max_{2\leq j\leq p} |r_j|$, *and* $r_j = \beta(\lambda_j)/\alpha^{(1)}(\lambda_j)$.

Remark: The case $p = 1$ and $q = 0$ is trivial as the necessary and sufficient condition for non-negativity is $\alpha_1 < 0$. Note that Condition (8) is necessary for u^* to be finite. If we define u^* to be ∞ when $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2)$, then the kernel g is non-negative if and only if Conditions (5)-(7) hold.

4 Numerical Examples

Below, we give two numerical examples demonstrating the use of Theorem 1 for checking the non-negativity of a kernel function. The key step of checking the non-negativity of a kernel function over a bounded interval is done via DIRECT, a global optimization technique that requires the evaluation of the function itself but not its derivatives; see Jones et al. (1993), Kelley (1999, pp. 149-152) and Gablonsky & Kelly (2001). Employing a global optimization procedure is pivotal as popular optimization methods such as the Newton-Raphson method may result in some local minimum, thereby leading to (possibly) fallacious conclusion about the non-negativity of the function. The name DIRECT is derived from one of its main features, *dividing rectangles*. For finding the global minimum, the algorithm consists of recursively dividing up the bounded domain into diminishing hyper-rectangles by (i) determining the "optimal" rectangles based on the sampled function values at the center of the existing rectangles, i.e. choosing rectangles that may contain the global minimum of the function and (ii) further subdividing the "optimal" rectangles. The search is generally stopped when a budgeted amount of function evaluations and/or number of sub-divisions of the rectangles is attained. There are several criteria for deciding which rectangles are optimal, all of which presuppose the Lipschitz continuity of the function, but differ on how they balance the search between local and possible global optima. The DIRECT method enjoys several convergence properties (see a technical report by D. E. Finkel & C. T. Kelley, available at `www.optimization-online.org/DB_FILE/2004/08/934.pdf`): (i) it is an exhaustive search, i.e. the centers of the rectangles are eventually dense in the domain (with unlimited number of function evaluations and rectangle divisions), and (ii) the intermediate set of optima, as obtained when the algorithm is stopped upon exhausting the budgeted number of function evaluations and/or rectangle sub-divisions, clusters around the true local and global optima. Moreover, the DIRECT method performs well empirically with benchmark examples and converges quickly.

The numerical examples below were conducted with a Pentium(R) IV 3.2 GHz. IBM machine using a FORTRAN program with IMSL Libraries in a Windows XP platform. The non-negativity of the kernel g over $[0, u^*]$ is checked by computing its global minimum via the modified DIRECT algorithm (the file titled "DIRECTv204.tar.gz", available at `http://plato.la.asu.edu/topics/problems/global.html`) of Gablonsky (2001, DIRECT version 2.0. User Guide, North Carolina University), with the stopping rule of no more than about 20,000 function evaluations or 6,000 rectangle subdivisions.

Example 1: Consider a CARMA(5,0) process with $\alpha(z) = z^5 + 9z^4 + 37z^3 + 81z^2 + 92z + 40$. The roots of $\alpha(z) = 0$ are -1, $-2 \pm 2i$, and $-2 \pm i$, and $u^* = 3.71405$. It takes 0.6406 seconds for the program to find the minimal value, which is $g(0.0000278) = 0.0000000$. Therefore, the kernel is non-negative.

Example 2: Consider a CARMA(5,4) process with the $\alpha(z)$ polynomial the same as the one considered in Example 1, and $\beta(z) = 1+0.5826351866z+2.027798934z^2 + 0.5712109673z^3 + 0.9520182788z^4$. The roots of $\beta(z) = 0$ are $-0.2 \pm i$ and $-0.1 \pm i$, and $u^* = 39.31429$. It takes 1.156 seconds for the program to find the minimal value, which is $g(0.4142046) = -0.2437303$. Therefore, the kernel is not always non-negative.

5 COGARCH Processes

We now briefly review the COGARCH processes. For further details, see Brockwell et al. (2006). Let m, s be integers such that $1 \leq m \leq s$, and $a_0, a_1, \ldots, a_m, b_1, \ldots, b_s \in \mathbb{R}$, $a_0 > 0$, $a_m \neq 0$, $b_s \neq 0$, and $a_{m+1} = \cdots = a_s =$

0. Define the $(s \times s)$-matrix B by

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_s & -b_{s-1} & -b_{s-2} & \cdots & -b_1 \end{bmatrix}, \ a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{s-1} \\ a_s \end{bmatrix}, \ e = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

with $B := -b_1$ if $q = 1$. Let $\{L_t\}_{t \geq 0}$ be a Lévy process with nontrivial Lévy measure and define the (left-continuous) volatility process $\{V_t\}_{t \geq 0}$ by

$$V_t = a_0 + a'Y_{t-}, \qquad t > 0, \qquad V_0 = a_0 + a'Y_0,$$

where $\{Y_t\}_{t \geq 0}$ is the unique càdlàg solution of the stochastic differential equation

$$dY_t = BY_{t-}dt + e(a_0 + a'Y_{t-})d[L, L]_t^{(d)}, \qquad t > 0, \tag{9}$$

with initial value Y_0, independent of the driving Lévy process $\{L_t\}_{t \geq 0}$. Here, $[L, L]^{(d)}$ denotes the discrete part of the quadratic covariation of $\{L_t\}_{t \geq 0}$. If the process $\{V_t\}_{t \geq 0}$ is strictly stationary and non-negative almost surly, we say that $\{G_t\}_{t \geq 0}$, given by

$$dG_t = V_t^{1/2} dL_t, \qquad t > 0, \qquad G_0 = 0,$$

is a COGARCH(m, s) process with parameters $a_0, a_1, \ldots, a_m, b_1, \ldots, b_s$ and the driving Lévy process. Brockwell et al. (2006) showed that if Y_0 is such that $\{V_t\}_{t \geq 0}$ is strictly stationary, and $a' \exp(Bt)e \geq 0$ for all $t \geq 0$, then $\{V_t\}_{t \geq 0}$ is non-negative with probability one. Note that $a' \exp(Bt)e$ is the kernel of a CARMA process with autoregressive coefficients $-b_s, ..., -b_1$, and moving average coefficients $a_1, ..., a_m$. Therefore, the results derived in Section 3 can be applied to a COGARCH process.

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Appendix

Proof of Theorem 1

We first prove the sufficiency of Conditions (5) - (8). First note that Condition (5) and the distinct eigenvalue condition of $\alpha(z) = 0$ imply $\alpha^{(1)}(\lambda_1) = \prod_{j=2}^{p}(\lambda_1 - \lambda_j) > 0$, which together with Condition (6) imply $r_1 = \beta(\lambda_1)/\alpha^{(1)}(\lambda_1) > 0$. Again, by the distinct eigenvalue condition of $\alpha(z) = 0$, Equation (4), Conditions (5) and (8), and the fact that $\exp(x) \geq 1 + x$ for all real x, we have, for $u \geq u^*$,

$$
\begin{aligned}
g(u) &= \sum_{k=1}^{p} r_k \exp(\lambda_k u) &(10)\\
&= \exp\{Re(\lambda_2)u\} \sum_{k=1}^{p} r_k \exp\left[\{\lambda_k - \mathrm{Re}(\lambda_2)\}u\right]\\
&\geq \exp\{Re(\lambda_2)u\} \Bigg(r_1 + r_1 u\{\mathrm{Re}(\lambda_1) - \mathrm{Re}(\lambda_2)\}\\
&\quad - \sum_{k=2}^{p} |r_k| \exp\left[\{\mathrm{Re}(\lambda_k) - \mathrm{Re}(\lambda_2)\}u\right]\Bigg)\\
&\geq \exp\{Re(\lambda_2)u\}\left[r_1 + r_1 u\{\mathrm{Re}(\lambda_1) - \mathrm{Re}(\lambda_2)\} - (p-1)r^*\right]\\
&\geq 0. &(11)
\end{aligned}
$$

Condition (7) and inequality (11) imply $g(u) \geq 0$ for all non-negative u. This completes the proof of the sufficiency. The necessity of (5) was shown in Tsai & Chan (2005). The necessity of (7) is trivial. For the necessity of (6), we note that the first term in the sum of the right hand side of (10) is dominating and has to be non-negative. Therefore, $r_1 = \beta(\lambda_1)/\alpha^{(1)}(\lambda_1)$ must be non-negative. But $\alpha^{(1)}(\lambda_1)$ is always positive, therefore $\beta(\lambda_1)$ must be non-negative. By the assumption that the polynomials $\alpha(\cdot)$ and $\beta(\cdot)$ have no common zeros, $\beta(\lambda_1)$ must be > 0. This proves the necessity of (6), and therefore, completes the proof of the Theorem.

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