A Note on Inequality Constraints in the GARCH Model

(SHORT RUNNING TITLE: GARCH Inequality Constraints)

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June 5, 2006

SUMMARY

We show that under some mild regularity conditions, a well-known sufficient condition of Nelson and Cai (1992) turns out to be also a necessary condition for a GARCH(p, q) model to have non-negative conditional variances, with probability one. For the case of $p \geq 3$, we, furthermore, derive readily verifiable sufficient (necessary) conditions for the non-negativity of the conditional variance process of a GARCH(p, q) model. The new inequality constraints are illustrated with an analysis of a daily USD/HKD exchange rate dataset.

Key words: Absolutely monotone; ARIMA model; exchange rate; generating function; volatility.

1 Introduction

Over the past two decades, there are many econometric models developed in the literature for modeling the volatility of asset returns. A popular model for modeling volatility is the Generalized Auto-Regressive Conditional Heteroscedastic (GARCH) model (Engle, 1982 and Bollerslev, 1986). For volatility modeling, the conditional variance process of the model must be non-negative almost surely. Hence, an important problem concerns the identification of necessary and sufficient conditions for a GARCH model to have non-negative conditional variances almost surely. For the ARCH model, the problem is trivial, see, e.g. Nelson and Cao (1992). The problem for the GARCH model is, on the other hand, more difficult. Nelson and Cao (1992) derived some necessary and sufficient conditions for the nonnegativity of GARCH(p, q) models with $p \leq 2$ and a sufficient condition for $p > 2$. In this article, we are interested in deriving necessary and sufficient conditions for the nonnegativity of GARCH(p, q) models for $p > 2$.

Under a mild regularity condition (Assumption (A1) below), the GARCH model can be re-written as an ARCH(∞) form, i.e. its conditional variance process admits a moving-average representation in terms of the convolution of the GARCH kernel and the squared error process. Hence, the conditional variance process of a GARCH model is always non-negative if the GARCH kernel is non-negative. Tsai and Chan (2005) studied the problem of characterizing the non-negativity of the kernel of a causal ARMA model by exploiting the idea that the non-negativity of the kernel is equivalent to the absolutely monotonicity of its generating function. Because of the ARCH(∞) representation of a GARCH model is similar to the moving-average representation of an ARMA model, many of the results of Tsai and Chan (2005) can be adapted here for the GARCH models.

The rest of the paper is organized as follows. In Section 2, we review the GARCH model and introduce some notations. The main results are stated in Section 3, and a numerical example is given in Section 4. We briefly conclude in

Section 5.

2 A Brief Review of GARCH Models

The GARCH(p, q) model is defined as

$$\epsilon_t = \sigma_t z_t, \tag{1}$$

$$\sigma_t^2 = \omega + \beta(L)\sigma_t^2 + \alpha(L)\epsilon_t^2, \tag{2}$$

where $\{z_t\}$ is a sequence of iid random variables with zero mean and unit variance, L is the lag (or backshift) operator (i.e. $L(X_t) \equiv X_{t-1}$), $\alpha(L) = \alpha_1 L + \alpha_2 L^2 + \cdots \alpha_q L^q$, and $\beta(L) = \beta_1 L + \beta_2 L^2 + \cdots + \beta_p L^p$. Thus, by definition, the $\{\epsilon_t\}$ process is serially uncorrelated with zero mean, but σ_t^2, the conditional variance of ϵ_t given $\epsilon_{t-1}, \epsilon_{t-2}, \epsilon_{t-3}...$, is changing over time. Under the assumption that

(A1) all the roots of $1 - \beta(z) = 0$ lie outside the unit circle,

equation (2) can be rewritten as an ARCH(∞) form:

$$\begin{aligned} \sigma_t^2 &= \{1 - \beta(1)\}^{-1}\omega + \{1 - \beta(L)\}^{-1}\alpha(L)\epsilon_t^2 \\ &= \omega^* + \sum_{k=1}^{\infty} \psi_k \epsilon_{t-k}^2 \qquad (3) \\ &= \omega^* + \Psi(L)\epsilon_t^2, \qquad (4) \end{aligned}$$

where

$$\Psi(z) = \sum_{k=0}^{\infty} \psi_k z^k = \frac{\alpha(z)}{1 - \beta(z)}, \tag{5}$$

with $\psi_0 = 0$. We also assume that

(A2) the polynomials $1 - \beta(z)$ and $\alpha(z)$ have no common roots,

which is needed for model identifiability. For the GARCH(p, q) model in equation (2) to be well-defined and the conditional variance to be positive almost surely for all t, all the coefficients in the ARCH (∞) representation in equation (3) must be non-negative, i.e. $\omega^* \geq 0$, and $\psi_k \geq 0$, for $k = 1, 2, ...$ In this paper, we are interested in studying conditions under which $\{\psi_i\}_{i=0}^{\infty}$ is non-negative. Finally, we note that the GARCH model is often combined with other time series models such as the ARIMA model to deal with the problem of conditional heteroscedascity, an example of which is given in the numerical illustration.

3 Main Results

To be a valid conditional variance, σ_t^2 must be non-negative with probability one. It is clear from (3) that, if $\omega^* \geq 0$ and $\psi_k \geq 0$ for all positive integer k, then $\sigma_t^2 \geq 0$. Nelson and Cao (1992) derived some necessary and sufficient conditions for the nonnegativity of GARCH(p,q) models with $p \leq 2$ and a sufficient condition for $p > 2$. In practice, given a particular set of GARCH parameters, checking the non-negativity of $\{\psi_k\}_{k=0}^{\infty}$ may be a numerically infeasible task. Nelson and Cao (1992) showed that, for $p > 2$ and under some mild conditions, the non-negativity of $\{\psi_k\}_{k=0}^{u^*}$ for some tractable integer u^* is sufficient for the non-negativity of $\{\psi_k\}_{k=0}^{\infty}$. In Theorem 1, we showed that, under the same mild conditions, the preceding condition of Nelson and Cao (1992) is not only a sufficient condition but it is also a necessary condition.

Let $\lambda_j, 1 \leq j \leq p$, be the roots of $1 - \beta(z) = 0$. With no loss of generality, we can and shall henceforth assume the following convention that

$$|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_p|. \tag{6}$$

Let $i = \sqrt{-1}$ and $\bar{\lambda}$ denote the complex conjugate of λ, $B(z) = 1 - \beta(z)$, and $B^{(1)}$ be the first derivative of B. We then have the following result.

THEOREM 1 *Consider a GARCH(p,q) model where $p \geq 2$. Let (A1) and (A2) be satisfied. Then the following holds:*

(a) $\omega^* \geq 0$ if and only if $\omega \geq 0$;

(b) Assuming the roots of $1 - \beta(z) = 0$ are distinct, and $|\lambda_1| < |\lambda_2|$, then Conditions (7) - (9) are necessary and sufficient for $\psi_k \geq 0$ for all positive integer k:

$$\lambda_1 \text{ is real, and } \lambda_1 > 1, \tag{7}$$

$$\alpha(\lambda_1) > 0, \tag{8}$$

$$\psi_k \geq 0, \text{ for } k = 1, \ldots, k^*, \tag{9}$$

where k^* is the smallest integer greater than or equal to $\{\log r_1 - \log((p-1)r^*)\}/(\log|\lambda_1| - \log|\lambda_2|)$,

$$r_j = -\frac{\alpha(\lambda_j)}{B^{(1)}(\lambda_j)}, \quad 1 \leq j \leq p, \quad \text{and} \quad r^* = \max_{2 \leq j \leq p} |r_j|.$$

The case $p = 1$ is trivial. See Theorem 1 of Nelson and Cao (1992). For $p = 2$, the k^* defined in Theorem 1 can be shown to be $q + 1$, cf. Theorem 2 of Nelson and Cao (1992). If the k^* defined in Theorem 1 is a negative number, then it can be seen from the proof that $\psi_k \geq 0$ for all positive k.

Requiring that $\psi_k \geq 0$ for all positive integer k imposes an infinite number of inequality constraints on $\{\alpha_j\}_{j=1}^q$ and $\{\beta_j\}_{j=1}^p$. For practical purposes (e.g. in estimation) it is necessary to reduce this to a finite number of inequalities. In Theorem 1 (b), we derive a set of verifiable necessary and sufficient conditions for the non-negativity of $\{\psi_j\}_{j=1}^\infty$ in terms of a finite number of inequalities under the weak condition that the characteristic equation $1 - \beta(z) = 0$ has distinct roots and the root of the smallest magnitude is unique. Indeed, the sufficiency part in part (b) has earlier been obtained by Nelson and Cao (1992), while the necessity of the conditions is a new result.

Tsai & Chan (2005) characterized the non-negativity of the kernel $\{\psi_k\}$ for any ARMA(p, q) model in terms of the absoulte monotonicity of its generating function. They made use of this characterization to show that a necessary condition for the kernel $\{\psi_k\}$ of a stationary ARMA(p, q) model to be non-negative is that λ_1 is real, and $\lambda_1 > 1$. Furthermore, for ARMA models of lower orders, they derived some readily verifiable necessary and sufficient conditions for the kernel $\{\psi_k\}$ to be non-negative.

We shall characterize the non-negativity of $\{\psi_i\}_{i=0}^\infty$ for any GARCH(p, q) model in terms of its generating function. For this purpose, we first recall the definition of the generating function (See Chapter XI of Feller, 1968). Let $\{p_i\}_{i=0}^\infty$ be a sequence of real numbers. If

$$u(x) = p_0 + p_1 x + p_2 x^2 + \cdots$$

converges in some interval $-x_0 < x < x_0$, where $x_0 > 0$, then $u(x)$ is called the generating function of the sequence $\{p_j\}$. For the $\{\psi_j\}$ defined by (3), its generating function is given by equation (5). The significance of the generating function of $\{\psi_i\}_{i=0}^\infty$ lies in the well-known result that the non-negativity of $\{\psi_i\}_{i=0}^\infty$ is equivalent to the absolutely monotonicity of its generating function (Feller, 1971, Theorem 2 of Chapter VII.2).

Now, we recall the definition of absolutely monotonicity; see Chapter VII of Feller (1971) and Chapter IV of Widder (1946) for further discussion. A continuous function $f(x)$ is absolutely monotone in the interval $a < x < b$ if it has non-negative derivatives of all orders there:

$$f^{(n)}(x) \geq 0, \qquad a < x < b, \qquad n = 0, 1, 2, \ldots.$$

Tsai and Chan (2005) exploited some properties of absolutely monotone functions to derive some necessary and sufficient conditions for an ARMA model to be non-negative. Now we state the non-negativity of $\{\psi_k\}$ in terms of the absolutely monotonicity of its generating function in the following theorem.

THEOREM 2 *Let (A1) and (A2) be satisfied. Then the following holds:*

(a) $\psi_k \geq 0$ for all positive integer k if and only if $\Psi(z) = \{1 - \beta(z)\}^{-1}\alpha(z), 0 \leq z < 1$, is absolutely monotone;

(b) if $\alpha_1 > 0$ and

$$\Psi_1(z) = \{1 - \beta(z)\}^{-1} \left\{ 1 + \frac{1}{\alpha_1} \sum_{j=1}^{q-1} \alpha_{j+1} z^j \right\}$$

is absolutely monotone, then $\psi_k \geq 0$ for all positive integer k.

Note that $\Psi_1(z)$ in (b) is the generating function of the kernel of an ARMA$(p, q-1)$ model. In particular, if $q = 1$, $\Psi_1(z) = \{1 - \beta(z)\}^{-1}$. Theorem 2 (b) provides a link with some recent results of Tsai and Chan (2005) about the non-negativity of ARMA models with that of GARCH models as stated in the following Theorem.

THEOREM 3 *Let (A1) and (A2) be satisfied. Then the following holds.*

(a) For a GARCH$(p, 1)$ model, if λ_j is real and $\lambda_j > 1$, for $j = 1, \ldots, p$, and $\alpha_1 \geq 0$, then $\psi_k \geq 0$ for all positive integer k.

(b) For a GARCH$(p, 1)$ model, if $\psi_k \geq 0$ for all positive integer k, then $\alpha_1 \geq 0$, $\sum_{j=1}^{p} \lambda_j^{-1} \geq 0$, λ_1 is real, and $\lambda_1 > 1$.

(c) For a GARCH$(3, 1)$ model, $\psi_k \geq 0$ for all positive integer k if and only if $\alpha_1 \geq 0$ and either of the following cases hold.

Case 1. all the λ_j's are real numbers, $\lambda_1 > 1$, and $\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} \geq 0$.

Case 2. $\lambda_1 > 1$, and $\lambda_2 = \bar{\lambda}_3 = |\lambda_2|e^{i\theta} = a + bi$, where a and b are real numbers, $b > 0$, and $0 < \theta < \pi$:

Case 2.1. $\theta = 2\pi/r$ for some integer $r \geq 3$, and $1 < \lambda_1 \leq |\lambda_2|$.

Case 2.2. $\theta \notin \{2\pi/r|\, r = 3, 4, ...\}$, and $|\lambda_2|/\lambda_1 \geq x_0 > 1$, where x_0 is the largest real root of $f_{n,\theta}(x) = 0$, and

$$f_{n,\theta}(x) = x^{n+2} - x\,\frac{\sin((n+2)\theta)}{\sin\theta} + \frac{\sin((n+1)\theta)}{\sin\theta}, \tag{10}$$

where n is the smallest positive integer such that $\sin((n+1)\theta) < 0$ *and* $\sin((n+2)\theta) > 0$.

(d) For a GARCH($3,1$) model, if $\lambda_2 = \bar{\lambda}_3 = |\lambda_2|e^{i\theta} = a + bi$, where a and b are real numbers, $b > 0$, and $a \geq \lambda_1 > 1$, then $\psi_k \geq 0$ for all positive integer k.

(e) For a GARCH($4,1$) model, if λ_j's are real, for $1 \leq j \leq 4$, then a necessary and sufficent condtion for $\{\psi_i\}_{i=0}^{\infty}$ to be non-negative is that $\alpha_1 \geq 0$, $\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} + \lambda_4^{-1} \geq 0$, and $\lambda_1 > 1$.

Remark 1: Note that x_0 is the only real root of equation (10) that is ≥ 1. See Tsai and Chan (2005).

Remark 2: Theorem 3 and the fact that the product of two absolutely monotone functions is again absolutely monotone (Theorem 2a of Widder, 1946, p. 145) can be used to construct simple sufficient conditions for GARCH(p, q) models from known results of GARCH$(p, 1)$ model. For example, if the ARCH coefficients (α's) of a GARCH(p, q) model are all non-negative, the model has non-negative conditional variances if the non-negativity property holds for the associated GARCH$(p, 1)$ models with a non-negative α_1 coefficient. As another example, consider a GARCH$(4, 1)$ model for which $\alpha_1 \geq 0$ and the roots of $1 - \beta(z) = 0$ satisfy condition (6); moreover, λ_1 and λ_4 are real numbers, $\lambda_4 > 1$, $\lambda_2 = \bar{\lambda}_3 = |\lambda_2|e^{i\theta} = a + bi$, where a and b are real numbers, and $a \geq \lambda_1 > 1$. Then by Theorem 3 (a), (b), and (d), $\{\psi_i\}_{i=0}^{\infty}$ is non-negative for this particular GARCH$(4, 1)$ model. These results complement the following well-known necessary and sufficient condition for the non-negativity of GARCH$(2, q)$ models obtained by Nelson and Cao (1992), which is repeated here for convenience of reference.

THEOREM 4 *(Nelson-Cao) Let (A1) and (A2) be satisfied. Then for a GARCH(2, q) model, $\psi_k \geq 0$ for all positive integer k if and only if the following conditions hold:*

$$\lambda_1 \text{ is real, and } \lambda_1 > 1, \tag{11}$$

$$\alpha(\lambda_1) > 0, \tag{12}$$

and

$$\psi_k \geq 0, \text{ for } k = 0, 1, ..., q. \tag{13}$$

4 An Empirical Example

As an illustration, we consider the daily USD/HKD (US dollar to Hong Kong dollar) exchange rate from January 1, 2005 to March 7, 2006, altogether 431 daily data; The data are available at `http://www.oanda.com/convert/fxhistory`. FXHistory is a user-friendly front-end for accessing the largest foreign exchange database on the Internet. These data are non-stationary, so we analyze the first difference of the logarithmically transformed daily exchange rates; see Fig. 1. After first differencing, the data appear to be stationary although it is clearly heteroscedastic with volaility clustering. An AR(1)-GARCH(3,1) model was fitted to the data, with an additive outlier on July 22, 2005, the date when China revalued the yuan by 2.1 percent and adopted a floating-rate system for the yuan. The intercept term in the conditional mean function was found to be not significantly different from zero, and hence it is omitted from the model; thus the returns have zero mean unconditionally. The fitted model has an AIC=-2070.9, being smallest among various competing (weakly) stationary models, see Table 1. Interestingly, for lower GARCH orders ($p \leq 2$), the fitted models are non-stationary but the fitted models are largely stationary when the GARCH order is higher than 2. As the data appear to be stationary, we choose the AR(1)-GARCH(3,1) model as the final model.

The models were fitted by Proc Autoreg of SAS. We used the default option of imposing the Nelson-Cao inequality constraints for the GARCH variance process to be non-negative. However, the inequality constraints so imposed are only necessary and sufficient for the non-negativity of the conditional variances of a GARCH(p, q) model for $p \leq 2$. For higher-order GARCH models, Proc Autoreg imposes the constraints that (1) $\psi_k \geq 0, 1 \leq k \leq \max(q-1, p)+1$ and (2) the non-negativity of the in-sample conditional variances, see SAS 9.1.3 Help and Documentation manual. Hence, higher-order GARCH models estimated by Proc Autoreg with the Nelson-Cao option need not have non-negative conditional variances with probability one. For the Hong Kong exchange rate data, the fitted model from Proc Autoreg is listed in Table 2, with the estimated conditional variances shown in Fig 2. Note that the GARCH2 (β_2) coefficient estimate is negative.

Since both the intercept and the ARCH coefficient are positive, we can apply Theorem 3(c) to check whether or not the conditional variance process defined by the fitted model is always non-negative. The characteristic equation $1 - \beta(z) = 0$ admits three roots equal to 1.153728, and $-0.483294 \pm 1.221474i$, so $\lambda_1 = 1.153728$ and $|\lambda_2|/\lambda_1 = 1.138579$. Based on numerical computations, n in Equation (10) turns out to be 2 and Equation (10) has one real root equal to 1.1385751 which is strictly less than $1.138579 = |\lambda_2|/\lambda_1$. Hence, we can conclude that the fitted model always results in non-negative conditional variances.

5 Conclusion

The preceding example illustrates the potential usefulness of the newly derived inequality constraints for verifying the nonnegativity of the conditional variance process of a higher order GARCH model. We have obtained in Theorem 1 a finite set of necessary and sufficient conditions for the nonnegativity of the conditional variance process of a GARCH model under the mild regularity conditions that $|\lambda_1| < |\lambda_2|$. This regulairty condition fails for a seasonal GARCH model, i.e. $\beta(z)$ is a polynomial in B^s for some positive integer s. Further research extending Theorems 1 and 3 for the seasonal case constitutes an interesting research problem.

Acknowledgements

We thank the National Science Council (NSC 94-2118-M-001-015), R.O.C., and the National Science Foundation (DMS-0405267) for partial support.

APPENDIX

Proof of Theorem 1.

We first prove part (a). By Equation (3), $\omega^* = \{1 - \beta(1)\}^{-1}\omega$. Furthermore, Condition (A1) on the root of $1 - \beta(1)$ implies $1 - \beta(1) > 0$. Thus, $\omega^* \geq 0$ if and only if $\omega \geq 0$. This proves part (a).

For part (b), the necessity of (9) is obvious. The necessity of (7) and (8) can be proved as follows. By Equations (4.8) and (4.9) of Feller (1968, p. 276 and p. 277), we have, for $n \geq \max\{p, q\} + 1$,

$$\psi_n = \sum_{i=1}^{p} \frac{r_i}{\lambda_i^{n+1}} \tag{14}$$

$$\sim \frac{r_1}{\lambda_1^{n+1}}, \tag{15}$$

where "$\sim$" means that the ratio of the two sides tends to 1, as $n \to \infty$. Thus, λ_1 must be real and > 1. Moreover, $r_1 = -\alpha(\alpha_1)/B^{(1)}(\lambda_1)$ must be ≥ 0. Note also that

$$-B^{(1)}(\lambda_1) = \frac{1}{\lambda_1} \prod_{j=2}^{p} \left(1 - \frac{\lambda_1}{\lambda_j}\right),$$

and by (6), $-B^{(1)}(\lambda_1) > 0$. Hence, $\alpha(\lambda_1) \geq 0$. But $\alpha(\lambda_1) \neq 0$ by assumption (A2). This proves the necessity of $\alpha(\lambda_1) > 0$. The proof of (b) for the sufficiency of Conditions (7) - (9) was given in Nelson and Cao (1992).

Proof of Theorem 2.

Part (a) follows from Feller (1971, Theorem 2 of Chapter VII.2).

Part (b) follows from the equality

$$\Psi(z) = \alpha_1 z \Psi_1(z).$$

and the fact that the product of two absolutely monotone functions is still absolutely monotone (Theorem 2a of Chapter IV, Widder, 1946).

Proof of Theorem 3.

Parts (a)-(e) are conditions on GARCH(p, q) models with $q = 1$, which can be proved as follows. For $q = 1$, $\alpha(z) = \alpha_1 z$, and by equation (4.8) of Feller (1968, p. 276), we have, for $n = 1, 2, 3, ...,$

$$\psi_n = \sum_{i=1}^{p} \frac{-\alpha(\lambda_i)}{B^{(1)}(\lambda_i)\lambda_i^{n+1}} = \alpha_1 \sum_{i=1}^{p} \frac{-1}{B^{(1)}(\lambda_i)\lambda_i^{n}} = \alpha_1 \psi_{n-1}^*,$$

where $\{\psi_n^*\}_{j=0}^{\infty}$ is the kernel of the AR(p) model:

$$X_t = B^{-1}(L)a_t = \sum_{k=0}^{\infty} \psi_k^* L^k a_t,$$

and $\{a_t\}$ is a sequence of iid random variables. Some necessary and/or sufficient conditions for the non-negativity of the AR(p) models are given by Theorem 2.1 of Tsai and Chan (2005). The proofs of parts (a)-(e) now follow from Theorem 2.1 (b), (d), (g), and (h) of Tsai and Chan (2005). This complets the proof of Theorem 3.

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AR order	GARCH order (p)	ARCH order (q)	AIC	Stationarity
0	3	1	-1915.3	non-stationary
1	1	1	-2054.3	non-stationary
1	1	2	-2072.5	non-stationary
1	1	3	-2051.0	non-stationary
1	2	1	-2062.2	non-stationary
1	2	2	-2070.5	non-stationary
1	2	3	-2059.2	non-stationary
1	3	1	-2070.9	stationary
1	3	2	-2064.8	stationary
1	3	3	-2062.8	stationary
1	4	1	-2061.7	non-stationary
1	4	2	-2054.8	stationary
1	4	3	-2062.4	stationary
2	3	1	-2066.6	stationary

Table 1: AIC of various models fitted to the daily returns of USD/HKD exchange rate.

coefficients	estimate	std. err.	t-ratio	p-value
AR1	0.1635	0.005892	-21.29	0.0022
ARCH0 (ω)	2.374×10^{-5}	6.93×10^{-6}	3.42	0.0006
ARCH1 (α_1)	0.2521	0.0277	9.09	$<.0001$
GARCH1 (β_1)	0.3066	0.0637	4.81	$<.0001$
GARCH2 (β_2)	-0.09400	0.0391	-2.41	0.0161
GARCH3 (β_3)	0.5023	0.0305	16.5	<0.0001
outlier	-0.1255	0.00589	-21.29	<0.0001

Table 2: Fitted AR(1)-ARCH(3,1) model for the daily returns of USD/HKD exchange rate. The model has an additive outlier on July 22, 2005.

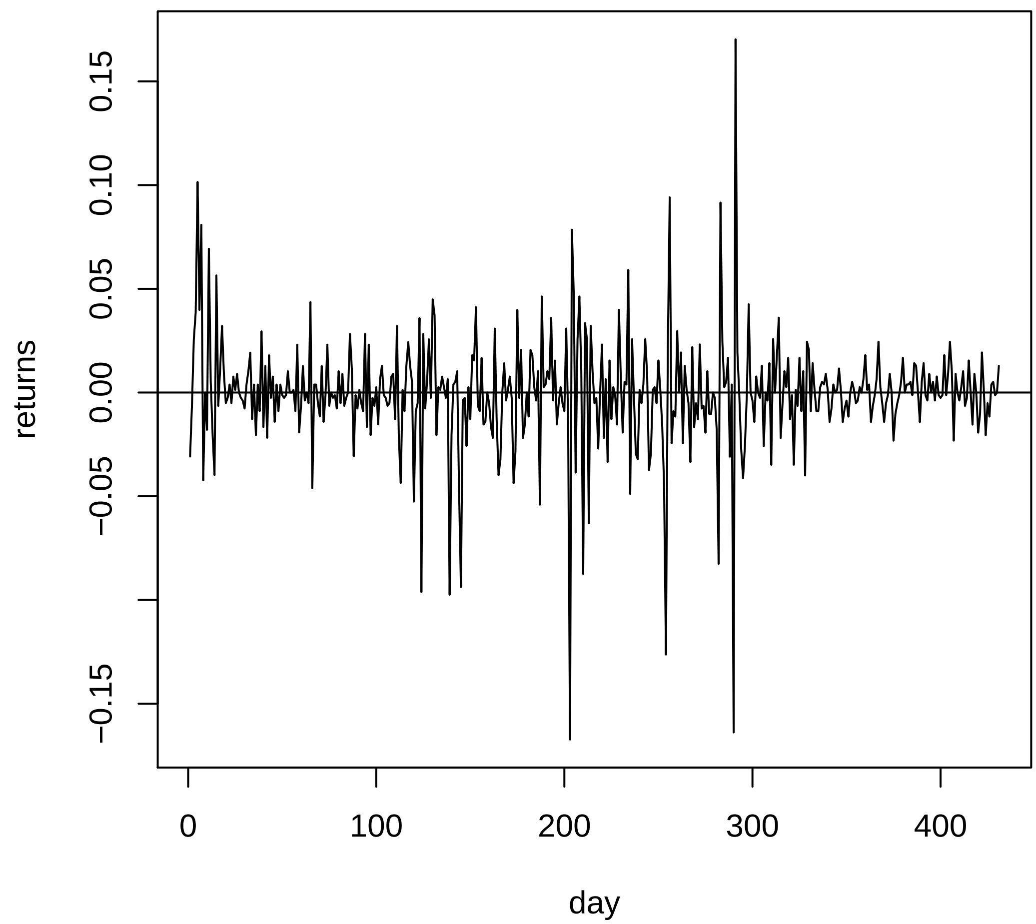

Figure 1: Daily returns of USD/HKD exchange rate from January 1, 2005 to March, 7, 2006.

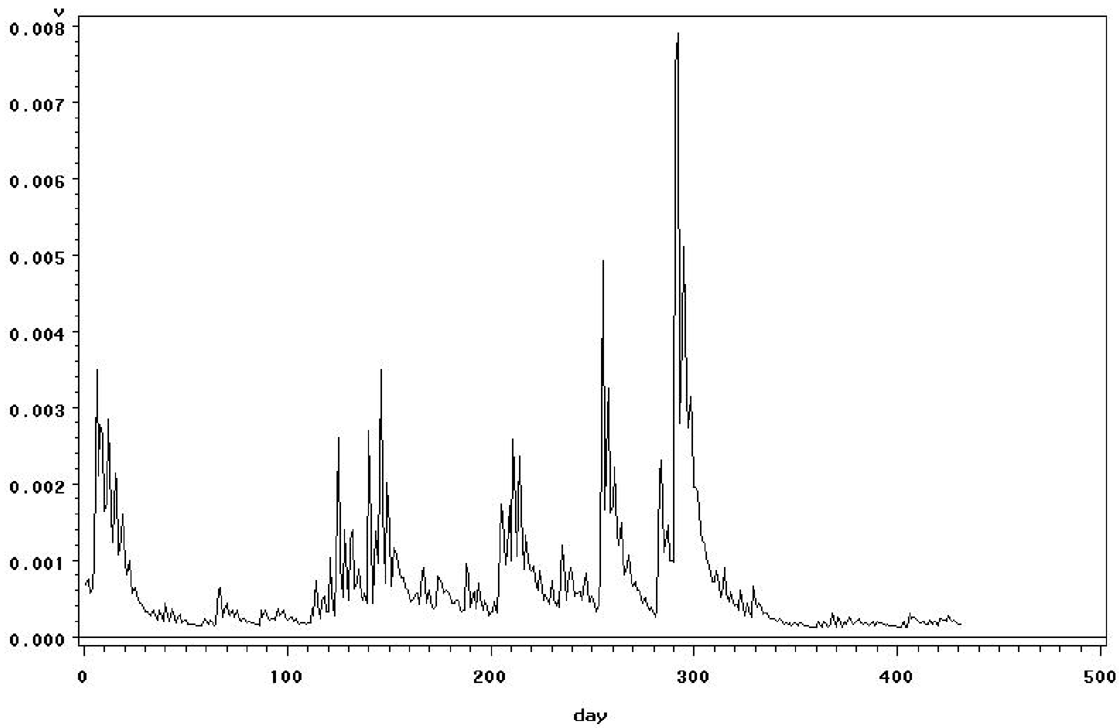

Figure 2: Estimated conditional variances of the daily returns of USD/HKD exchange rate data from the fitted AR(1)-GARCH(3,1) model.